# THE MINKOWSKI $?(x)$ FUNCTION, A CLASS OF SINGULAR MEASURES, QUASI-MODULAR AND MEAN-MODULAR FORMS. I

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Abstract. The Minkowski question mark function is a rich object which can be explored from the perspective of dynamical systems, complex dynamics, metric number theory, multifractal analysis, transfer operators, integral transforms, and as a function itself via analysis of continued fractions and convergents. Our permanent target, however, was to get an arithmetic interpretation of the moments of $?(x)$ (which are relatives of periods of Maass wave forms) and to relate the function $?(x)$ to certain modular objects. In this paper we establish this link, embedding $?(x)$ not into the modular-world itself, but into a space of functions which are generalizations and which we call *mean-modular forms*. For this purpose we construct a wide class of measures; in real case these were already defined by Denjoy in 1936. From this perspective, the modular forms for the whole modular group as well as the Stieltjes transform of $?(x)$, the so-called *dyadic period function*, minus the Eisenstein series of weight 2, fall under the same uniform definition. The main result is the construction of the canonical isomorphism between the $\mathbb{C}$-vector spaces of quasi-modular forms and mean-modular forms. This gives $?(x)$-related interpretation of quasi-modular forms. Via this construction, we find another canonical epimorphism from the space of mean-modular forms of weight $k$ to the space of modular forms of weight $k-2$. The fourth power of the Dedekind $\eta$-function does appear naturally in this construction.

## 1. Introduction

The relation between continued fractions and modular functions is and old and deep subject; see, for example, [11, 14, 18]. In this paper we provide yet another example of this relation of a very different sort.

The Minkowski question mark function $?(x):[0,1]\mapsto[0,1]$ is defined by

$$?([0,a_1,a_2,a_3,\ldots])=2\sum_{\ell=1}^{\infty}(-1)^{\ell+1}2^{-\sum_{j=1}^{\ell}a_j},\quad a_j\in\mathbb{N};$$

$x=[0,a_1,a_2,a_3,\ldots]$ stands for a representation of $x$ by a regular continued fraction. In view of the current paper, note that the Minkowski question mark function can be defined also in

*Date*: April 3, 2018.

2010 *Mathematics Subject Classification.* Primary 11F11, 26A30, 11F03.

*Key words and phrases.* Minkowski question mark function, quasi-modular forms, singular measures, dyadic period function.

The research of the author was supported by the Research Council of Lithuania grant No. MIP-072/2015. The author sincerely acknowledges hospitality of Prof. J. Steuding in Würzburg University, Prof. D. Mayer in Clausthal Technical University, Max Planck Institute for Mathematics in Bonn, and also thanks Prof. N. Diamantis and Prof. K. Bringmann for showing interest in these results.

terms of semi-regular continued fractions. These are given by

$$[[b_1,b_2,b_3,\ldots]]=\cfrac{1}{b_1-\cfrac{1}{b_2-\cfrac{1}{b_3-\ddots}}},$$

where integers $b_i\geq 2$. Each real irrational number $x\in(0,1)$ has a unique representation in this form, and rationals $x\in(0,1)$ have two representations: one finite and one infinite which ends in $[[2,2,2,\ldots]]=1$. It was proved in [4] that

$$?([[b_1,b_2,b_3,\ldots]])=\sum_{\ell=1}^{\infty}2^{\ell-\sum_{j=1}^{\ell}b_j}. \tag{1}$$

The function $?(x)$ is continuous, strictly increasing, and singular. For $x\in[0,1]$, it satisfies functional equations

$$?(x)=\begin{cases}1-?(1-x),\\ 2?\big(\frac{x}{x+1}\big).\end{cases}$$

These equations are responsible for the rich arithmetic nature of $?(x)$ and its relations (at least analogies) to the objects in the modular-world [1, 2]: for example, if we define

$$G(z)=\int\limits_0^1\frac{x}{1-xz}\,\mathrm{d}?(x),$$

then $G(z)=o(1)$ if $z\to\infty$ and the distance to $\mathbb{R}_+$ remains bounded away from 0, and

$$\frac{1}{z}+\frac{1}{z^2}G\Big(\frac{1}{z}\Big)+2G(z+1)=G(z),\quad z\in\mathbb{C}\setminus[1,\infty).$$

In this paper we exhibit explicitly the connection of $?(x)$ to the modular world. The factor "2" in the above formula - an intrinsic constant which comes from the dyadic nature of $?(x)$ - was always an obstacle which prevented an application of many techniques (Hecke operators, modularity, Fourier series) to the theory of $?(x)$. Now it appears that there exists a natural way to integrate $?(x)$ into the modular world, and this factor "2" is no longer an obstacle but rather the reason why this integration is possible. For this purpose, first, we construct a wide generalization of $?(x)$.

## 2. A class of functions

Here we present a new way to construct a wide class of continuous fractal functions which encode the self-similarity via semi-regular continued fractions.

**Proposition 1.** *Let $\mathbf{q}=\{q_\ell:2\leq\ell<\infty\}$ be the sequence of complex numbers such that*

$$\sum_{\ell=2}^{\infty}q_\ell=1,\quad\sum_{\ell=2}^{\infty}|q_\ell|<+\infty,\quad\sup_{\ell}|q_\ell|<1.$$

*Then there exists the function $\mu=\mu_{\mathbf{q}}:[0,1]\mapsto\mathbb{C}$ with the following properties.*

1) *It is continuous, $\mu(0)=0$, $\mu(1)=1$.*

2) *The function* $\mu$ *is of bounded variation. If all* $q_\ell$ *are real and non-negative, then* $\mu$ *is non-decreasing; if all* $q_\ell$ *are strictly positive, then* $\mu$ *is strictly increasing.*
3) *The function* $\mu$ *has the following self-similarity property:*

$$\mu\Big(\frac{1}{\ell - x}\Big) = q_\ell \cdot \mu(x) + \sum_{j=\ell+1}^{\infty} q_j, \quad 2 \leq \ell < \infty, \quad x \in [0,1].$$

4) *if* $q_\ell = 2^{1-\ell}$, $\ell \geq 2$, *then* $\mu(x) = ?(x)$.

*Proof.* To construct such a function, we use iterations. As an initial state, set $\mu_0(x) = x$, $x \in [0,1]$. Then define $\mu_{w+1}$ piecewise recurrently by

$$\mu_{w+1}(x) = q_\ell \cdot \mu_w\Big(\ell - \frac{1}{x}\Big) + \sum_{j=\ell+1}^{\infty} q_j, \quad x \in \Big[\frac{1}{\ell}, \frac{1}{\ell-1}\Big], \quad w \geq 0.$$

By induction we see that $\mu_{w+1}(0) = 0$, $\mu_{w+1}(1) = 1$, and that $\mu_w$ is continuous. Indeed, at the points $x = \frac{1}{\ell}$, $\ell \geq 2$, where this definition is ambiguous, both values coincide, since

$$\mu_{w+1}\Big(\frac{1}{\ell}\Big) = q_\ell \cdot \mu_w(0) + \sum_{j=\ell+1}^{\infty} q_j = q_{\ell+1}\mu_w(1) + \sum_{j=\ell+2}^{\infty} q_j.$$

Now, consider the following series

$$\mu_0(x) + \sum_{w=0}^{\infty} \big(\mu_{w+1}(x) - \mu_w(x)\big). \tag{2}$$

Let $\sup_\ell |q_\ell| = \delta < 1$, and $\sup_{[0,1]} |\mu_1(x) - \mu_0(x)| = M$. By the very construction,

$$\mu_{w+1}(x) - \mu_w(x) = q_\ell \cdot \Big(\mu_w\Big(\ell - \frac{1}{x}\Big) - \mu_{w-1}\Big(\ell - \frac{1}{x}\Big)\Big), \quad x \in \Big[\frac{1}{\ell}, \frac{1}{\ell-1}\Big], \quad w \geq 1.$$

So, for $w \geq 1$,

$$\sup_{x\in[0,1]} |\mu_{w+1}(x) - \mu_w(x)| \leq \delta \cdot \sup_{x\in[0,1]} |\mu_w(x) - \mu_{w-1}(x)|.$$

Thus, the series (2) is majorized by the series $\sum_w M\delta^w$, and so the function

$$\mu(x) = \lim_{w\to\infty} \mu_w(x)$$

is continuous and satisfies all of the needed properties, as can be checked. □

We call this function $\mu_{\mathbf{q}}$ the $\mathbf{q}-$*question mark function*. For example, the Figures 1,2,3 shows the graph of these in cases $\mathbf{q} = (\frac{2}{3}, \frac{1}{3}, 0, 0, \ldots)$, $\mathbf{q} = (\frac{4}{7}, \frac{2}{7}, \frac{1}{7}, 0, 0, \ldots)$, $\mathbf{q} = (\frac{4}{7}, \frac{4}{7}, -\frac{1}{7}, 0, 0, \ldots)$.

For a given collection $\mathbf{q} = \{q_\ell : 2 \leq \ell < \infty\}$, which satisfies the requirements of Proposition 1, let $\sum\limits_{j=\ell+1}^{\infty} q_j = Q_\ell$. Let $x \in [0,1]$ be given by a semi-regular continued fraction $[[b_1, b_2, b_3, \ldots]]$. Then the explicit expression for $\mu_{\mathbf{q}}$, according to the property 3) of Proposition 1 , is given by

$$\mu_{\mathbf{q}}(x) = Q_{b_1} + q_{b_1}Q_{b_2} + q_{b_1}q_{b_2}Q_{b_3} + q_{b_1}q_{b_2}q_{b_3}Q_{b_4} + \cdots . \tag{3}$$

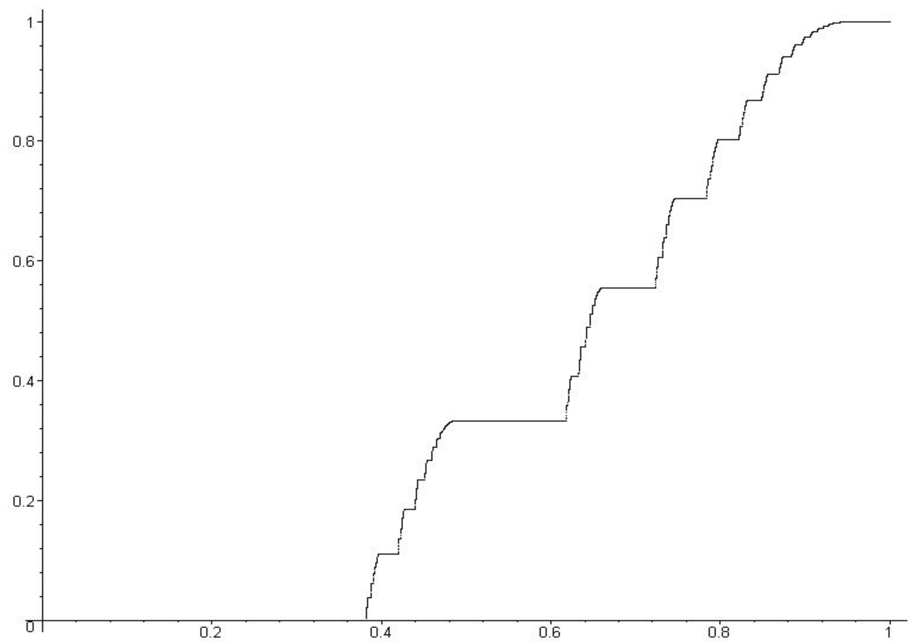



FIGURE 1. $(\frac{2}{3}, \frac{1}{3})$-question mark function, $x \in [0,1]$

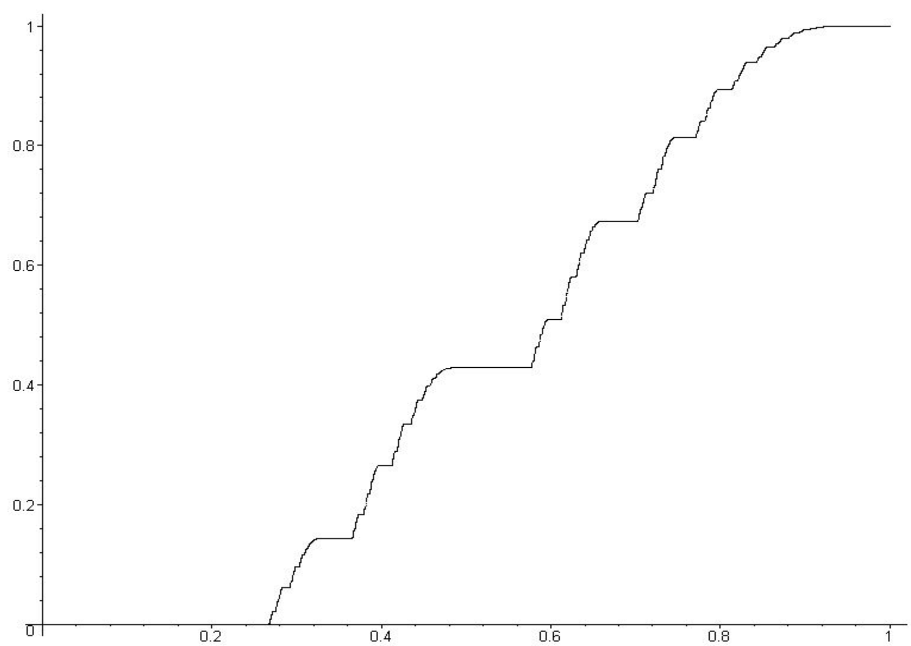



FIGURE 2. $(\frac{4}{7}, \frac{2}{7}, \frac{1}{7})$-question mark function, $x \in [0,1]$

As an aside, let us define

$$\mathfrak{m}_{\mathbf{q}}(s) = \int\limits_{0}^{1} e^{xs}\,\mathrm{d}\mu_{\mathbf{q}}(x), \quad p_{\mathbf{q}}(s) = \sum_{\ell=2}^{\infty} q_{\ell} e^{-i\ell s}.$$

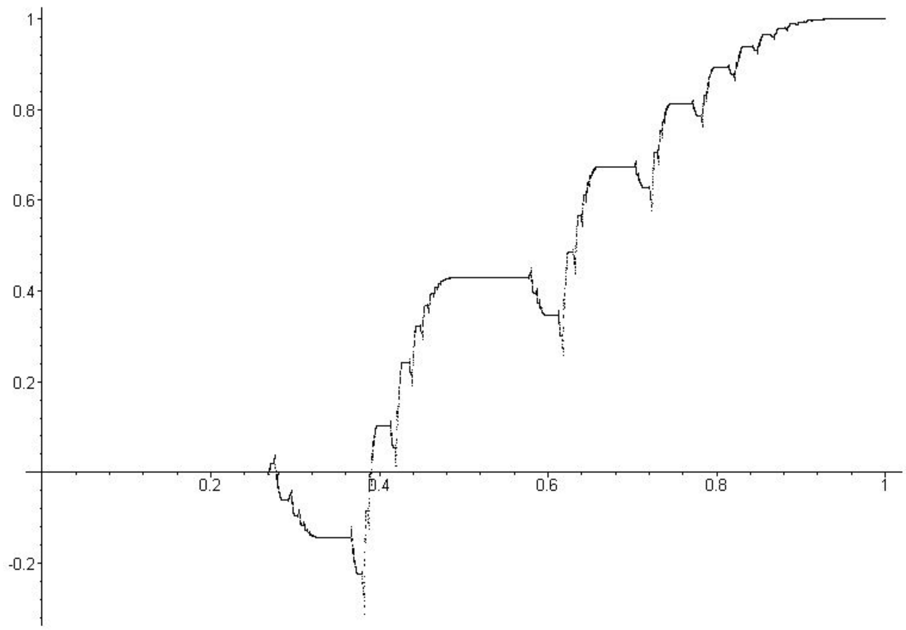



FIGURE 3. $(\frac{4}{7}, \frac{4}{7}, -\frac{1}{7})$-question mark function, $x \in [0, 1]$

FIGURE 4. The Minkowski question mark function, $x \in [0, 1]$

It is unknown whether $\mathfrak{m}_{\mathbf{q}}(is)$ vanishes at infinity for $s \in \mathbb{R}$ in case of the Minkowski question mark function - this is *the Salem's problem* [5, 6]. Most likely, all $\mathfrak{m}_{\mathbf{q}}(is)$ vanish at infinity. It is out of the scope of the current paper, but we mention that the integral equation for the

Laplace-Stieltjes transform of $?(x)$, defined by [1]

$$\mathfrak{m}(s)=\int\limits_0^1 e^{xs}\,\mathrm{d}?(x),\quad s\in\mathbb{C},$$

is compatible with this much more general construction. So, the function $\mathfrak{m}_{\mathbf{q}}(s)$ is entire, and it satisfies the following integral equation

$$i\,\mathfrak{m}_{\mathbf{q}}(is)p_{\mathbf{q}}(s)=\int\limits_0^\infty \mathfrak{m}'_{\mathbf{q}}(it)J_0(2\sqrt{st})\,\mathrm{d}t,\quad s>0;$$

the integral converges conditionally. On the other hand, the three term functional equation for the Stieltjes transform of $?(x)$ is compatible only with a narrow one parameter subclass of such $\mathbf{q}$'s which we introduce now, since this is our main object.

## 3. A SPECIAL SUBCLASS

3.1. **Semi-regular versus regular continued fractions.** We will henceforth focus on the important sequence $\mathbf{q}$ given by $q_\ell=(1-\varkappa)\varkappa^{\ell-2}$, $\ell\geq 2$, $\varkappa\in\mathbb{C}$, $|\varkappa|<1$, $|1-\varkappa|<1$. Let therefore $\mu_{\mathbf{q}}=\mu_\varkappa$ in this case. Note that now we have

$$Q_\ell=\sum_{j=\ell+1}^\infty q_j=\varkappa^{\ell-1}.$$

So, the explicit formula (3) gives

$$\mu_\varkappa([[b_1,b_2,b_3,\ldots]])=\sum_{\ell=1}^\infty(1-\varkappa)^{\ell-1}\varkappa^{\sum\limits_{j=1}^{\ell}b_j-2\ell+1}. \tag{4}$$

In particular, for $\varkappa=\frac{1}{2}$ we recover the expression (1). Also,

$$\mu_\varkappa(1)=\mu_\varkappa([[2,2,2,\ldots]])=\sum_{\ell=1}^\infty(1-\varkappa)^{\ell-1}\varkappa=1.$$

Let $x=[0,a_1,a_2,a_3,\ldots]$ be given by a regular continued fraction. As was proved in [20], and used in [4], one has

$$x=[0,a_1,a_2,a_3,\ldots]=[[a_1+1,2_{a_2-1},a_3+2,2_{a_4-1},a_5+2,2_{a_6-1},\ldots]].$$

(Note that $a_1$ is slightly exceptional). Here $2_n=\underbrace{2,2,\ldots,2}_{n}$. We plug now the right hand side into (4). This gives

$$\mu_\varkappa(x)=\varkappa^{a_1-1}-\varkappa^{a_1-1}(1-\varkappa)^{a_2}+\varkappa^{a_1-1}(1-\varkappa)^{a_2}\varkappa^{a_3}-\varkappa^{a_1-1}(1-\varkappa)^{a_2}\varkappa^{a_3}(1-\varkappa)^{a_4}+\cdots.$$

So, we here once again rediscover the generalization of $?(x)$ which was first given by Denjoy, and then explored by Tichy and Uitz [22] and Zhabitskaya [23]. Moreover, we see that this function is well-defined, is continuous, is bounded and is of bounded variation (the latter can be checked with some work) on the boundary of $\mathcal{D}=\{|\varkappa|<1,|1-\varkappa|<1\}$, except, possibly, at the two points $\varkappa = e^{\pm\frac{\pi i}{3}}$. We will soon see that these two points are crucial in modular interpretation of the function $\mu_{\varkappa}$. For, example, the last formula gives

$$\begin{aligned}
\mu_k([0,1,1,1,\ldots]) &= \mu_{\varkappa}\Big(\frac{\sqrt{5}-1}{2}\Big) = \frac{\varkappa}{1-\varkappa+\varkappa^2},\\
\mu_{\varkappa}([0,2,2,2,\ldots]) &= \mu_{\varkappa}(\sqrt{2}-1) = \frac{2\varkappa^2-\varkappa^3}{(1-\varkappa+\varkappa^2)(1+\varkappa-\varkappa^2)},\\
\mu_{\varkappa}([0,3,3,3,\ldots]) &= \mu_{\varkappa}\Big(\frac{\sqrt{13}-3}{2}\Big) = \frac{3\varkappa^3-3\varkappa^4+\varkappa^5}{(1-\varkappa+\varkappa^2)(1+\varkappa-2\varkappa^3+\varkappa^4)},\\
\mu_{\varkappa}([0,1,2,1,2,1,\ldots]) &= \mu_{\varkappa}(\sqrt{3}-1) = \frac{2\varkappa^2-\varkappa^3}{1-\varkappa+2\varkappa^2-\varkappa^3}.
\end{aligned}$$

We see that for all points $\theta \in (0,1)$ which are equivalent to $\frac{\sqrt{n^2+4}-n}{2}$ for a certain $n \in \mathbb{N}$, meaning that their continued fraction terminates at $\bar{n} = [n,n,n,\ldots]$, $\mu_{\varkappa}(\theta)$ is an expression which has $1-\varkappa+\varkappa^2$ in the denominator, and so

$$\lim_{\substack{\varkappa\to\varrho\\ |\varkappa|\leq 1, |1-\varkappa|\leq 1}} \mu_{\varkappa}(\theta) = \infty.$$

Therefore we should be careful in investigating the Stieltjes trasnform of $\mu_{\varkappa}$, which is to come next, at $\varkappa = \varrho$.

Now we will describe the symmetry property of $\mu_{kk}(x)$. First, let $\ell \geq 2$. Then

$$\begin{aligned}
\mu_{\varkappa}\Big(\frac{1}{\ell}\Big) &= \mu_{\varkappa}([0,\ell]) = \mu^{\ell-1},\\
\mu_{\varkappa}\Big(1-\frac{1}{\ell}\Big) &= \mu_{\varkappa}([0,1,\ell-1]) = 1-(1-\varkappa)^{\ell-1}.
\end{aligned}$$

Therefore, based on the example of the Minkowski $?(x)$ function, we can conjecture that in general,

$$\mu_{\varkappa}(x) + \mu_{1-\varkappa}(1-x) \equiv 1. \tag{5}$$

This indeed does hold, and this can be demonstrated with analysis of quotients. We will show that this identity holds using another method, by employing the Stieltjes transform, which is to come next.

3.2. **Stieltjes transform.** Let us define

$$G(\varkappa,z) = \int\limits_0^1 \frac{1}{\frac{1}{x}-z}\,\mathrm{d}\mu_{\varkappa}(x), \quad z \in \mathbb{C}\setminus[1,\infty). \tag{6}$$

Note that this implies

$$\frac{\partial^s}{\partial z^s} G(\varkappa,z) = s! \int\limits_0^1 \frac{1}{(\frac{1}{x}-z)^{s+1}}\,\mathrm{d}\mu_{\varkappa}(x).$$

So, $G(\varkappa,z)$ is a holomorphic function in both variables. Let $\mathcal{D} = \{\varkappa \in \mathbb{C} : |\varkappa| < 1, |1-\varkappa| < 1\}$. This is the definition domain of the function $G(\varkappa,z)$ in variable $\varkappa$. If $\varkappa \in \mathcal{D}$ and $\varkappa \to 0_+$, then the function $\mu_\varkappa$ tends pointwise to the function which is 0 in $[0,1)$ and 1 at $x=1$. Thus,

$$\lim_{\varkappa\to 0_+} G(\varkappa,z)=G(0,z)=\frac{1}{1-z}. \tag{7}$$

This satisfies the functional equation (9) in case $\varkappa=0$. On the other hand, if $\varkappa\to 1_-$, then the function $\mu_\varkappa$ tends pointwise to the function which is 0 at $x=0$ and 1 in the interval $(0,1]$. Thus, we also get

$$\lim_{\varkappa\to 1_-} G(\varkappa,z)=G(1,z)\equiv 0. \tag{8}$$

**Theorem 1.** *The function $G(\varkappa,z)$ satisfies the three term functional equation (we don not count the crucial non-homogeneous part)*

$$G(\varkappa,z+1)-\varkappa G(\varkappa,z)=\frac{(1-\varkappa)}{(1-z)^2}G\Big(\varkappa,\frac{1}{1-z}\Big)+\frac{1-\varkappa}{1-z},\quad z\in\mathbb{C}\setminus[1,\infty). \tag{9}$$

*In particular,*

$$G(\varkappa,1)-G(\varkappa,0)=\frac{1-\varkappa}{\varkappa}.$$

*Moreover, $G(\varkappa,z)=o(1)$ if $z\to\infty$ remains bounded away from $[1,\infty)$. More precisely: for $z\to\infty$ under the same condition, we have*

$$G(\varkappa,z)\sim-\frac{1}{z}+\frac{\alpha}{z^2},\ \text{where } \alpha=G(\varkappa,0)-\frac{2-\varkappa}{1-\varkappa},\quad \frac{\partial^s}{\partial z^s}G(\varkappa,z)=O\Big(z^{-(s+1)}\Big).$$

If the function $f(\varkappa,z)$ satisfies ...

*Proof.* First, we note the identity

$$\int\limits_0^1 f(x)\,\mathrm{d}\mu_\varkappa(x)=\sum_{\ell=2}^{\infty}(1-\varkappa)\varkappa^{\ell-2}\int\limits_0^1 f\Big(\frac{1}{\ell-x}\Big)\,\mathrm{d}\mu_\varkappa(x),$$

provided that all integrals are absolutely convergent. This follows from Proposition 1, the Property 3. In the special case, for $f(x)=(\frac{1}{x}-z)^{-1}$, this reduces to

$$G(\varkappa,z)=\int\limits_0^1\frac{1}{\frac{1}{x}-z}\,\mathrm{d}\mu_\varkappa(x)=\sum_{\ell=2}^{\infty}(1-\varkappa)\varkappa^{\ell-2}\int\limits_0^1\frac{1}{\ell-x-z}\,\mathrm{d}\mu_\varkappa(x).$$

Thus,

$$G(\varkappa,z+1)-\varkappa G(\varkappa,z)=(1-\varkappa)\int\limits_0^1\frac{1}{1-x-z}\,\mathrm{d}\mu_\varkappa(x).$$

Now, let us use the identity

$$\frac{1}{1-x-z}=\frac{1}{(1-z)^2}\cdot\frac{1}{\frac{1}{x}-\frac{1}{1-z}}+\frac{1}{1-z}.$$

This gives the functional equation (9). The regularity property is immediate. □

Let $U,S,I,T,R$ be the standard $2\times 2$ matrices:

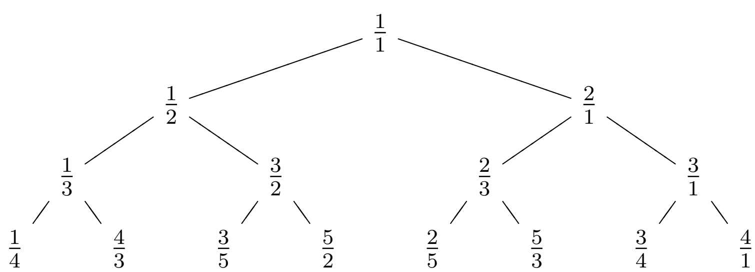
$$U=\begin{pmatrix}0&1\\-1&1\end{pmatrix},\quad S=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad T=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad R=\begin{pmatrix}1&0\\1&1\end{pmatrix}.$$

The matrices $U,S$ satisfy $U^3=S^2=I$, and freely generate the modular group, subject to these two relations, while $T=U^2S$, $R=US$ (all relations are considered modulo $\pm I$), and $T,R$ freely generate the monoid, with no relations.

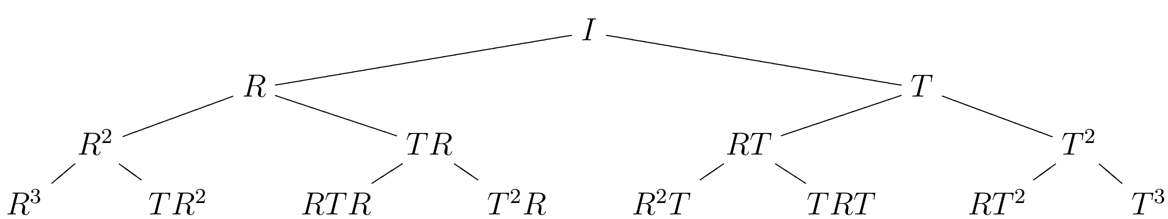
We will also need the matrix

$$W=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

of determinant $-1$, $W^2=I$, and which thus does not belong to the modular group. Recall that the Calkin-Wilf tree $\mathscr{T}$ is a binary tree which enumerates all positive rational numbers [12]. It stars from the node $\frac{1}{1}$, and each node $x$ generates two offsprings - the left one $\frac{x}{x+1}$, and the right one $x+1$:




Equally, we can construct the tree $\mathscr{T}_{\mathbf{M}}$ that enumerates the free monoind $\mathbf{M}=\{R,T\}$. It starts from the matrix $I$, and each node-matrix $M$ generates two offsprings - the left one $RM$, and the right one $TM$:




The bijection between both trees is given by $t:\mathscr{T}_{\mathbf{M}}\mapsto\mathscr{T}$, $t:\begin{pmatrix}a&b\\c&d\end{pmatrix}\mapsto\frac{a+b}{c+d}$.

3.3. **The symmetry property.** Let $f(\varkappa,z)=G(\varkappa,z+1)$, and let us rewrite the functional equation (19) as

$$f=\varkappa f|T^{-1}+(1-\varkappa)f|R^{-1}+\frac{1-\varkappa}{1-z}.$$

Let us apply $|W$ to this, and write the result as

$$f|W=\varkappa f|W^2T^{-1}W+(1-\varkappa)f|W^2R^{-1}W+\frac{1-\varkappa}{z(z-1)}.$$

But now,

$$WT^{-1}W = R^{-1}, \quad WR^{-1}W = T^{-1}.$$

So, if we denote $g = f|W$, we obtain the functional equation

$$g = \varkappa g|R^{-1} + (1-\varkappa)g|T^{-1} + \frac{1-\varkappa}{z(z-1)}.$$

This is almost the same, as (19), only $\varkappa$ and $1-\varkappa$ are swapped, and also the non-homogeneous term is different. Let, finally $g(\varkappa, z) = h(\varkappa, z) - \frac{1}{z}$. Then by a direct calculation,

$$h = \varkappa h|R^{-1} + (1-\varkappa)h|T^{-1} + \frac{\varkappa}{z-1}.$$

By a uniqueness property, $h(\varkappa, z) = -f(1-\varkappa, z)$, and so we have proved

**Proposition 2.** *The function $G(\varkappa, z)$ satisfies the following symmetry property*

$$\frac{1}{z^2}G\Big(\varkappa, \frac{1}{z}+1\Big) = -G(1-\varkappa, z+1) - \frac{1}{z}.$$

So, as expected, the case of the Minkowski question mar function, namely, $\varkappa = 1$, is the only one where we can compute

$$G(\varkappa, 0) = \int\limits_0^1 x \,\mathrm{d}\mu_{\varkappa}(x)$$

in closed form $G(\frac{1}{2}, 0) = \frac{1}{2}$. Also, this Proposition is compatible with (7) and (8).

Now we will demonstrate the symmetry property. Let us calculate

$$\begin{aligned}\frac{1}{z^2}G\Big(\varkappa, \frac{1}{z}+1\Big) &= \frac{1}{z^2}\int\limits_0^1 \frac{1}{\frac{1}{x}-\frac{1}{z}-1}\,\mathrm{d}\mu_{\varkappa}(x) \\ &= \frac{1}{z^2}\int\limits_0^1 \frac{1}{\frac{1}{1-x}-\frac{1}{z}-1}\,\mathrm{d}(1-\mu_{\varkappa}(1-x)) = -\int\limits_0^1 \Big(\frac{1}{\frac{1}{x}-z-1}+\frac{1}{z}\Big)\,\mathrm{d}\mu_{1-\varkappa}(x).\end{aligned}$$

The last equality comes from Proposition 2. But now, we use another algebraic identity

$$\frac{1}{z^2}\frac{1}{\frac{1}{1-x}-\frac{1}{z}-1} = -\frac{1}{\frac{1}{x}-z-1} - \frac{1}{z}.$$

This shows that the Stieltjes transform of $1-\mu_{\varkappa}(1-x)-\mu_{1-\varkappa}(x)$ is identically zero, and thus this proves (5).

On the other side, this symmetry property will be of no importance in two modular connections we will establish. The reason for this is that $\det(W) = -1$. In order this involution to be applicable, we must solve the system

$$A^{-1}R^{-1}A = T^{-1}, \quad A^{-1}T^{-1}A = T^{-1}.$$

Unfortunately, the only solution is $A \pm W$.

3.4. **A double series for $\eta^4(z)$.** We will soon need it, and it is apt now to consider the following problem.

Let us define the double series

$$\Omega(z)=\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{c(m,n)}{(mz+n)^2},$$

where coefficients $c(m,n)$ have the following properties:

T) $c(m,n+m)=\varrho^{-1}c(m,n)$ for $m,n\geq 0$;
R) $c(m+n,n)=\varrho c(m,n)$ for $m,n\geq 0$.
S) $c(-n,m)=c(n,-m)=-c(m,n)=-c(-m,-n)$.

First, note that if these are satisfied, then the first sum defining $\Omega$ vanishes. Indeed,

$$c(0,-1)=-c(1,0)=-\varrho c(1,1)=\varrho c(-1,1)=\varrho^2(-1,0)=\varrho^2 c(0,1).$$

Such coefficients are unique, and they can be defined as follows. Let $(m,n)=d$. Put $c(m,n)=c\big(\frac{m}{d},\frac{n}{d}\big)$. Suppose now, $(n,m)=1$, $n,m>0$. Note that in the modular group one has $ST^{-1}S=R$. Let us find $0<m_0\leq m$, $0\leq n_0<n$ such that (11) holds for $\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} m_0 & n_0\\ m & n\end{pmatrix}$. The we put

$$c(m,n)=\varrho^{\sum\limits_{r=1}^{s}\beta_r-\sum\limits_{r=1}^{s}\alpha_r}.$$

Let $\varrho=e^{\frac{\pi i}{3}}$, and define

$$\Psi(z)=\sum_{n\in\mathbb{Z}}\frac{\varrho^{-n}}{z+n}.$$

This is understood in the usual wayof Cauchy principal value, i.e. $\lim_{N\to\infty}\sum_{-N}^{N}$. Divide now this into 6 disjoint sums corresponding to $n \bmod 6$. Let us use the identity

$$\sum_{n\in\mathbb{Z}}\frac{1}{z+n}=\frac{\pi}{\tan\pi z}=-\pi i\frac{1+q}{1-q}.$$

Thus, this gives

$$\frac{\pi}{\tan\frac{\pi z}{6}}=-2\pi i\Big(\frac{1}{2}+\sum_{r=1}^{\infty}e^{\frac{\pi i z}{3}r}\Big).$$

Therefore,

$$\Psi(z)=\sum_{j=0}^{5}\frac{\pi\varrho^{-j}}{6\tan\frac{\pi(z+i)}{6}}=-\frac{2\pi i}{6}\sum_{j=0}^{5}\Big(\frac{1}{2}\varrho^{-j}+\sum_{r=1}^{\infty}\varrho^{-j+rj}e^{\frac{\pi i z}{3}r}\Big)=-2\pi i e^{\frac{\pi i z}{3}}\sum_{r=1}^{\infty}q^r.$$

Thus,

$$\sum_{n\in\mathbb{Z}}\frac{\varrho^{-n}}{(z+n)^2}=-4\pi^2 e^{\frac{\pi i z}{3}}\sum_{r=1}^{\infty}\Big(r+\frac{1}{3}\Big)q^r.$$

## 4. THE SERIES FOR THE FUNCTION $G(\varkappa, z)$

Using the same method as in [3], we see that the functional equation and boundedness property in Proposition 1 implies

$$G(\varkappa, z+1) = (1-\varkappa) \sum_{\substack{a,b,c,d\geq 0,\\ ad-bc=1}} \frac{\varkappa^{\imath(\frac{a+b}{c+d})}(1-\varkappa)^{\jmath(\frac{a+b}{c+d})}}{\big[(a+c)z-(b+d)\big](cz-d)}; \tag{10}$$

here $\imath$ and $\jmath$ stand for the number of maps $T$ and $R$, respectively, needed to obtain the rational number $\frac{a+b}{c+d}$ from the root $\frac{1}{1}$ in the Calkin-Wilf tree [12]. Equivalently, if

$$R^{\alpha_1}T^{\beta_1}R^{\alpha_2}T^{\beta_2}\cdots R^{\alpha_s}T^{\beta_s} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{11}$$

(where all the exponents are positive, except, possibly, $\alpha_1$ and $\beta_s$ can be 0), then

$$\imath\Big(\frac{a+b}{c+d}\Big) = \sum_{r=1}^{s}\beta_s, \quad \jmath\Big(\frac{a+b}{c+d}\Big) = \sum_{r=1}^{s}\alpha_s.$$

To see that the series (10), subject that it is absolutely convergent, is immediate, just look at Proposition 1. The terms $G(\varkappa, z+1)$, $\frac{1-\varkappa}{1-z}$, $\varkappa G(\varkappa, (z-1)+1)$, $\frac{1-\varkappa}{(1-z)^2}G(\varkappa, \frac{z}{1-z}+1)$ correspond to the identity $\mathbf{M} = I \cup \mathbf{M}T \cup \mathbf{M}R$, where $\mathbf{M}$, as before, stands for the free moniod $\{R, T\}$. More precisely, these four elements of the identity correspond to (the inverse of) the whole monoid $\mathbf{M}$, the node $I^{-1} = I$, the (inverse of) right branch $T^{-1}\mathbf{M}^{-1} = (\mathbf{M}T)^{-1}$, and (the inverse of) the left brach $R^{-1}\mathbf{M}^{-1} = (\mathbf{M}R)^{-1}$.

Let us fix the notations

$$\varrho = e^{\frac{\pi i}{3}}, \quad \mathcal{D} = \{\varkappa \in \mathbb{C} : |\varkappa| < 1, |1-\varkappa| < 1\}.$$

**Theorem 2.** *Let $\epsilon > 0$. The series (10) for $G(\varkappa, z)$ converges absolutely and uniformly for $\{\Im(z) > \epsilon, |\varkappa| < 1-\epsilon, |1-\varkappa| < 1-\epsilon\}$. So, $G(\varkappa, z)$ is an analytic function for $\Im(z) > 0$, $\varkappa \in \mathcal{D}$. Moreover, $G(\varkappa, z)$ is also defined for $\varkappa \in \overline{\mathcal{D}}$, and is a continuous function there for $\Im(z) > 0$.*

As an aside, this Theorem implies that

$$G(1, z+1) = 0, \quad G(0, z+1) = \sum_{n=0}^{\infty}\frac{1}{((n+1)z-1)\cdot(nz-1)} = -\frac{1}{z},$$

the facts that we already know from (7) and (8). Moreover,

$$\frac{\partial}{\partial\varkappa}G(\varkappa, z) \underset{\varkappa\to 1}{\sim} \sum_{n=1}^{\infty}\frac{\varkappa^n}{n-z}.$$

To calculate the same limit as $\varkappa \to 0$, we need to take all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that are of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} = \begin{pmatrix} m+1 & 1 \\ mn+m+n & n+1 \end{pmatrix}, \quad n, m \geq 0.$$

$$\frac{\partial}{\partial\varkappa}G(\varkappa,z)\underset{\varkappa\to 0}{\sim}\sum_{n,m=0}^{\infty}\frac{(1-\varkappa)^{n+m+1}}{\Big[\big((m+1)(n+2)-1\big)z-(n+2)\Big]\cdot\Big[\big((m+1)(n+1)-1\big)z-(n+1)\Big]}.$$

*Proof.* Fix positive co-prime integers $c,d$, and let $c_0,d_0$ be the unique pair such that $c_0d-d_0c=1$, $1\leq c_0\leq c$, $0\leq d_0<d$. Then the series (10) reads as

$$\begin{aligned}
G(\varkappa,z+1) &= (1-\varkappa)\sum_{\substack{c,d>0\\(c,d)=1}}\sum_{n=1}^{\infty}\frac{\varkappa^{\imath(\frac{c_0+d_0+(n-1)(c+d)}{c+d})}(1-\varkappa)^{\jmath(\frac{c_0+d_0+(n-1)(c+d)}{c+d})}}{\big[(c_0+nc)z-(d_0+nd)\big](cz-d)}\\
&+ (1-\varkappa)\sum_{n=1}^{\infty}\frac{\varkappa^{n-1}}{n-z}\\
&= \sum_{n=1}^{\infty}(1-\varkappa)\varkappa^{n-1}\sum_{\substack{c,d>0\\(c,d)=1}}\frac{\varkappa^{\imath(\frac{c_0+d_0}{c+d})}(1-\varkappa)^{\jmath(\frac{c_0+d_0}{c+d})}}{\big[(c_0+nc)z-(d_0+nd)\big](cz-d)}\\
&+ (1-\varkappa)\sum_{n=1}^{\infty}\frac{\varkappa^{n-1}}{n-z}. \qquad (12)
\end{aligned}$$

Note that $\sum_{n=1}^{\infty}\frac{\varkappa^{n-1}}{n}$ converges absolutely. We apply this to both sums.

The main difficulty (and the source for the richness of the theory of quasi-modular forms, and, *a posteriori*, the reason of this paper) is the fact that $\sum\limits_{\substack{c,d>0\\(c,d)=1}}\frac{1}{(cz-d)^2}$ does not converge.

So, let $z=x+iy$, $y>0$, $\mathbb{R}_+=[0,\infty)$, and let us define

$$A=A(z)=\inf_{\substack{c,d\in\mathbb{R}_+\\c+d=1}}(cx-d)^2+(cy)^2.$$

Obviously, $A>0$. Thus, for $c,d>0$,

$$\frac{1}{|cz-d|^2}=\frac{1}{(cx-d)^2+(cy)^2}\leq\frac{1}{A(c+d)^2},$$

where $A=A(z)>0$. The same reasoning as in ([3], Corrigenda) shows that it suffices to prove an absolute an uniform convergence of

$$\Upsilon(\varkappa,z)=\sum_{\substack{c,d>0\\(c,d)=1}}\frac{\varkappa^{\imath(\frac{c_0+d_0}{c+d})}(1-\varkappa)^{\jmath(\frac{c_0+d_0}{c+d})}}{(cz-d)^2}.$$

We can do it more generally, in any of the two regions $\{|\varkappa|\leq 1,|1-\varkappa|<1-\epsilon\}$, and $\{|\varkappa|<1-\epsilon,|1-\varkappa|\leq 1\}$. Suppose, it is the first case. The second case is dealt with analogously.

So, to finish, we only need to show that

$$S=\sum_{\substack{c,d>0\\(c,d)=1}}\frac{(1-\epsilon)^{\jmath(\frac{c_0+d_0}{c+d})}}{(c+d)^2} \qquad (13)$$

converges.

Suppose $a=c_0$, $b=d_0$, and for the matrix $\begin{pmatrix} c_0 & d_0 \\ c & d \end{pmatrix}$ we have the product (11). Then $\alpha_1 \geq 1$. Express the rational number $\frac{c_0+d_0}{c+d}$ as a continued fraction $[0,a_1,a_2,\ldots,a_t]$, $a_i \in \mathbb{N}$, $a_t \geq 2$. Then [1, 12]

$$\sum_{j=1}^{t} a_j = \sum_{j=1}^{s}(\alpha_j+\beta_j)+1, \quad t=2s \text{ or } 2s-1, \text{ depending on whether } \beta_s>0 \text{ or } \beta_s=0.$$

Now, let $p=c_0+d_0$, $q=c+d$, and $p_0$, $q_0$ be co-prime non-negative integers given by $\frac{p_0}{q_0}=[0,a_1,a_2,\ldots,a_{t-1}]$. Then the sum of all intervals of rank $t$ is equal to 1 [16]:

$$\sum_{t \text{ is fixed}} \frac{1}{q(q+q_0)} = 1.$$

For example, the general rank 2 interval is given by $[\frac{1}{n+\frac{1}{m}}, \frac{1}{n+\frac{1}{m+1}})$, and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(nm+1)(nm+n+1)} = \sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1.$$

So,

$$\sum_{t \text{ is fixed}} \frac{1}{q^2} < 2.$$

Now, $\jmath(\frac{c_0+d_0}{c+d}) \geq s \geq \frac{t}{2}$. So, if we group in the sum (13) elements according to the depth of the continued fraction expansion, we obtain

$$S \leq \sum_{t=1}^{\infty}(1-\epsilon)^{t/2} \sum_{\frac{p}{q} \text{ is of depth } t} \frac{1}{q^2} < 2\sum_{t=1}^{\infty}(1-\epsilon)^{t/2},$$

and we are done.

Now we will prove the second part of the Theorem. Consider, for example, $|1-\varkappa|<1-\epsilon$, $|\varkappa|\leq 1$. We will show that one can regroup the sum for $G(\varkappa,z+1)$ so that when $|\varkappa|<1$ approaches $\varkappa_0$ on the arc $|\varkappa|=1$, $|1-\varkappa|<1-\epsilon$, the function $G(\varkappa,z+1)$ tends to a uniquely defined limit $G(\varkappa_0,z+1)$. First, note that

$$(1-\varkappa)\varkappa^{n-1} = (1-\varkappa^n)-(1-\varkappa^{n-1}).$$

So, since the series (12) for $G(\varkappa,z)$ is absolutely convergent for $\varkappa \in \mathcal{D}$, after regrouping both sums we obtain

$$\begin{aligned} G(\varkappa,z+1) &= \sum_{n=1}^{\infty}(1-\varkappa^n)\sum_{\substack{c,d>0\\(c,d)=1}} \frac{\varkappa^{\imath(\frac{c_0+d_0}{c+d})}(1-\varkappa)^{\jmath(\frac{c_0+d_0}{c+d})}}{\big[(c_0+nc)z-(d_0+nd)\big]\big[(c_1+nc)z-(d_1+nd)\big]} \\ &+ \sum_{n=1}^{\infty}\frac{1-\varkappa^n}{(n-z)(n+1-z)}, \end{aligned}$$

where $c_1=c_0+c$, $d_1=d_0+d$. Now, $\sum_{n=1}^{\infty}\frac{1-\varkappa^n}{n^2}$ converges absolutely. Using the result, which we just proved, about the series $\Upsilon(\varkappa,z)$, we get the needed result.

What is left to prove is that $G(\varkappa,z+1)$ remains bounded and continuous for $\varkappa\in\overline{\mathcal{D}}\setminus\{\varrho\}$, $\varkappa\to\varrho$.

Indeed,

$$\frac{1}{\big[(c_0+nc)z-(d_0+nd)\big](cz-d)}-\frac{1}{n(cz-d)^2}=$$

meaning that if $z$ belongs to the compact region of the upper half-plane, the ratio of the left and right sides remains bounded and ounded away from 0 for all coprime $c,d$. □

The function $G(\varkappa,z)$ at $\varkappa=\frac{1}{2}$ is our main interest. However, due to analyticity, we can explore this function at any articular $\varkappa$ to get all the information via analytic continuation. The exceptional role is played by $\varkappa=\varrho$, and let therefore

$$G(\varrho,z+1)-\frac{i}{2\pi}G_2(z)=\psi(z).$$

Note that $1-\varrho=\varrho^{-1}$. Let us apply the functional equation (9) for $z\mapsto z$, as well as for $z\mapsto\frac{1}{1-z}$. We get

$$\begin{aligned}0 &= \psi(z)-\varrho\psi(z-1)-\frac{\varrho^{-1}}{(1-z)^2}\psi\Big(\frac{z}{1-z}\Big)\\ &- \frac{\varrho^{-2}}{(1-z)^2}\Big[\psi\Big(\frac{1}{1-z}\Big)-\varrho\psi\Big(\frac{z}{1-z}\Big)-\frac{\varrho^{-1}(1-z)^2}{z^2}\psi\Big(-\frac{1}{z}\Big)\Big]\\ &= \Big[\psi(z)+\frac{\varrho^{-3}}{z^2}\psi\Big(-\frac{1}{z}\Big)\Big]-\varrho\Big[\psi(z-1)+\frac{\varrho^{-3}}{(1-z)^2}\psi\Big(\frac{1}{1-z}\Big)\Big].\end{aligned}\tag{14}$$

First, $\varrho^{-3}=-1$. Next, let

$$\mathscr{W}(z)=\psi(z)-\frac{1}{z^2}\psi\Big(-\frac{1}{z}\Big).$$

We have

$$\mathscr{W}(z)=G(\varrho,z+1)-\frac{1}{z^2}G\Big(\varrho,-\frac{1}{z}+1\Big)+\frac{1}{z}.\tag{15}$$

Via the algebraic identity (14) we obtain that

$$\mathscr{W}(z)=\varrho\mathscr{W}(z-1)\Longrightarrow\mathscr{W}(z+6)=\mathscr{W}(z).\tag{16}$$

Writing $\mathscr{W}(z)=\sum_{n\in\mathbb{N}_0}p_n\exp(\frac{2\pi in}{6})$, we immediately get that

$$\mathscr{W}(z)=\sum_{n\equiv 1\ (\mathrm{mod}\ 6)}p_n e^{\frac{2\pi inz}{6}}.$$

Now we will show the opposite - if $\mathscr{W}$ satisfies But now, directly from (15) we infer that

$$\frac{1}{z^2}\mathscr{W}\Big(-\frac{1}{z}\Big)=-\mathscr{W}(z).\tag{17}$$

So, the function $\mathscr{W}^6$ is a modular form of weight 12 for the full modular group. As the Fourier expansion shows, it is the cups form. And so $\mathscr{W}^6(z) = c\Delta(z)$. Moreover, one possible solutions to (16) and (17) is given by

$$\eta^4(z).$$

Indeed, $\eta(z)$ satisfies

$$\begin{cases} \eta(z+1) = e^{\frac{\pi i}{12}}\eta(z), \\ \eta\Big(-\frac{1}{z}\Big) = \sqrt{-iz}\,\eta(z). \end{cases}$$

Thus,

$$\mathscr{W}(z) = c\eta^4(z).$$

If we know it, we can partially recover $\psi$ from two equations

$$\begin{cases} \psi - \psi|_S = \mathscr{W}, \\ \psi - \varrho\psi|_{T^{-1}} - \varrho^{-1}\psi|_{R^{-1}} = 0. \end{cases}$$

We will soon investigate how this determines $\psi$.

4.1. **Fourier series.** Now we can see direclt that the function $\mathscr{W}(z)$, defined by (15), satsifies $\mathscr{W}(z+1) = \varrho\mathscr{W}(z)$, and we will calculate its Taylor series. This will allow to calculate the constant $c$.

...

We can rewrite the last identity as follows, to comply more with standard appearance of such identities satisfied by polynomial and rational period functions. Since $T^{-1} = SU$, $R^{-1} = SU^2$, using the first identity, we get

$$\begin{aligned} \psi - \varrho\psi|_{SU} - \varrho^{-1}\psi|_{SU^2} &= \psi - \varrho(\psi - \mathscr{W})|_U - \varrho^{-1}(\psi - \mathscr{W})|_{U^2} \\ &= \psi - \varrho\psi|_U - \varrho^{-1}\psi|_{U^2} + \varrho\mathscr{W}|_U + \varrho^{-1}\mathscr{W}|_{U^2}. \end{aligned}$$

But

$$\begin{aligned} \mathscr{W}|_U &= \mathscr{W}|_{T^{-1}S} = \varrho^{-1}\mathscr{W}|_S = -\varrho^{-1}\mathscr{W}; \\ \mathscr{W}|_{U^2} &= \mathscr{W}|_{TS} = \varrho\mathscr{W}|_S = -\varrho\mathscr{W}. \end{aligned}$$

So, we have proved

**Proposition 3.** *The function $\psi$ satisfies*

$$\begin{cases} \psi - \psi|_S = \mathscr{W}, \\ \psi - \varrho\psi|_U - \varrho^{-1}\psi|_{U^2} = 0. \end{cases}$$

## 5. MEAN-MODULAR FORMS

Let $\mathfrak{h}$ be the upper half plane, and let $G_2(z)$ stands for the holomorphic quasi-modular Eisenstein series of weight 2 [10, 21]:

$$G_2(z) = \frac{\pi^2}{3} - 8\pi^2\sum_{n=1}^{\infty}\sigma_1(n)e^{2\pi i n z}.$$

We will also use the standard normalization

$$E_2(z) = \frac{3}{\pi^2}\,G_2(z) = 1 - 24e^{2\pi i z} - 72e^{4\pi i z} - \cdots.$$

Also, let $E_4$ and $E_6$, as usual, be normalized Eisenstein series of weights 4 and 6. We know that for $z\in\mathfrak{h}$,

$$G_2(z+1)=G_2(z),\quad G_2(-1/z)=z^2G_2(z)-2\pi iz.$$

There exist several extensions of the space $M_k$ of modular forms of weight $k$. One of the extensions is the space $\widetilde{M}_k$ of the so called *quasi-modular forms*, which are weight $k$ elements of the ring $\mathbb{C}[E_2,E_4,E_6]$. Another extension is the space $\widehat{M}_k$ of *almost holomorphic modular forms.* The latter are functions which transform like modular forms of weight $k$, but instead of being holomorphic, they are polynomials in $\frac{1}{y}=\frac{1}{\Im(z)}$ with coefficients being holomorphic functions of moderate growth. The result by M. Kaneko and D. Zagier [15] claims that there is a canonical isomporphism between $\widehat{M}_k$ and $\widetilde{M}_k$. It just assigns to an element of $\widehat{M}_k$, as a polynomial in $\frac{1}{y}$, its constant term. The expanded version of the proof of is presented in [9].

Now we describe another $?(x)$-related extension of $M_k$, and will later prove that all these three extensions are canonically isomorphic. This gives unexpected view on quasi-modular forms, where $?(x)$ essentially enters the picture.

We will need the following differential identities of Ramanujan [10]:

$$\frac{1}{2\pi i}E_2'=\frac{E_2^2-E_4}{12},\quad \frac{1}{2\pi i}E_4'=\frac{E_2E_4-E_6}{3},\quad \frac{1}{2\pi i}E_6'=\frac{E_2E_6-E_4^2}{2}. \tag{18}$$

A direct calculation shows that $\frac{i}{2\pi}G_2(z)$ (as a constant function in variable $\varkappa$) satisfies the functional equation (9) for $z\in\mathfrak{h}$. Let, as before, the number $\varkappa$ belong to $\mathcal{D}$. If $G(\varkappa,z)$ is the function from the previous subsection, then, if we set

$$Q(\varkappa,z)=-\frac{6}{\pi i}\Big(G(\varkappa,z+1)-\frac{i}{2\pi}G_2(z)\Big)=-\frac{6}{\pi i}G(\varkappa,z+1)+E_2(z),$$

we see that for every $\epsilon>0$ this function is uniformly bounded for $\Im(z)>\epsilon$, $\varkappa\in\overline{\mathcal{D}}$, and it satisfies the functional equation

$$\boxed{f(\varkappa,z)=\varkappa f(\varkappa,z-1)+\frac{1-\varkappa}{(1-z)^k}f\Big(\varkappa,\frac{z}{1-z}\Big)} \tag{19}$$

for $k=2$. The function $Q$ is the fundamental function which plays the same role among mean-modular forms (see below) as $E_2$ plays in the theory of quasi-modular forms.

Our main interest is the equation (19) in case $\varkappa=\frac{1}{2}$, $k=2$, since this, as we have seen, is directly related to the Minkowski question mark function. Also, we have seen a modular connection at $\varkappa=\varrho$. Nevertheless, suppose $f(\varkappa,z)$ satisfies (19), and let us consider this identity as the one for the function of two complex variables $\varkappa$ and $z$. At the one end of the real interval $\varkappa\in[0,1]$, the function $f(\varkappa,z)$ is $T$-periodic:

$$f(1,z)=f(1,z)|T^n,\quad n\in\mathbb{Z}.$$

The $R$-periodicity holds at the other end:

$$f(0,z)=f(0,z)|R^n,\quad n\in\mathbb{Z}.$$

The two matrices $T$ and $R$ generate the whole modular group, and $R$ and $T$ are primitive elements there (i.e. not powers of other matrices) of infinite order. For example,

$$Q(1,z)=E_2(z),\quad Q(0,z)=\frac{6}{\pi i z}+E_2(z),$$

which are, respectively, $T$- and $R$-periodic.

**Definition 1.** *Let $k\in 2\mathbb{N}$. The function $f(\varkappa,z)$ is called* a weight $k$ mean-modular form *or* MMF *for short, if*

i) *it is a bivariate holomorphic function and satisfies the functional equation (19) for $z\in\mathfrak{h}$, $\varkappa\in\overline{\mathcal{D}}$; the latter means that it is holomorphic in $\mathcal{D}$ and extends continuously to its closure;*

ii) *for every $\epsilon>0$ there exist a constant $C(\epsilon)$ such that $|f(\varkappa,z)|<C(\epsilon)$ for $\Im(z)>\epsilon$, $\varkappa\in\overline{\mathcal{D}}$.*

Let $\mathcal{H}$ stand for the ring of functions in variable $\varkappa$, which are holomorphic in $\mathcal{D}$ and continuous in its closure. The set of weight $k$ mean-modular forms make an $\mathcal{H}$-module, which we denote by $\mathsf{MMF}_k$.

The first main result of this paper reads as follows.

**Theorem 3.** *Let $f$ be a MMF of weight $k$. Then the following limit*

$$\mathfrak{E}f(\varkappa,z)=\lim_{\substack{n\to\infty\\ n\in\mathbb{Z}}}f(\varkappa,z-n)$$

*exists, and*

$$\mathfrak{E}:\mathsf{MMF}_k\mapsto\widetilde{M}_k\otimes_{\mathbb{C}}\mathcal{H}.$$

*Moreover, $\mathfrak{E}$ is an isomorphism.*

*There exists a canonical $\mathbb{C}$-subspace of $\mathsf{MMF}_k$, call it $\mathsf{Mmf}_k$, consisting of mean-modular forms $f$, characterized by a fact that $\mathfrak{E}(f)$ is a constant function in $\varkappa$. Then one has an isomorphism*

$$\mathfrak{E}:\mathsf{Mmf}_k\mapsto\widetilde{M}_k.$$

Thus, when we talk about mean-modular forms as forming a $\mathbb{C}$-space or $\mathcal{H}$-module, me mean $\mathsf{Mmf}_k$ or $\mathsf{MMF}_k$, respectively.

**Definition 2.** *We call a function $T(z)$* a mean-modular section, *or* MMS, *of weight $k$, if there exists a mean-modular form $f(\varkappa,z)$ of weight $k$ such that*

$$T(z)=f\Big(\frac{1}{2},z\Big).$$

Mean-modular section is a very natural definition, since the point $\varkappa=\frac{1}{2}$ lies in the very center of the region $\mathcal{D}$.

Denote the $\mathbb{C}$-linear space of MMS of weight $k$ by $\mathsf{Mms}_k$. The main motivation of this paper are the following three facts:

- ◊ if $T(z)$ is a modular form for $\mathsf{PSL}_2(\mathbb{Z})$, then $T(z)$ is a MMS of the same weight; indeed, $T(z)$ is a MMF as a constant function in $\varkappa$.

- ♢ "Sporadic" solutions of the three term functional equation (19) (for a specific $\varkappa$), which are also in $M_k(\Gamma(N))$, do not qualify MMS (see Appendix B).
- ♢ Most importantly,

$$\int\limits_0^1 \frac{x}{1-x(z+1)}\,\mathrm{d}?(x) - \frac{i}{2\pi}G_2(z).$$

is a MMS of weight 2.

Let, correspondingly, define the map $S:\mathsf{MMF}_k\mapsto\mathsf{Mms}_k$, by

$$S\Big(f(\varkappa,z)\Big)=f\Big(\frac{1}{2},z\Big).$$

This restricts also to the map $S:\mathsf{Mmf}_k\mapsto\mathsf{Mms}_k$, which we denote by the same letter $S$. The map $\mathfrak{E}:\mathsf{Mms}_k\mapsto\widetilde{M}_k$, which we denote by the same letter $\mathfrak{E}$ again (this should not cause a confusion) is defined by

$$\mathfrak{E}(T)=\lim_{\substack{n\to\infty\\ n\in\mathbb{Z}}} T(z-n).$$

**Corollary 1.** *The map $\mathfrak{E}:\mathsf{Mms}_k\mapsto\widetilde{M}_k$ is an isomorphism.*

Let us define another map $\Gamma$, which maps from $\mathsf{MMF}_k$ or $\mathsf{Mmf}_k$, by the following formula:

$$\Gamma\Big(f(\varkappa,z)\Big)=f(\varrho,z)-\frac{1}{z^k}f\Big(\varrho,-\frac{1}{z}\Big).$$

The second main result of this paper reads as follows.

**Theorem 4.** *For any $f\in\mathsf{MMF}_k$ or $f\in\mathsf{Mmf}_k$, there exists a modular form $g\in M_{k-2}$, such that*

$$\Gamma(f)=\eta^4 g.$$

In particular, here we recover a structural constant $C$, wired canonically into the Minkowski question mark function, which is defined by

$$\tau(Q)=C\eta^4(z).$$

More generally, let us limit ourselves to $\mathsf{Mmf}_k$, and let us define the following maps (see Table 1):

- $\tau=\Gamma\circ S^{-1}$.
- $Pr=\Gamma\circ S^{-1}\circ\mathfrak{E}^{-1}$.
- $t:M_{k-2}\cdot\eta^4\mapsto M_{k-2}$ by $t(f)=f\cdot\eta^{-4}$.
- Lastly, define $\widehat{Pr}=t\circ Pr$.

So, the diagram in Table 1 is commutative.

**Corollary 2.** *There exists the canonical Minkowski $?(x)$-function related canonical surjective homomorphism*

$$\widehat{Pr}:\widetilde{M}_k\mapsto M_{k-2}.$$

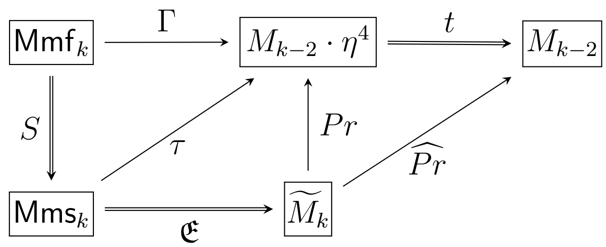



TABLE 1. Commutative diagram of homomorphisms of the following $\mathbb{C}$-vector spaces: mean-modular forms $\mathsf{Mmf}_k$; mean-modular sections $\mathsf{Mms}_k$; quasi-modular forms $\widetilde{M}_k$; and modular forms $M_{k-2}$. Double lines denote an isomorphism.

As will be clear later from Proposition 6, this map is uniquely defined by the constant $C$.

Note that our results imply that if the two conditions i) and ii) are satisfied by the function $f(\varkappa, z)$, then there always exists

$$\lim_{n\to\infty} f(\varkappa, z-n) = A(\varkappa, z) \sim \sum_{j=0}^{\infty} a_j(z)(\varkappa-1)^j;$$

(see Definition 3 for the meaning). Each $a_j(z)$ is then $1-$periodic, and the function $f(\varkappa, z)$ can be uniquely reconstructed as

$$f(\varkappa, z) = \sum_{j=0}^{\infty} \widehat{a_j}(\varkappa, z)(\varkappa-1)^j;$$

(so, a lift can be constructed for each of the function $a_j(z)$ separately). This corresponds to the fact that if $f(\varkappa, z)$ satisfies the functional equation (19), then so does $f(\varkappa, z)p(\varkappa)$, where $p(\varkappa)$ is holomorphic function in $\mathcal{D}$, continuous on $\overline{\mathcal{D}_\epsilon}$. To eliminate the question of convergence, we therefore put the requirement that $a_j(z) = 0$ for $j \geq 1$ in the very definition itself.

One can ask the following question - which periodic function in $\mathfrak{h}$ can be lifted to RT forms? Since this is out of scope of the current paper, we only note that the series (30) implies the following

**Proposition 4.** *If $A_0(z)$ is a bounded periodic function in $\mathfrak{h}$, it can be lifted (uniquely) to an RT form.*

Our aim of this paper is to integrate the Minkowski question mark function into the modular world. Thus, the main result of the current paper is the following Theorem; the definition of the linear map $\vartheta = \vartheta_k : \mathsf{Mmf}_k \mapsto \mathsf{Mmf}_{k+2}$ will be given in the Section 6. According to Definition 1, one can define the linear map

$$\mathfrak{E} : \mathsf{Mms}_k \mapsto \widetilde{M}_k$$

as follows. I

Also, if $f(\varkappa, z)$ is a MMF of weight $k$, let us define

$$\mathfrak{D} f(\varkappa, z) = f(\varrho, z) - \frac{1}{z^k} f\Big(\varrho, -\frac{1}{z}\Big)$$

Thus, if $f \in M_k$ (treated as a constant function in $\varkappa$), then $\mathfrak{E} f = f$.

**Theorem.** *The map $\mathfrak{E}$ is an isomorphism, and every quasi-modular form $E(z)$ of weight $k$ can be lifted (uniquely) to a mean-modular form. We have the decomposition*

$$\mathsf{Mmf}_k = M_k \oplus \bigoplus_{r=1}^{k/2} \vartheta^{r-1}(Q) \cdot M_{k-2r}.$$

*(Here $\vartheta^0 = \mathrm{id}$). This gives $\dim_{\mathbb{C}}(\mathsf{Mmf}_k) = \dim_{\mathbb{C}}(\widetilde{M_k})$.*

The relation of $Q$ with $?(x)$ is our main motivation for an introduction of the space $\mathsf{Mmf}_k$. If $f(\varkappa, z) = E(z)$ is a constant function in $\varkappa$, ten $E(z)$ is a modular form. In fact, there are many functions, constant in variable $\varkappa$, which satisfy the functional equation (19)) but fail the regularity condition. For example, when $k = 2$ such functions are $j'(z)P(j(z))$, where $j(z)$ is the $j$-invariant ("Hauptmodul" for the full modular group), and $P$ is any polynomial.

Let us denote the arc of the circle $|1 - \varkappa| = \epsilon > 0$, which is inside the disc $|\varkappa| \leq 1$, by $\Omega(\epsilon)$. When $\epsilon \to 0_+$, the length of the arc $\Omega(\epsilon)$ is $\pi\epsilon + O(\epsilon^2)$.

**Definition 3.** *Let $Q \subset \mathfrak{h}$ be a compact set. For $j \in \mathbb{N}_0$, the $j$-th coordinate of the mean-modular form $f(\varkappa, z)$ is an analytic function in $z$ which is defined recurrently by*

$$A_j(z) = \lim_{\epsilon \to 0_+} \frac{1}{\pi i} \int\limits_{\Omega(\epsilon)} \frac{1}{(\varkappa - 1)^{j+1}} \Big( f(\varkappa, z) - \sum_{s=0}^{j-1} (\varkappa - 1)^s A_s(z) \Big) \,\mathrm{d}\varkappa, \quad j \in \mathbb{N}_0, \quad z \in Q,$$

*where the integral is taken via the arc $\Omega(\epsilon)$ from the bottom upwards, and an empty sum is 0 by convention. The definition of $A_j(z)$ is extended to $\mathfrak{h}$ by expanding $Q$. Symbolically, we write*

$$f(\varkappa, z) \sim \sum_{j=0}^{\infty} (\varkappa - 1)^j A_j(z).$$

Comparing the corresponding coefficients at powers of $\varkappa$ we obtain the basic relations among these functions:

$$A_j(z) - A_j(z-1) = A_{j-1}(z-1) - \frac{1}{(1-z)^k} A_{j-1}\Big(\frac{z}{1-z}\Big). \tag{20}$$

In other words,

$$A_j|(I - T^{-1}) = A_{j-1}|(T^{-1} - R^{-1}).$$

This holds for $j \geq -1$, assuming $A_{-1}(z) \equiv 0$.

### 5.1. **Mean-modular sections.**

## 6. Homomorphisms of MMF

6.1. **Serre's derivative.** Similarly as in ([10], Section 5.1), we prove the following

**Proposition 5.** *If $f$ is a MMF of weight $k$, then*

$$\vartheta_k(f) = \frac{1}{2\pi i} \frac{\partial}{\partial z} f(\varkappa, z) - \frac{k}{12} \cdot E_2(z) \cdot f(\varkappa, z)$$

*is a MMF of weight $k + 2$.*

If we apply the operator $\vartheta_k$ to weight $k$ MMF and it is unambiguous, we may drop the subscript $k$.

*Proof.* Let, for simplicity, $f' = \frac{\partial}{\partial z} f$, and we ommit the first variable $\varkappa$. Then, if $f$ is a MMF, then

$$f'(z) - \varkappa f'(z-1) - \frac{1-\varkappa}{(1-z)^{k+2}} f'\Big(\frac{z}{1-z}\Big) = \frac{(1-\varkappa)k}{(1-z)^{k+1}} f\Big(\frac{z}{1-z}\Big). \quad (21)$$

Further, let $u(z) = f(z)E_2(z)$. Then, according to the properties of $G_2(z)$, we have:

$$\begin{aligned} u(z) &= f(z)E_2(z), \\ \varkappa u(z-1) &= \varkappa f(z-1)E_2(z), \\ \frac{1-\varkappa}{(1-z)^{k+2}} u\Big(\frac{z}{1-z}\Big) &= \frac{1-\varkappa}{(1-z)^{k}} f\Big(\frac{z}{1-z}\Big)E_2(z) + \frac{6i(1-\varkappa)}{\pi(1-z)^{k+1}} f\Big(\frac{z}{1-z}\Big). \end{aligned}$$

Thus,

$$u(z) - \varkappa u(z-1) - \frac{1-\varkappa}{(1-z)^{k+2}} u\Big(\frac{z}{1-z}\Big) = -\frac{(1-\varkappa)6i}{\pi(1-z)^{k+1}} f\Big(\frac{z}{1-z}\Big). \quad (22)$$

Comparing (21) and (22), we get the needed property that $\frac{1}{2\pi i} f' - \frac{k}{12} E_2 f$ is a MMF. The second assertion of the proposition is obvious. □

We can see indeed that the operator $\vartheta$ is indeed a "derivation". Let $f$ be weight $k$ mean modular form, and $g$ be weight $\ell$ modular form. Then $fg$ is weight $k+\ell$ MMF, and

$$\vartheta_{k+\ell}(fg) = \frac{1}{2\pi i}(f'g + fg') - \frac{k+\ell}{12} E_2 fg = \vartheta_k(f)g + f\vartheta_\ell(g); \quad (23)$$

So, $\vartheta$ satisfies the Leibniz rule.

Next, based on (18), we have

$$\begin{aligned} \mathfrak{E}Q &= E_2; \\ \mathfrak{E}\vartheta(Q) &= -\frac{1}{12}E_2^2 - \frac{1}{12}E_4; \\ \mathfrak{E}\vartheta^2(Q) &= \frac{1}{72}E_2^3 + \frac{1}{72}E_2E_4 + \frac{1}{36}E_6. \end{aligned}$$

6.2. **Commutativity.** In this subsection we establish that the map $\vartheta$ commutes with maps $\mathfrak{E}$, $S$, and $t$ from Table 1. Of course, on both sides of commutativity identities $\vartheta$ is defined on different spaces. This should not cause a confusion.

$\mathfrak{E}$) The map $\mathfrak{E}$ commutes with the derivation $\vartheta$. That is, let $f \in \mathsf{Mmf}_k$. Then

$$\mathfrak{E}\vartheta(f) = \vartheta(\mathfrak{E}f), \quad (24)$$

where the second $\vartheta$ is the standard (Serre's) derivation $\widetilde{M_k} \mapsto \widetilde{M_{k+2}}$. This is checked directly.

$S$) Of course, $S$ and $\vartheta$ commute.

$\Gamma$) Next, let $f(\varkappa, z)$ be in $\mathsf{MMF}_k$. We have:

$$\begin{aligned}\vartheta(f)(\varrho,z) &= \frac{1}{2\pi i}f'(\varrho,z)-\frac{k}{12}E_2(z)f(\varrho,z).\\ \frac{1}{z^{k+2}}\vartheta(f)\Big(\varrho,-\frac{1}{z}\Big) &= \frac{1}{2\pi i z^{k+2}}f'\Big(\varrho,-\frac{1}{z}\Big)-\frac{k}{12z^{k+2}}E_2\Big(-\frac{1}{z}\Big)f\Big(\varrho,-\frac{1}{z}\Big).\end{aligned}$$

Subtracting, and using the property that relates $E_2(z)$ with $E_2(-1/z)$, we get

$$\Gamma\circ\vartheta(f)=\vartheta\circ\Gamma(f).$$

Here the first "$\vartheta$" is a map $\mathsf{Mmf}_k\mapsto\mathsf{Mmf}_{k+2}$, an d the second "$\vartheta$" is the map $M_{k-2}\eta^4\mapsto M_k\eta^4$, defined by the same formula of Serre's derivation. The fact that this is well-defined follows as a side result, but it can be checked independently. Indeed, let $f\in M_{k-2}\eta^4$, $f=g\eta^4$. Then it satisfies

$$f(z+1)=\varrho f(z),\quad f\Big(-\frac{1}{z}\Big)=-z^k f(z),$$

and by the same reasoning as above, $\vartheta_k f\in M_k\eta^4$.

$t$) Let $f\in M_{k-2}\eta^4$, $f=g\eta^4$. As already defined, $t(f)=g$. We have:

$$\begin{aligned}t\circ\vartheta(f) &= \frac{1}{2\pi i}f'\eta^{-4}-\frac{k}{12}E_2 f\eta^{-4},\\ \vartheta\circ t(f) &= \vartheta(f\eta^{-4})=\frac{1}{2\pi i}(f'\eta^{-4}-4f\eta^{-5}\eta')-\frac{k-2}{12}E_2(f\eta^{-4}).\end{aligned}$$

Note that $t(f)\in M_{k-2}$, and thus in Serre's derivative we must use $k-2$ instead of $k$. But yet again, the last two expressions are equal, since the function $\eta$ satisfies the differential equation

$$\frac{\eta'}{\eta}=\frac{\pi i}{12}E_2.$$

Combining all these properties, we get

**Proposition 6.** *The maps $\widehat{Pr}$ and $\vartheta$ commute, as shown in Table 2.*

*Proof.* Indeed,

$$\widehat{Pr}\circ\vartheta=t\circ\Gamma\circ S\circ\mathfrak{E}^{-1}\circ\vartheta=\vartheta\circ t\circ\Gamma\circ S\circ\mathfrak{E}^{-1}=\vartheta\circ\widehat{Pr}.$$

□

Let us define

$$\mathsf{Mmf}=\bigoplus_{k\in 2\mathbb{N}_0}\mathsf{Mmf}_k.$$

Since we will show in the next Section that $\mathfrak{E}$ is indeed an isomorphism of $\mathbb{C}-$vector spaces, this provides the product structure inside MMF by the following construction. If $f\in\mathsf{Mmf}_k$, $g\in\mathsf{Mmf}_\ell$, then we define $f\star g\in\mathsf{Mmf}_{k+\ell}$ by

$$f\star g=\mathfrak{E}^{-1}\big(\mathfrak{E}(f)\cdot\mathfrak{E}(g)\big).\qquad(25)$$

The product "$\star$" turns $\mathsf{Mmf}$ into the graded algebra. If $g\in M_k$, then

$$f\star g=f\cdot g.$$

And finally, the map $\vartheta$ and the product $\star$ are compatible with the Leibniz rule; that is, if $f\in\mathsf{Mmf}_k$, $g\in\mathsf{Mmf}_\ell$, then

$$\vartheta(f\star g)=\vartheta(f)\star g+f\star\vartheta(g).$$

Indeed, let $f\in\mathsf{Mmf}_k$, $g\in\mathsf{Mmf}_\ell$. Then we have:

$$\begin{aligned}\vartheta(f\star g)&\overset{(25)}{=}\vartheta\Big(\mathfrak{E}^{-1}\big(\mathfrak{E}f\cdot\mathfrak{E}g\big)\Big)\overset{ii)}{=}\mathfrak{E}^{-1}\Big(\vartheta\big(\mathfrak{E}f\cdot\mathfrak{E}g\big)\Big)\\&\overset{(23)}{=}\mathfrak{E}^{-1}\Big(\vartheta(\mathfrak{E}f)\cdot\mathfrak{E}g+\mathfrak{E}f\cdot\vartheta(\mathfrak{E}g)\Big)\\&\overset{(24)}{=}\mathfrak{E}^{-1}\Big(\mathfrak{E}\vartheta(f)\cdot\mathfrak{E}g\Big)+\mathfrak{E}^{-1}\Big(\mathfrak{E}f\cdot\vartheta(\mathfrak{E}g)\Big)\\&\overset{(25)}{=}\vartheta(f)\star g+f\star\vartheta(g).\end{aligned}$$

## 7. THE PROOF

So, we will now list the elements of $\mathsf{Mmf}_k$ which we already know. For simplicity, we omit the first variable $\varkappa$.

Weight 2, $\dim(\mathsf{Mmf}_2)=1$: $Q$.

Weight 4, $\dim=2$: $\vartheta(Q)$, $E_4$. These are linearly independent, since the first MMF is non-periodic, while the second is.

Weight 6, $\dim=3$: $\vartheta^2(Q)$, $E_4Q$, $E_6$. These three MMF are also linearly independent. Indeed suppose the contrary,

$$a\vartheta^2(Q)+bE_4Q+cE_6=0.$$

Substitute $z\mapsto z-n$ and take now the limit $n\to\infty$. Thus, in fact we are applying the $\mathfrak{E}$ operator:

$$a\mathfrak{E}\vartheta^2(Q)+bE_4\mathfrak{E}Q+cE_6=0. \tag{26}$$

This is the combination of the products of $E_2$, $E_4$ and $E_6$. The coefficient at $E_2^3$ is $aq_2$, so the algebraic independence of $E_2$, $E_4$ and $E_6$ implies that $a=0$. Then the coefficient at $E_2E_4$ of the remaining terms in (26) is $bq_0$, and this again yields $b=0$. We therefore find that $a=b=c=0$. Essentially the same method works to show that for every weight, the below constructed MMF are linearly independent. We can also calculate

$$\mathfrak{E}Q\cdot\mathfrak{E}\vartheta(Q)=-\frac{1}{12}E_2^3-\frac{1}{12}E_2E_4=-6\mathfrak{E}\vartheta^2(Q)+\frac{1}{6}\mathfrak{E}E_6.$$

So,

$$Q\star\vartheta(Q)=-6\vartheta^2(Q)+\frac{1}{6}E_6.$$

Weight 8, $\dim=4$: $\vartheta^3(Q)$, $E_4\vartheta(Q)$, $E_6Q$, $E_8$. Note that, for example, $\vartheta(E_4Q)$ does not give anything new, since using the properties (18) and (23), we have:

$$\vartheta(E_4Q)=\vartheta(E_4)Q+E_4\vartheta(Q)=-\frac{1}{3}E_6Q+E_4\vartheta(Q).$$

Weight 10, $\dim=5$: $\vartheta^4(Q)$, $E_4\vartheta^2(Q)$, $E_6\vartheta(Q)$, $E_8Q$, $E_{10}$.

Weight 12, dim $=7$: $\vartheta^5(Q)$, $E_4\vartheta^3(Q)$, $E_6\vartheta^2(Q)$, $E_8\vartheta(Q)$, $E_{10}Q$, $E_{12}$, $\Delta$.

Weight 14, dim $=8$: $\vartheta^6(Q)$, $E_4\vartheta^4(Q)$, $E_6\vartheta^3(Q)$, $E_8\vartheta^2(Q)$, $E_{10}\vartheta(Q)$, $E_{12}Q$, $\Delta Q$, $E_{14}$.

Weight 16, dim $=10$: $\vartheta^7(Q)$, $E_4\vartheta^5(Q)$, $E_6\vartheta^4(Q)$, $E_8\vartheta^3(Q)$, $E_{10}\vartheta^2(Q)$, $E_{12}\vartheta(Q)$, $\Delta\vartheta(Q)$, $E_{14}Q$, $E_{16}$, $\Delta E_4$. These 10 MMF are linearly independent, which is proven by the same method. Indeed, we take the "$\mathfrak{E}$" operator of the linear dependeancy of the above 10 MMF. That all coefficients vanish, we prove by inspecting first the coeffcient at $E_2^8$, then at $E_4E_2^6$, and so on. In fact, there are already two terms which contain $E_2^2$; these are $E_{12}\mathfrak{E}\vartheta(Q)$, $\Delta\mathfrak{E}\vartheta(Q)$. But $E_{12}$ and $\Delta$ are linearly independent.

So, we see that, for even $k\geq 2$,

$$\dim(\mathsf{Mmf}_k)=\sum_{\ell=0}^{k/2}\dim_{\mathbb{C}}(M_{k-2\ell}).$$

To prove the main Theorem suppose that we have already proven that

$$\mathfrak{E}\vartheta^s(g)=q_sE_2^{s+1}+\{\text{terms involving } E_2 \text{ and at least one of } E_4,E_6 \text{ of total weight } 2s+2\}.$$

Then from (18) and the definition of $\vartheta$ we derive that

$$q_0=1,\quad q_{s+1}=-\frac{s+1}{12}q_s\Longrightarrow q_s\neq 0. \tag{27}$$

The $\mathfrak{E}$ operation helps to rule out linear dependence among the set of MMF of weight $k$, which we are thus construct. The crucial ingredient is a well-known and important fact which claims that the Eisenstein series $E_2,E_4$ and $E_6$ are algebraically independent [10]. Since (27) shows that $\mathfrak{E}$ is injective and surjective, this proves the main Theorem.

In fact, the map $\mathfrak{E}$, when considered as the map from RT forms to periodic functions, is also injective.

**Proposition 7.** *Let $f(\varkappa,z)\sim\sum_{j=0}^{\infty}(\varkappa-1)^jA_j$ be a weight $k$ RT form. This means that there exists a limit*

$$\lim_{n\to\infty}f(\varkappa,z-n)=A_0(z);$$

*(A constant function in $\varkappa$). Then $f$ is uniquely determined by $A_0(z)$.*

The assumption, as we know, implies that $\lim_{n\to\infty}A_j(z-n)=0$ for $j\geq 1$. We recall that (20) imply that $A_0(z)$ is necessarily a $1-$periodic function.

*Proof.* Indeed, suppose, we have already proved that $A_{i-1}(z)$ is uniquely determined for $i\leq j$. Then from (20) we have

$$A_j|(T^{-n}-T^{-n-1})=A_{j-1}|(T^{-n-1}-R^{-1}T^{-n}),\quad n\in\mathbb{Z}. \tag{28}$$

Since for every fixed $z$, $A_j|T^{-n}\to 0$, as $n\to\infty$, by the assumption of the lemma, the series

$$\sum_{n=0}^{\infty}A_j|(T^{-n}-T^{-n-1})$$

converges to the value $A_j(z)$. So must converge the series made of the right side of (28) for $n = 1, 2, \ldots$, and thus $A_j(z)$ is uniquely determined from

$$A_j(z) = \sum_{n=0}^{\infty} A_{j-1}|(T^{-n-1} - R^{-1}T^{-n}). \tag{29}$$

□

**Proposition 8.** *Again, suppose the assumptions of the previous lemma hold. Let $A_0(z)$ be a $1-$periodic function, and*

$$A_0|(I - R^{-1}) = A(z).$$

*Then the unique MMF $f(\varkappa, z)$, satisfying the condition $\lim_{n\to\infty} f(\varkappa, z-n) = A_0(z)$, is given by*

$$f(\varkappa, z) = A_0(z) - (1-\varkappa) \sum_{\substack{a,b,c,d\geq 0,\\ ad-bc=1}} \frac{\varkappa^{\imath(\frac{a+b}{c+d})}(1-\varkappa)^{\jmath(\frac{a+b}{c+d})}}{(cz-d)^k} A\Big(\frac{az-b}{-cz+d}\Big). \tag{30}$$

*Proof.* The non-homogenic functional equation

$$g(\varkappa, z) = \varkappa g(\varkappa, z)|T^{-1} + (1-\varkappa)g(\varkappa, z)|R^{-1} + (1-\varkappa)A(z)$$

has at least two solutions. The first one is given by $g_0(\varkappa, z) = A_0(z)$, and the second one (at least formally) is given by

$$g_1(\varkappa, z) = (1-\varkappa) \sum_{M=\begin{pmatrix} a & -b \\ -c & d\end{pmatrix} \in \{T^{-1}, R^{-1}\}} \varkappa^{\imath(\frac{a+c}{b+d})}(1-\varkappa)^{\jmath(\frac{a+c}{b+d})} A(z)|M.$$

Here, as before, $\{T^{-1}, R^{-1}\}$ is the semigroup generated by the two matrices $T^{-1}$ and $R^{-1}$. Note that the condition implies $a, b, c, d \geq 0$, $ad - bc = 1$. So, $g_0(\varkappa, z) - g_1(\varkappa, z)$ is formally a MMF. Now, comparison of (29) and (30) with some matrix calculations shows that in fact it is the only possible MMF with the given first coefficient $A_0(z)$ and satisfying the requirements of the lemma. □

So, let, for example, $\mathbf{e}_n(z) = e^{2\pi i z}$. We thus can construct unique functions $\widehat{\mathbf{e}_{\mathbf{n},\mathbf{k}}}(\varkappa, z)$, which satisfy the functional equation (19), and for which $\mathfrak{E}(\widehat{\mathbf{e}_{\mathbf{n},\mathbf{k}}}) = e^{2\pi i n z}$.

## APPENDIX A. ANALYTIC CONTINUATION

To fully solve our ultimate problem, that is, to arithmetically describe the moments of ?$(x)$, we need the full information about $G(\varkappa, z+1)$ at $\varkappa = \frac{1}{2}$. Via Taylor series, we may recover the latter if we know all the derivatives of $G(\varkappa, z+1)$ at $\varkappa = \varrho$. In this Appendix we briefly outline our method to tackle this problem. The full details will be presented in the second part of this work [8].

A.1. **Higher order Taylor coefficients at $\varkappa=\varrho$.** Let

$$G(\varkappa,z+1)-\frac{i}{2\pi}G_2(z)\sim\sum_{k=0}^{\infty}(\varkappa-\varrho)^k\psi_k(z), \tag{31}$$

so that $\psi_0(z)=\psi(z)$. The coefficients $\psi_k$ are defined exactly in the same manner as we did with $A_s(z)$ in Definition 3. Note that

$$\varkappa=\varrho+(\varkappa-\varrho),\quad 1-\varkappa=\varrho^{-1}-(\varkappa-\varrho).$$

Comparing coefficients at $(\varkappa-\varrho)^k$, we immediately obtain

$$\psi_k(z)-\varrho\psi_k(z-1)-\frac{\varrho^{-1}}{(1-z)^2}\psi_k\Big(\frac{z}{1-z}\Big)=\psi_{k-1}(z-1)-\frac{1}{(1-z)^2}\psi_{k-1}\Big(\frac{z}{1-z}\Big).$$

In [8] we will see how this, with some additional informations, defines $\psi_k$ uniquely. We will then explore whether the series in (31) converges in the usual sense, or we need to use, say, the Borel summation. The latter was used in [3, 4] to give the conjectural series (verified numerically) which converges to the moments of $?(x)$.

A.2. **Beyond $\varkappa=\varrho$.** Let for convenience $g(\varkappa,z)=G(\varkappa,z+1)$, and let us rewrite the functional equation (9) in three different ways as follows. The first one is the same as (9). The second one is obtained by dividing it by $\varkappa$ and applying $|T$, and the second is obtained by dividing by $1-\varkappa$, and then applying $|R$. Thus we obtain

$$\begin{aligned}\mathbf{FE}_{01}\ :g &= \varkappa g|T^{-1}+(1-\varkappa)g|R^{-1}+\frac{1-\varkappa}{1-z},\\ \mathbf{FE}_{1\infty}:g &= \frac{1}{\varkappa}g|T+\frac{\varkappa-1}{\varkappa}g|R^{-1}T+\frac{1-\varkappa}{\varkappa z},\\ \mathbf{FE}_{\infty 0}:g &= \frac{1}{1-\varkappa}g|R-\frac{\varkappa}{1-\varkappa}g|T^{-1}R-\frac{1}{z+1}.\end{aligned}$$

Note that

$$\varkappa+(1-\varkappa)=\frac{1}{\varkappa}+\frac{\varkappa-1}{\varkappa}=\frac{1}{1-\varkappa}-\frac{\varkappa}{1-\varkappa}=1.$$

Let us define

$$\begin{aligned}\mathcal{D}_{01} &= \mathcal{D},\\ \mathcal{D}_{1\infty} &= \Big\{\varkappa\in\mathbb{C}:|\varkappa|>1,\Re(\varkappa)>\frac{1}{2}\Big\}=\Big\{\varkappa\in\mathbb{C}:\Big|\frac{1}{\varkappa}\Big|<1,\Big|\frac{\varkappa-1}{\varkappa}\Big|<1\Big\},\\ \mathcal{D}_{\infty 0} &= \Big\{\varkappa\in\mathbb{C}:|\varkappa-1|>1,\Re(\varkappa)<\frac{1}{2}\Big\}=\Big\{\varkappa\in\mathbb{C}:\Big|\frac{1}{1-\varkappa}\Big|<1,\Big|\frac{\varkappa}{1-\varkappa}\Big|<1\Big\}.\end{aligned}$$

The functional equation $\mathbf{FE}_{ab}$, $\{a,b\}\subset\{0,1,\infty\}$ will be the key to extend the function $g(\varkappa,z)$ analytically to the region $\mathcal{D}_{ab}$. However, $\mathbf{FE}_{1\infty}$ and $\mathbf{FE}_{\infty 0}$ differ from $\mathbf{FE}_{01}$ in the following two ways. Consider $\mathbf{FE}_{1\infty}$. While $T^{-1}$ and $R^{-1}$ freely generate the monoid $\mathbf{M}$, the matrix $A:=R^{-1}T=SUS=\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}$ is of order 3, and $TA=S$ is of order 2. Thus, $A$ and $T$ generate the whole modular group, subject to two conditions $(TA)^2=A^3=I$. Equally, consider $\mathbf{FE}_{\infty 0}$. The matrix $T^{-1}R=A^{-1}=SU^2S$ is of order 3, and $RA^{-1}=S$ is of order 2. This will have two consequences. We have seen i Proposition 1 that $g(\varkappa,z)$ can be defined for $\varkappa=0$ and $\varkappa=1$. However, we will see that $\varkappa=\infty$ is a singularity. And second consequence is that for $\varkappa\in\mathcal{D}_{1\infty}\cup\mathcal{D}_{\infty 0}$, the function $g(\varkappa,z)$, as a function in variable $z$, is defined only for $\Im(z)>0$, while for $\varkappa\in\mathcal{D}_{1\infty}$ it is defined in the cut-plane $\mathbb{C}\setminus[0,\infty)$.

Proceeding in the same way as with $\mathbf{FE}_{01}$, which lead to the series (10), we may derive that $\mathbf{FE}_{1\infty}$ implies the series

$$g(\varkappa,z) = \frac{1-\varkappa}{\varkappa}\sum_{\substack{s\in\mathbb{N}\\ \alpha_1\geq 0,\alpha_2,\alpha_3,\ldots,\alpha_s\geq 1\\ \beta_1,\beta_2,\ldots,\beta_{s-1}\geq 1,\beta_s\geq 0}}\frac{1}{(az+b)(cz+d)}\Big(\frac{\varkappa-1}{\varkappa}\Big)^{\sum\alpha_i}\Big(\frac{1}{\varkappa}\Big)^{\sum\beta_i};$$

$$\text{here }\begin{pmatrix} a & b\\ c & d\end{pmatrix} = A^{\alpha_1}T^{\beta_1}A^{\alpha_2}T^{\beta_2}\cdots A^{\alpha_s}T^{\beta_s}.$$

Note that now, contrary to $\mathbf{FE}_{01}$, each matrix from the modular group appears in the sum infinitely often. For example, we may wonder what is the part of this sum which corresponds to the identity matrix $I$. Thus, we get the partial sum

$$\frac{1-\varkappa}{\varkappa z}\sum_{A^{\alpha_1}T^{\beta_1}A^{\alpha_2}T^{\beta_2}\cdots A^{\alpha_s}T^{\beta_s}=I}\Big(\frac{\varkappa-1}{\varkappa}\Big)^{\sum\alpha_i}\Big(\frac{1}{\varkappa}\Big)^{\sum\beta_i}. \qquad (32)$$

At this point of investigation, our research has ramified. This lead to a supplementary paper [7]. The latter might be considered as an appendix to [8].

To calculate the sum (32), note that $A$ and $T$ are subject to two conditions $A^3=(TA)^2=I$. So, if in the product $A^{\alpha_1}T^{\beta_1}A^{\alpha_2}T^{\beta_2}\cdots A^{\alpha_s}T^{\beta_s}$ we encounter two $T's$ in a row, this never reduces to the unity. Thus, for such a word to reduce to a unity, it is necessary that it is of the form

$$A^{\alpha_1}(TA)^{\beta_1}A^{\alpha_2}(TA)^{\beta_2}\cdots A^{\alpha_s}(TA)^{\beta_s}.$$

So we in fact have a sum

$$\frac{1-\varkappa}{\varkappa z}\sum_{A^{\alpha_1}(TA)^{\beta_1}A^{\alpha_2}(TA)^{\beta_2}\cdots A^{\alpha_s}(TA)^{\beta_s}=I}\Big(\frac{\varkappa-1}{\varkappa}\Big)^{\sum(\alpha_i+\beta_i)}\Big(\frac{1}{\varkappa}\Big)^{\sum\beta_i}.$$

The sum thus is equal to the $3rd$ degree algebraic function

$$Q\Big(\frac{\varkappa-1}{\varkappa},\frac{\varkappa-1}{\varkappa^2}\Big),$$

where $Q(x,y)=1+y^2+x^3+\cdots$ is a bivariate 3rd degree algebraic function defined by [7], Theorem 1:

$$(y^6-x^6+6y^2x^3-3y^4+2x^3+3y^2-1)Q^3+(x^3y^2-y^4+x^3+2y^2-1)Q^2+(x^3-y^2+1)Q+1=0.$$

To fully understand $\mathbf{FE}_{01}$, we need analogous results not only for the unity, but for all elements of the modular group. This can be done using the second method in [7]; namely, using a *push-down automaton*.

A.3. **The more general approach.** However, we will see that there is a method how to tackle a problem $\varkappa=\varrho$ and $|\varkappa|>1$, $|1-\varkappa|>1$ using a uniform approacm.

We will soon see that a three-term functional equation (9) can be rewritten in an alternative form as a 4-term of 5-term functional equation.

Let $G(\varkappa, z+1) = f(\varkappa, z)$ for simplicity, and us rewrite (9) as

$$f + \frac{1-\varkappa}{z-1} = \varkappa f|SU + (1-\varkappa)f|SU^2,$$

where $|$ is a weight 2 "slash"-operator. Let us apply now $|S$, $|US$, and $|U^2S$ to this. We get a linear system

$$\begin{cases} f|S - \frac{1-\varkappa}{z(z+1)} &= \varkappa f|SUS + (1-\varkappa)f|SU^2S, \\ f|US - \frac{1-\varkappa}{z+1} &= \varkappa f|SU^2S + (1-\varkappa)f, \\ f|U^2S + \frac{1-\varkappa}{z} &= \varkappa f + (1-\varkappa)f|SUS. \end{cases}$$

Any one of these equations is equivalent to (9), and implies the other two. The determinant of this system

$$D = \begin{vmatrix} 0 & \varkappa & 1-\varkappa \\ 1-\varkappa & 0 & \varkappa \\ \varkappa & 1-\varkappa & 0 \end{vmatrix} = 3\varkappa^2 - 3\varkappa + 1.$$

It vanishes at the point $\varkappa = \tau$, where we put

$$\tau = \frac{1}{2} + \frac{\sqrt{3}}{6}i,$$

and at $\varkappa = \bar{\tau}$. Suppose now, $\varkappa \neq \tau, \bar{\tau}$. Solving the linear system, we obtain

$$(3\varkappa^2 - 3\varkappa + 1)f = \varkappa(\varkappa-1)f|S + (1-\varkappa)^2 f|US + \varkappa^2 f|U^2S + \frac{\varkappa(1-\varkappa)}{z} - \frac{(1-\varkappa)^2}{z+1}. \quad (33)$$

We can show that for all $\varkappa \neq \tau, \bar{\tau}$, this is equivalent to the initial functional equation. Indeed, let us apply $SUS$ to (33), multiply by $(1-\varkappa)$, and add (33) multiplied by $\varkappa$. We get the third equation of (33) multiplied by $D$. Suppose, a function $f(\varkappa, z)$ satisfies all the regularity requirements needed for a mean-modular form of weight $k$, and it satisfies (**??**). Then for $\varkappa \neq \tau, \bar{\tau}$ we recover (**??**). Due to continuity, this holds for $\varkappa = \tau, \bar{\tau}$. Thus,

**Proposition 9.** *The 4-term functional equation (33) is equivalent to (19).*

Now we will solve the following elementary problem: find a region in the complex plane such that

$$\Big|\frac{\varkappa^2}{D}\Big| < 1, \quad \Big|\frac{(1-\varkappa)^2}{D}\Big| < 1.$$

Let $\varkappa = x + iy$. Then the boundary of the domain in question described by the firs inequality is an algebraic quartic

$$\mathcal{Q} = \{x + iy \in \mathbb{C} : 8x^4 + 8y^4 + 16x^2y^2 - 18x^3 - 18xy^2 + 15x^2 + 3y^2 - 6y + 1 = 0\}.$$

The second curve is obtained from this by a reflection $\mathcal{Q} \mapsto 1 - \mathcal{Q}$. These two curves are inside $\overline{\mathcal{D}}$, and the boundary of the region, which satisfies these two inequalities, is an "$\infty$" shape as shown in Figure **??**. Thes two curves intersect at thre points

$$\varkappa = \frac{1}{2}, \quad \varkappa = \frac{1}{2} + \frac{1}{2}i, \quad \varkappa = \frac{1}{2} - \frac{1}{2}i.$$

## APPENDIX B. MODULAR SOLUTIONS

We will now show that the requirement that a mean-modular form is holomorphic in variable $\varkappa$ is essential and strong, since there exists too many functions which satisfy (19) for certain particular fixed $\varkappa$. Moreover, such functions can even be modular forms for congruence subgroups. Consequently, such "sporadic" solutions do not qualify as MMF.

Let $N \in \mathbb{N}$, $k \in 2\mathbb{N}$. Consider the space of modular forms $M_k(\Gamma(N))$. Let $\mathbf{u}(z) = (u_1(z), u_2(z), \dots, u_\ell(z))$ be the basis of this space. We know that for any $u(z) \in M_k(\Gamma(N))$, both $u(z-1)$ and $(1-z)^{-k}u(z/(1-z))$ belong to $M_k(\Gamma(N))$. This simply follows from the fact that $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$. So there exists two matrices $A$ and $B$ such that

$$\mathbf{u}(z-1)^T = A\mathbf{u}(z)^T, \quad (1-z)^{-k}\mathbf{u}(z/(1-z))^T = B\mathbf{u}(z)^T.$$

We want a function

$$\sum_{i=1}^{\ell} a_i u_i(z), \quad a_i \in \mathbb{C},$$

to satisfy (19). There exists a non-zero vector $(a_1, \dots, a_\ell)$ if and only if the determinant of the matrix $I - \varkappa A - (1-\varkappa)B$ vanishes:

$$P_{N,k}(\varkappa) := \det\big(I - B + \varkappa(B - A)\big) = 0.$$

So, each pair $N \geq 2$, $k \in 2\mathbb{N}$, generates the polynomial $P_{N,k}(\varkappa)$, and each root of this polynomial produces the element of $M_k(\Gamma(N))$ that also satisfies (19). For example, let $N = 2$, $k = 2$. The space $M_2(\Gamma(2))$ is 2$-$dimensional and is spanned by $\vartheta^4(0, 1/2; z)$ and $\vartheta^4(1/2, 0; z)$, the Jacobi's theta functions (see further). The polynomial $P_{2,2}(\varkappa) = 3\varkappa(1-\varkappa)$. So, in this case only $\varkappa = 0$ belongs to $\mathcal{D}$. Anyway, using approach via theta constants, we have calculated many possible $\varkappa$, and there are plenty of whose which belong to $\mathcal{D}$; for example, $\varkappa = \frac{1}{2} + \frac{1}{2}i$ is one of them. The approach via theta consists consists of the following.

Let us define, for $a, b \in \mathbb{R}$, $k \in \mathbb{N}$ (no relation to the weight!), $z \in \mathfrak{h}$, the theta-constants [13, 19]

$$\begin{aligned}
\vartheta(a, b; z)_k &= \sum_{n \in \mathbb{Z}} e^{k\pi i[(a+n)^2 z + 2b(a+n)]} = \vartheta(a, kb; kz)_1; \\
\vartheta(a, b; z)'_k &= 2k\pi i \sum_{n \in \mathbb{Z}} (a+n) e^{k\pi i[(a+n)^2 z + 2b(a+n)]} = k\vartheta(a, kb; kz)'_1.
\end{aligned}$$

(No relation to the "$\vartheta$" map!) The next identities are checked directly; they are either immediate, or follow from the Poisson summation formula.

**Proposition 10.** *The functions $\vartheta(a, b; z)_k$ and $\vartheta(a, b; z)'_k$ for rational $a, b$ are modular forms of weights $1/2$ and $3/2$, respectively. Further, we have*

1-1') $\vartheta(a+1, b; z)_k = \vartheta(a, b; z)_k$;
2-2') $\vartheta(a, b + \frac{1}{k}; z)_k = e^{2\pi i a}\vartheta(a, b; z)_k$;
3-3') $\vartheta(a, b; z+1)_k = e^{-k\pi i(a^2+a)}\vartheta(a, b + a + \frac{1}{2}; z)_k$;
4) $\vartheta(-a, -b; z)_k = \vartheta(a, b; z)_k$;
4') $\vartheta(-a, -b; z)'_k = -\vartheta(a, b; z)'_k$;

5) $\vartheta(a,b;-\frac{1}{z})_k = k^{-1/2}(-iz)^{1/2}e^{2k\pi iab}\sum\limits_{s=0}^{k-1}\vartheta(b+\frac{s}{k},-a;z)_k.$

5') $\vartheta(a,b;-\frac{1}{z})'_k = k^{-1/2}i(-iz)^{3/2}e^{2k\pi iab}\sum\limits_{s=0}^{k-1}\vartheta(b+\frac{s}{k},-a;z)'_k.$

*1-1', 2-2' and 3-3' mean that the same transformation rules hold for $\vartheta(a,b;z)_k$ and $\vartheta(a,b;z)'_k$.*

So, we start from any product of these theta constants, that include only rational parameters $a,b$, and which amount to the total weight of, say, 2. This function satisfies transformation properties under $z\mapsto z+1$, $z\mapsto -z^{-1}$. It belongs to the finite orbit, and thus this also reduces to the condition for the determinant. For example, let us consider the simplest case of weight 2 and when these products are in fact 4th powers of theta constants.

B.1. **Theta functions $\vartheta^4(a,b;z)_1$ for $4a,4b\in\mathbb{Z}$.** There are three orbits in this case. First, the orbit-singleton $(\frac{1}{2},\frac{1}{2})$, which produce a zero theta constant. Further, the 3-element orbit $(0,0)$, $(\frac{1}{2},0)$, $(0,\frac{1}{2})$, which was already investigated; these three functions are related via the Jacobi identity:

$$\vartheta^4(0,0;z)=\vartheta^4(1/2,0;z)+\vartheta^4(0,1/2;z).$$

The third orbit consists of 6 elements $(0,\frac{1}{4})$, $(\frac{1}{4},0)$, $(\frac{1}{4},\frac{1}{4})$, $(\frac{3}{4},\frac{1}{4})$, $(\frac{1}{4},\frac{1}{2})$, and $(\frac{1}{2},\frac{1}{4})$. Therefore,

$$\mathbf{u}(z)^T=\begin{pmatrix}\vartheta^4(0,1/4;z)\\ \vartheta^4(1/4,0;z)\\ \vartheta^4(1/4,1/4;z)\\ \vartheta^4(3/4,1/4;z)\\ \vartheta^4(1/4,1/2;z)\\ \vartheta^4(1/2,1/4;z)\end{pmatrix},$$

and the space generated by all six components is invariant under the action of $T$ and $S$.

B.2. **Theta functions $\vartheta^4(a,b;z)_1$ for $(6a,6b)\in\mathbb{Z}^2$, $(2a,2b)\notin\mathbb{Z}^2$.** In this case the theta functions split into three orbits: $Q_1$, consisting of 4 functions with rational pairs $(a,b)=(\frac{1}{6},\frac{1}{6})$, $(\frac{5}{6},\frac{1}{6})$, $(\frac{1}{6},\frac{1}{2})$, $(\frac{1}{2},\frac{1}{6})$; $Q_2$, consisting of 4 rational pairs $(\frac{1}{3},\frac{1}{3})$, $(\frac{1}{3},\frac{1}{2})$, $(\frac{1}{2},\frac{1}{3})$, $(\frac{2}{3},\frac{1}{3})$, and $Q_3$, consisting of 8 pairs $(0,\frac{1}{6})$, $(0,\frac{1}{3})$, $(\frac{1}{6},0)$, $(\frac{1}{6},\frac{1}{3})$, $(\frac{1}{3},0)$, $(\frac{1}{3},\frac{1}{6})$, $(\frac{2}{3},\frac{1}{6})$, $(\frac{5}{6},\frac{1}{3})$. For example,

$$\mathbf{u}(z)^T=\begin{pmatrix}\vartheta^4(1/6,1/6;z)\\ \vartheta^4(5/6,1/6;z)\\ \vartheta^4(1/6,1/2;z)\\ \vartheta^4(1/2,1/6;z)\end{pmatrix},$$

and the space generated by all four components is invariant under the action of $T$ and $S$; this is the subspace of $M_2(\Gamma_2(18))$. In fact, we can use not only the fourth powers but products of different theta constants, this produces the plethora of solutions to (19) with many different algebraic $\varkappa$.

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