

# TRANSFER OPERATOR FOR THE GAUSS' CONTINUED FRACTION MAP. I. STRUCTURE OF THE EIGENVALUES AND TRACE FORMULAS

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ABSTRACT. Let  $\mathcal{L}$  be the transfer operator associated with the Gauss' continued fraction map, known also as the *Gauss-Kuzmin-Wirsing operator*, acting on a Banach space. In this work we prove an asymptotic formula for the eigenvalues of  $\mathcal{L}$ . This settles, in a much stronger form, the conjectures of D. Mayer and G. Roepstorff (1988), A. J. MacLeod (1992), Ph. Flajolet and B. Vallée (1995), also supported by several other authors. Further, we find an exact series for the eigenvalues, which also gives the canonical decomposition of trace formulas due to D. Mayer (1976) and K. I. Babenko (1978). This crystallizes the contribution of each individual eigenvalue in the trace formulas.

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## 1. THE OPERATOR AND THE CONJECTURES

Our first attack of this problem, based on the techniques developed in [5], is presented in [6]. Though the obtained result is highly supported by numerical computations, the series for the eigenvalues in [6] is too complex and gives neither the structure, nor asymptotics. Moreover, the latter technique involves complex dynamics rather explicitly, and therefore this gives even more additional complications.

Here we present another approach (the project spanning 6 years) which does answer these questions and does not use complex dynamics. In the second part of this paper [1] we delve deeper into the nature of the functions  $W_\ell(\mathbf{X})$  and investigate their refined arithmetic structure, and in the third part [2] we explore this problem for *the Mayer-Ruelle operator* for any complex  $s$ , not just  $s = 1$  which corresponds to the current paper (see Subsection 4.2).

**1.1. Introduction.** Let  $\mathbb{D}$  be the disc  $\{z \in \mathbb{C} : |z - 1| < \frac{3}{2}\}$ . Let  $\mathbf{V}$  be the Banach space of functions which are analytic in  $\mathbb{D}$  and are continuous in its closure, with the supremum norm. *The Perron-Frobenius, or the transfer operator for the Gauss' continued fraction map*, also called *the Gauss-Kuzmin-Wirsing operator*, is defined for functions  $f \in \mathbf{V}$  by [27, 34, 35, 43, 51]

$$\mathcal{L}[f(t)](z) = \sum_{m=1}^{\infty} \frac{1}{(z+m)^2} f\left(\frac{1}{z+m}\right). \quad (1)$$

Our chief interest is the point spectrum of this operator. The operator

$$\mathcal{M}[f(t)](z) = \sum_{m=1}^{\infty} \frac{z+1}{(z+m)(z+m+1)} f\left(\frac{1}{z+m}\right),$$

considered on  $\mathbf{V}$ , has the same point spectrum, counting with algebraic multiplicities. It is essential in the study of transfer operators for the Gauss map [31, 42, 43]. However, we will not use it.

As was shown in [8, 9], the operator  $\mathcal{L}$  is then of trace class and is nuclear of order 0. Thus, it possesses the eigenvalues  $\lambda_n$ ,  $n \in \mathbb{N}$ , which are real numbers,  $|\lambda_n| \geq |\lambda_{n+1}|$ ,  $\lambda_1 = 1$ , and

$$\sum_{n=1}^{\infty} |\lambda_n|^\epsilon < +\infty \text{ for every } \epsilon > 0.$$

In fact, the Hilbert-Schmidt operator  $\mathcal{K}$ , defined by

$$\mathcal{K}[u(t)](x) = \int_0^{\infty} \frac{J_1(2\sqrt{xy})}{\sqrt{(e^x - 1)(e^y - 1)}} u(y) dy,$$

for  $u$  belonging to the Hilbert space  $L^2(\mathbb{R}_+, m)$ ,  $dm(y) = \frac{y}{e^y - 1} dy$ , has the same point spectrum, counting algebraic multiplicities [8, 42, 43]; here  $J_1(\star)$  is the Bessel function with an index 1. Numerically, eigenvalues are presented in the Table 1. Pascal Sebah [48] provided high-precision numerical calculations. We note that sometimes other authors present not accurate numerical results (concerning 10<sup>th</sup> or higher significant digits).

$n$	$(-1)^{n+1}\lambda_n$	$n$	$(-1)^{n+1}\lambda_n$	$n$	$(-1)^{n+1}\lambda_n \cdot 10^6$
1	1.0000000000000000	9	0.0002441314655245158	17	0.1010905532214992
2	0.3036630028987326	10	0.00009168908376859330	18	0.03834969795026564
3	0.1008845092931040	11	0.00003451654616385425	19	0.01455613838668023
4	0.03549615902165984	12	0.00001301769787702303	20	0.005527567937997608
5	0.01284379036244026	13	0.000004916782302464491	21	0.002099913582972687
6	0.004717777511571031	14	0.000001859307351509042	22	0.007980457682720196
7	0.001748675124305511	15	0.0000007038113430870398	23	0.0003033862949098575
8	0.0006520208583205029	16	0.0000002666413434479564	24	0.0001153695418144668

TABLE 1. The eigenvalues  $\lambda_n$  for  $1 \leq n \leq 24$ , with 16 significant digits.

The transfer operator  $\mathcal{L}$  arises from the Gauss map  $F(x) = \{1/x\}$ ,  $x \in (0, 1]$ ,  $F(0) = 0$ , in the following way: for functions  $f \in L^1[0, 1]$  it is defined by

$$\mathcal{L}[f(t)](z) = \sum_{y \in F^{-1}(z)} \frac{1}{|F'(y)|} \cdot f(y),$$

which agrees with (1). Let further  $F^{(1)} = F$ , and  $F^{(k)} = F \circ F^{(k-1)}$  for  $k \geq 2$ . As is now well-known, due to important contributions by C. F. Gauss, R. O. Kuzmin, P. Lévy, E. Wirsing, K. I. Babenko, K. I. Babenko and S. P. Jur'ev, D. Mayer, we have

$$\mu(a \in [0, 1] : F^{(k)}(a) < z) = \frac{\log(1+z)}{\log 2} + \sum_{n=2}^{\infty} \lambda_n^k \Phi_n(z). \quad (2)$$

Here  $\mu(\star)$  stands for the Lebesgue measure, and for each  $n \geq 2$ , the eigenfunction  $\Phi_n(z)$  is defined in the cut plane  $\mathbb{C} \setminus (-\infty, -1]$ , it satisfies the conditions  $\Phi_n(0) = \Phi_n(1) = 0$ , the regularity condition

$$\sup_{\Re(z) \geq -\frac{1}{2}} |(z+1)U(z)| < +\infty. \quad (3)$$

where  $U(z) = \Phi'_n(z)$ , and the functional equation

$$\Phi_n(z+1) - \Phi_n(z) = \frac{1}{\lambda_n} \cdot \Phi_n\left(\frac{1}{z+1}\right).$$

Thus,  $\Phi_1(z) = \frac{\log(1+z)}{\log 2}$ ,  $\lambda_1 = 1$ . The eigenfunctions of  $\mathcal{L}$  are then given by  $\Phi'_n(z)$ ,  $n \in \mathbb{N}$ . Moreover, every function  $U(z)$  which satisfies, for a certain  $\lambda \in \mathbb{R} \setminus \{0\}$ , the functional equation

$$U(z) = U(z+1) + \frac{1}{\lambda(z+1)^2} U\left(\frac{1}{z+1}\right), \quad z \in \mathbb{C} \setminus (-\infty, -1], \quad (4)$$

and the regularity property (3), is the eigenfunction of  $\mathcal{L}$  with the eigenvalue  $\lambda$ . More details can be found in [8, 35, 51].

The nature of the eigenvalues  $\lambda_n$  for  $n \geq 2$  is unknown. It is widely believed that these constants are unrelated to other most important constants in mathematics. In particular, it is expected that they are neither algebraic numbers nor periods [36]. We remind that a *period* is a value of an absolutely convergent integral of a rational function with algebraic coefficients, over a domain in  $\mathbb{R}^n$

given by polynomial inequalities with algebraic coefficients. All periods form a ring  $\mathcal{P}$ . However, we will soon see in the item (ii) of Theorem 2 that these eigenvalues are deeply related to an *extended ring of periods*  $\mathcal{P}[\frac{1}{\pi}]$  in a direct way.

**1.2. Trace formulas.** Now, more than 480 digits of  $\lambda_2$  have been calculated [12], but one can get rigorous certificates only for the several first few digits of  $\lambda_2$  and  $\lambda_3$  [19, 35, 40, 42, 52]. On the other hand, the trace of the operator  $\mathcal{L}$  can be given explicitly. As was shown in [41] (see also [17, 18, 37, 42, 43]), we have

$$\mathrm{Tr}(\mathcal{L}) = \sum_{n=1}^{\infty} \lambda_n = \int_0^{\infty} \frac{J_1(2x)}{e^x - 1} dx = \sum_{\ell=1}^{\infty} \frac{1}{\xi_{\ell}^{-2} + 1} \quad (5)$$

$$= \frac{1}{2} - \frac{1}{2\sqrt{5}} + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \binom{2k}{k} (\zeta(2k) - 1)$$

$$= 1 - \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{5}} + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \binom{2k}{k} \left( \zeta(2k) - 1 - \frac{1}{2^{2k}} \right),$$

$$\mathrm{Tr}(\mathcal{L}^2) = \sum_{n=1}^{\infty} \lambda_n^2 = \int_0^{\infty} \int_0^{\infty} \frac{J_1(2\sqrt{xy})^2}{(e^x - 1)(e^y - 1)} dx dy = \sum_{i,j=1}^{\infty} \frac{1}{(\xi_{i,j} \xi_{j,i})^{-2} - 1}. \quad (6)$$

Here

$$\xi_{\ell} = \frac{1}{\ell+} \frac{1}{\ell+} \frac{1}{\ell+} \cdots = \frac{\sqrt{\ell^2 + 4} - \ell}{2}, \quad \xi_{\ell}^{-2} + 1 = \frac{\ell^2 + 4 + \ell\sqrt{\ell^2 + 4}}{2}, \quad \ell \in \mathbb{N},$$

$$\xi_{i,j} = \frac{1}{i+} \frac{1}{j+} \frac{1}{i+} \frac{1}{j+} \cdots, \quad (\xi_{i,j} \cdot \xi_{j,i})^{-2} - 1 = \frac{D}{2} + \frac{ij + 2}{2} \sqrt{D}, \quad D = (ij + 2)^2 - 4, \quad i, j \in \mathbb{N}.$$

These are quadratic irrationals whose continued fraction expansion are strictly 1 and 2-periodic. That is, they are all fixed points of  $F$  and  $F^2$ , respectively. The formula for  $\mathrm{Tr}(\mathcal{L})$  which involves zeta values was derived in [17, 19]. In Subsection 11.1 we will derive similar formula for  $\mathrm{Tr}(\mathcal{L}^2)$ , and this is very useful computationally.

**Proposition 1.** *We have an identity:*

$$\begin{aligned} \mathrm{Tr}(\mathcal{L}^2) &= \frac{2}{5 + 3\sqrt{5}} + \frac{1}{3 + 2\sqrt{3}} + \frac{4}{21 + 5\sqrt{21}} + \frac{3}{16 + 12\sqrt{2}} \\ &+ \sum_{k=2}^{\infty} (-1)^k \binom{2k-2}{k-2} \left( \zeta^2(k) - 1 - \frac{2}{2^k} - \frac{2}{3^k} - \frac{3}{4^k} \right) \\ &= 1.103839653617_+. \end{aligned}$$

*The  $k$ th term of this series is asymptotically equal to  $\frac{1}{2}(\frac{4}{5})^k(\pi k)^{-1/2}$ .*

We have many formulas to calculate zeta values with very high accuracy, and this gives a fast method to calculate  $\mathrm{Tr}(\mathcal{L}^2)$ . As is clear from [19], the authors of that paper had already derived this formula, though this was not written down explicitly. The importance of this formula is apparent in numerical calculations, when one extracts approximations to the eigenvalues  $\lambda_n$  as eigenvalues of a certain high-order matrix; usually about the order 300-400. While increasing this order some

eigenvalues do stabilize, thus it can be guessed that they correspond to some real  $\lambda_n$ , while others (“spurious” ones) tend to disappear. The invalidation of the formula for  $\text{Tr}(\mathcal{L}^2)$  also shows that these “spurious” eigenvalues should be ruled out - this fact was noticed by all mathematicians who tried numerically to operate with similar matrices.

In general, for  $a_i \in \mathbb{N}$ ,  $1 \leq i \leq k$ , we put

$$\xi_{a_1, a_2, \dots, a_k} = [0, \overline{a_1, a_2, \dots, a_k}].$$

This number is a fixed point of  $F^{(k)}$ . Then we have the fundamental result [43]:

$$\sum_{n=1}^{\infty} \lambda_n^k = \text{Tr}(\mathcal{L}^k) = \sum_{i_1, i_2, \dots, i_k=1}^{\infty} \left[ \prod_{s=1}^k \xi_{i_s, i_{s+1}, \dots, i_k, i_1, \dots, i_{s-1}}^{-2} - (-1)^k \right]^{-1}. \quad (7)$$

We will see in Subsection 12.1 that for the number in the square brackets, denoted there by  $\varrho$ , we have a very simple formula. Indeed, let  $p = \xi_{i_1, i_2, \dots, i_k}$ , and

$$\frac{ap + b}{cp + d} = p, \text{ where } a, b, c, d \in \mathbb{N}_0, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} P_{k-1} & P_k \\ Q_{k-1} & Q_k \end{pmatrix},$$

where  $\frac{P_s}{Q_s}$  is the  $s$ th convergent to  $p$ ,  $\frac{P_1}{Q_1} = \frac{1}{i_1}$ . Then

$$\prod_{s=1}^k \xi_{i_s, i_{s+1}, \dots, i_k, i_1, \dots, i_{s-1}}^{-2} - (-1)^k = \frac{D}{2} + \frac{a+d}{2} \sqrt{D}, \quad D = (a+d)^2 - 4(-1)^k.$$

This, of course, does not depend on the cyclic permutation of  $\{i_1, i_2, \dots, i_k\}$ , since the matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is then replaced by a similar matrix. Indeed,

$$\begin{pmatrix} P_{k-1} & P_k \\ Q_{k-1} & Q_k \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & i_1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & i_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & i_k \end{pmatrix},$$

and the fact becomes obvious.

Without going into details, we note that these trace formulas and the field itself is deeply and intricately related to the Selberg zeta function [32], the Riemann zeta function, Maass wave forms and modular forms for the full modular group [2, 37, 39, 44, 52] - see Subsection 4.2.

**1.3. Previous conjectures.** Throughout this paper, we fix the notation

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

The constants  $\lambda_n$  have received a considerable amount of attention in recent decades. Nevertheless, there remained three outstanding unresolved problems. No theoretical progress was made towards any of them, only a computational one. As was said before, we henceforth arrange the eigenvalues according to their absolute value:

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots$$

Of course, in case  $\lambda_n = \pm\lambda_{n+1}$  for some  $n$ , this arrangement is not uniquely defined. Despite this, we have

**Conjecture.** *The following three statements are true:*

- i) *Simplicity. The eigenvalues are simple,  $|\lambda_n|$  strictly decreases.*
- ii) *Sign. The eigenvalues have alternating sign:  $(-1)^{n+1}\lambda_n > 0$ .*
- iii) *Ratio. There exists  $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\lambda_{n+1}} = -\frac{3+\sqrt{5}}{2} = -\phi^2$ .*

The first conjecture can be attributed to Babenko [8] and Mayer and Roepstorff [43], the second - to Mayer and Roepstorff too, also reiterated by MacLeod [40]. The last was raised by MacLeod (only with a constant  $\approx -2.6$ ), and seconded by Flajolet and Vallée [17, 19, 20, 21], with the constant  $-\phi^2$ . Several other authors also claimed to believe these conjectures, or state that “for the time being [2014], no way for obtaining [formula for the eigenvalues] can be expected” [30].

The ratio conjecture has the following explanation. The spectrum of the operator

$$\mathcal{L}_0[f(t)](z) = \frac{1}{(z+1)^2} f\left(\frac{1}{z+1}\right), \quad f \in \mathbf{V},$$

is given by  $\lambda_n^0 = (-1)^{n+1}\phi^{-2n}$ ,  $n \in \mathbb{N}$ , with the corresponding eigenfunction being

$$u_0(n, z) = \frac{(z - \phi^{-1})^{n-1}}{(z + \phi)^{n+1}}. \quad (8)$$

(see Section 6). It is expected that the terms in (1) for  $m \geq 2$  act only as small perturbations to  $\mathcal{L}_0$ ; we will show (using algebra and combinatorics, but no functional analysis) that this is indeed the case. Of course, the “Sign” and “Simplicity” conjectures follow from the “Ratio” conjecture for sufficiently large  $n$ , provided it is effective and we can verify these conjectures for the first initial values of  $n$ .

## 2. MAIN RESULTS

**2.1. Formulation.** It is surprising that in fact the real asymptotics of the sequence  $\lambda_n$ , minding the above remark about  $\mathcal{L}_0$ , is the simplest imaginable from the ratio conjecture.

**Theorem 1** (Asymptotics). *We have the formula*

$$(-1)^{n+1}\lambda_n = \phi^{-2n} + C \cdot \frac{\phi^{-2n}}{\sqrt{n}} + d(n) \cdot \frac{\phi^{-2n}}{n},$$

$$\text{where the constant } C = \frac{\sqrt[4]{5} \cdot \zeta(3/2)}{2\sqrt{\pi}} = 1.1019785625880999_+;$$

here  $\zeta(\star)$  is the Riemann zeta function, and the function  $d(n)$  is bounded.

Based on high precision numerical computations of P. Sebah [48], we calculated the function  $d(n)$  in Table 2. Since  $\frac{\phi^{-300}}{150} \approx 0.13 \cdot 10^{-64}$ , this table overwhelmingly convinces in the validity of Theorem 1.

Indeed, suppose we change the constant “2” in the denominator of the fraction which defines  $C$  with 1.9 and 2.1, respectively. The values of the so obtained functions  $d_{1.9}(n)$  and  $d_{2.1}(n)$  for  $n = 148, 149$ , and 150 would then be, respectively,

$$-0.346715, \quad -0.349104, \quad -0.351485; \quad \text{and} \quad 0.997259, \quad 0.999403, \quad 1.001539.$$

In the first case, the value for  $d_{1.9}(n)$  decreases by approx. 6.8‰ (per mille) at each step, and in the second  $d_{2.1}(n)$  increases by approx. 2.1‰. Whereas for the correct constant 2, we see the decrease of  $d(n) = d_{2.0}(n)$  by only 0.025‰ at each step.

$n$	$d(n)$	$n$	$d(n)$	$n$	$d(n)$	$n$	$d(n)$
1	0.51605	5	0.43430	40	0.36268	130	0.359061
2	0.60424	10	0.38504	50	0.36159	148	0.358871
3	0.52221	20	0.36884	70	0.36040	149	0.358862
4	0.46629	30	0.36460	100	0.35952	150	0.358852

TABLE 2. The function  $d(n)$ .

Theorem 1 is much stronger than the ratio conjecture. Seemingly, the apparent distance from  $\lambda_1 \phi^2 = 2.61803_+$  to 1 prevented other mathematicians from posing a much stronger conjecture on the asymptotics. Nevertheless, for example, we have:  $|\lambda_{150}| \cdot \phi^{300} = 1.0923_+$ .

Our second result gives the much more refined structure of the eigenvalues. Let  $P_m^{(\alpha, \beta)}(x)$  stand for the classical Jacobi polynomials [49]; see Subsection 4.1 for more details.

**Theorem 2** (Arithmetic and decomposition of trace formulas). *The following holds.*

(i) *There exist functions  $W_v(\mathbf{X})$ ,  $v \geq 0$ , defined by  $W_0(\mathbf{X}) = 1$ ,  $W_1(\mathbf{X}) = \frac{5}{4} \cdot \phi^{-2\mathbf{X}} P_{\mathbf{X}-1}^{(0,1)}(3/2)$ , and then by a certain explicit recurrence - this will be given later, see (35) - such that*

$$(-1)^{n+1} \lambda_n = \phi^{-2n} \sum_{\ell=0}^{\infty} W_{\ell}(n).$$

For the functions  $W_{\ell}(n)$  we have an asymptotic formula

$$W_{\ell}(n) = \frac{\sqrt[4]{5}}{2\sqrt{\pi} \cdot \ell^{3/2} \sqrt{n}} + \frac{B}{\ell^{3/2} n} \text{ for } n, \ell \geq 1,$$

where the function  $B = B(\ell, n)$  is bounded.

(ii) This decomposition is compatible with (7) and gives the decomposition of trace formulas for the powers of  $\mathcal{L}$ : for the first, the second, and the third powers we have, respectively,

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^{-2n} W_{\ell-1}(n) &= \frac{1}{\xi_{\ell}^{-2} + 1}, \quad \ell \geq 1, \\ \sum_{i+j=\ell} \sum_{n=1}^{\infty} \phi^{-4n} W_{i-1}(n) W_{j-1}(n) &= \sum_{i+j=\ell} \frac{1}{(\xi_{i,j} \xi_{j,i})^{-2} - 1}, \quad \ell \geq 2, \\ \sum_{i+j+k=\ell} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^{-6n} W_{i-1}(n) W_{j-1}(n) W_{k-1}(n) &= \sum_{i+j+k=\ell} \frac{1}{(\xi_{i,j,k} \xi_{j,k,i} \xi_{k,i,j})^{-2} + 1}, \quad \ell \geq 3. \end{aligned} \quad (9)$$

Analogously for higher powers of  $\mathcal{L}$ , for  $\mathcal{L}^k$ .

(iii)<sup>★</sup> Further, let  $\mathbf{a} = \{\ell_1, \ell_2, \dots, \ell_s\}$ ,  $\ell_i \in \mathbb{N}$ . Let us define

$$\Omega_{\mathbf{a}}(w) = \sum_{n=1}^{\infty} \left( \prod_{i=1}^s W_{\ell_i}(n) \right) w^n, \quad |w| < 1.$$

Then all  $\Omega_{\mathbf{a}}(w) = \Omega_{\ell_1, \ell_2, \dots, \ell_s}(w)$  are arithmetic functions.

(iv) The solution to (4) for the  $n$ th eigenvalue, the  $n$ th eigenfunction, can be canonically normalized, so that there exists a sequence of structural constants  $\{U_n(\phi^{-1}), n \in \mathbb{N}\}$ .

So, the fundamental trace formulas (7) are split into a countable number of formulas each. Moreover, we will see in the Subsection 3.4 that, due to combinatoric reasons alone, one further refined decomposition occurs. Concerning the item (iv), one canonical normalization of  $U_n$  comes from (2), but it is ineffective. The normalization we propose is effective and the most natural in deriving formula in the item (i), as will be soon clear; see Proposition 2.

The simplest examples of (iii) are

$$\begin{aligned} \Omega_{\emptyset}(w) &= \frac{w}{1-w}, \\ \Omega_1(w) &= \frac{\phi^2 + w}{2((\phi^4 - w)(1-w))^{1/2}} - \frac{1}{2}. \end{aligned} \quad (10)$$

So, in the trace formulas now we are able to crystallize the contribution of each individual eigenvalue. Thus, this defines an infinite matrix

$$\left( (-1)^{n+1} \phi^{-2n} W_{\ell-1}(n) \right)_{n, \ell=1}^{\infty},$$

whose elements in rows add up to eigenvalues, elements in columns add up to  $(\xi_{\ell}^{-2} + 1)^{-1}$ , and the sum of all real numbers in the matrix is equal to  $\text{Tr}(\mathcal{L})$ . At the same time we are able to define a

<sup>★</sup>This part will be explained, proved and expanded in [1].



unique two variable function which governs the whole collection of eigenvalues and trace formulas:

$$\mathfrak{G}(w, \omega) = \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} w^n \omega^{\ell-1} W_{\ell-1}(n) = \sum_{\ell=0}^{\infty} \Omega_{\ell}(w) \omega^{\ell}, \quad |\omega| \leq 1, \quad |w| < 1.$$

For example,

$$-\mathfrak{G}(-\phi^{-2}, \omega) = \sum_{n=1}^{\infty} \lambda_n(\omega) = \text{Tr}(\mathcal{L}_{\omega}) = \sum_{\ell=1}^{\infty} \frac{\omega^{\ell-1}}{\xi_{\ell}^{-2} + 1} = \int_0^{\infty} \frac{J_1(2x)}{e^x - \omega} dx.$$

The operator  $\mathcal{L}_{\omega}$  will be soon defined in Subsection 2.2. It is important to observe that Theorem 2, for the first time, gives rigorous certificates (though considerable computational time and space resources are needed) to calculate numerically the first few digits of an eigenvalue with any index.

**2.2. The approach.** Our method of proving these results is constructive. Consider a function  $f(\omega, z)$ , which is analytic in  $\omega$  for  $|\omega| \leq 1$ , and for every such  $\omega$ ,  $f(\omega, z) \in \mathbf{V}$  as a function in  $z$ . Denote the set of all such functions by  $\tilde{\mathbf{V}}$ . Then

$$\|f\|_{\tilde{\mathbf{V}}} = \sup_{\substack{|\omega| \leq 1 \\ |z-1| \leq \frac{3}{2}}} |f(\omega, z)|$$

makes  $\tilde{\mathbf{V}}$  into a Banach space. Let us define the operator  $\mathcal{L}_{\omega} : \tilde{\mathbf{V}} \mapsto \tilde{\mathbf{V}}$  by

$$\mathcal{L}_{\omega}[f(\omega, t)](z) = \sum_{m=1}^{\infty} \frac{\omega^{m-1}}{(z+m)^2} f\left(\omega, \frac{1}{z+m}\right). \quad (11)$$

If  $G(\omega, z)$  is an eigenfunction of this operator, then a function  $G(\omega, z)$  is defined up to a “scalar” multiple, which (in this case) is an arbitrary function in  $\omega$ , holomorphic for  $|\omega| \leq 1$ . Thus, if  $\lambda(\omega)$  is an eigenvalue of this operator, then

$$\lambda(\omega)G(\omega, z) = \omega\lambda(\omega)G(\omega, z+1) + \frac{1}{(z+1)^2} \cdot G\left(\omega, \frac{1}{z+1}\right), \quad z \in \mathbb{C} \setminus (-\infty, -1], \quad |\omega| \leq 1. \quad (12)$$

We note that though  $G(\omega, z)$  is initially defined for  $|z-1| < \frac{3}{2}$ , eigenfunctions automatically extend to the region  $\{\mathbb{C} \setminus (-\infty, -1]\}$ .

For a two variable analytic function  $g(\omega, x)$ , let  $g(\omega, x)[x^m] = \beta_m(\omega)$  is the coefficient at  $x^m$  in a Taylor series expansion of  $g(\omega, x)$  around  $x = 0$ . All our results follow from the following explicit construction.

**Proposition 2.** *For every  $n \in \mathbb{N}$ , there exists the unique analytic function  $\lambda_n(\omega)$  and the unique analytic function  $G_n(\omega, z)$ ,  $|\omega| \leq 1$ ,  $\Re z > -\frac{1}{2}$ , whose Taylor coefficients in the variable  $\omega$  are explicitly constructable (see Section 6), which satisfies in conjunction the functional equation (12) and the regularity condition (3) (uniformly in  $\omega$ ), with the following initial value and normalization properties:*

i)

$$\lambda_n(0) = (-1)^{n+1} \phi^{-2n},$$

ii)

$$G_n(0, z) = \frac{(z - \phi^{-1})^{n-1}}{(z + \phi)^{n+1}},$$

iii)

$$\frac{5}{(1-x)^2} G_n\left(\omega, \frac{x\phi + \phi^{-1}}{1-x}\right)[x^{n-1}] = 1;$$

here  $1 = 1\omega^0 + 0\omega^1 + 0\omega^2 + \dots$  is a constant function in  $\omega$ .

In terms of Section 6, the last item will correspond to the choice  $q_0^{(n)} = 1$ ,  $q_v^{(n)} = 0$  for  $v \geq 1$ . We include the second property for clarity only, since it is the consequence of the first property, the regularity condition, and the three term functional equation (12). Thus, for every  $n$ , we construct explicitly  $\lambda_n = \lambda_n(1)$  and the solution  $U_n(z) = G_n(1, z)$ , therefore canonically normalized which together satisfy (4). The existence of this canonical normalization, alternative to the one implied by (2), is also a novel feature in the theory of the transfer operator  $\mathcal{L}$ . This means that for each eigenvalue  $\lambda_n$ , there exists the canonical solution to (4); for example, this reveals a set of new structural constants  $\{U_n(\phi^{-1}) : n \in \mathbb{N}\}$ , as claimed by the item (iv) of Theorem 2. The specific choice of  $z = \phi^{-1}$  will become clear from Subsection (4.4).

Comparing trace formulas we will see that there are no other eigenvalues and eigenfunctions apart from those explicitly constructed. *A priori*, eigenvalue  $\delta(\omega)$  of some operator  $T_\omega : \tilde{\mathbf{V}} \mapsto \tilde{\mathbf{V}}$ , which is analytic in  $\omega$ , can be expressed as a *Puiseux series*, thus containing fractional powers of  $\omega$ :

$$\delta(\omega) = \sum_{k=0}^{\infty} a_k \omega^{k/L}, \quad a_k \in \mathbb{C}, \text{ for some } L \in \mathbb{N}.$$

**Corollary.** *Puiseux series for the eigenvalues and eigenfunctions of (12) contain only integral powers of  $\omega$ .*

Thus, collecting everything together, we obtain

**Corollary.** *All three claims of the Conjecture are true.*

Note that we prove the *eigenvalue simplicity conjecture* independently of the asymptotic result: the proof of simplicity, as well as the decomposition formulas are based on Proposition 2, but follow independently. Indeed, this proposition shows that for one particular eigenvalue only one eigenfunction can be constructed, and there are no other, since that would invalidate the trace formulas. For the sign conjecture there is no direct proof, so we must rely on computer resources and explicit bounds for the function  $B(\ell, n)$  from the Theorem 2.

**2.3. The case  $\omega = \frac{1}{2}$ .** Note that, in relation to the *Minkowski question mark function*, the eigenvalues of the following operator were investigated in [3, 4]. For  $f \in \mathbf{V}$ , it is defined by

$$\tilde{\mathcal{L}}[f(t)](z) = \sum_{n=1}^{\infty} \frac{1}{2^n (z+n)^2} f\left(\frac{1}{z+n}\right).$$

The first four eigenvalues are

$$\tilde{\lambda}_1 = 0.25553210_+, \quad \tilde{\lambda}_2 = -0.08892666_+, \quad \tilde{\lambda}_3 = 0.03261586_+, \quad \tilde{\lambda}_4 = -0.01217621_+.$$

Based on the results of the current work and the formula (43), we can therefore claim that

$$(-1)^{n+1}\tilde{\lambda}_n = \frac{1}{2}\Lambda\left(\frac{1}{2}, n\right) = \phi^{-2n} \sum_{\ell=0}^{\infty} W_{\ell}(n)2^{-\ell-1}, \quad n \in \mathbb{N}.$$

In particular,

$$(-1)^{n+1}\tilde{\lambda}_n = \frac{\phi^{-2n}}{2} + \tilde{C} \cdot \frac{\phi^{-2n}}{\sqrt{n}} + \tilde{d}(n) \cdot \frac{\phi^{-2n}}{n},$$

$$\text{where the constant } \tilde{C} = \frac{\sqrt[4]{5} \cdot \text{Li}_{3/2}(1/2)}{4\sqrt{\pi}} = 0.1317875324465590_+,$$

and the function  $\tilde{d}(n)$  is bounded.

### 3. EXAMPLES AND DECOMPOSITIONS

**3.1. Initial cases of decomposition.** In the next examples  $k$  always means which power of the operator  $\mathcal{L}$  we are looking at, and  $\ell$  - the sum of partial quotients in the period of the quadratic irrational.

The decomposition identities of Theorem 1, (ii) can be checked to hold true for, say,  $\ell = 1, 2$  (in the first case which describes  $\mathcal{L}^1$ ),  $\ell = 3$  (in the second case which describes  $\mathcal{L}^2$ ), and  $\ell = 4$  (in the third case which describes  $\mathcal{L}^3$ ), if we use the generating function for Jacobi polynomials (45):

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^{-2n} &= \frac{1}{\phi^2 + 1} = \frac{1}{\xi_1^{-2} + 1} = (-1)^1 \Omega_{\emptyset}(-\phi^{-2}), \\ \frac{5}{4} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^{-4n} P_{n-1}^{(0,1)}(3/2) &= \frac{1}{4 + 2\sqrt{2}} = \frac{1}{\xi_2^{-2} + 1} = (-1)^1 \Omega_1(-\phi^{-2}), \\ \frac{5}{4} \sum_{n=1}^{\infty} \phi^{-6n} P_{n-1}^{(0,1)}(3/2) &= \frac{1}{6 + 4\sqrt{3}} = \frac{1}{(\xi_{1,2}\xi_{2,1})^{-2} - 1} = (-1)^2 \Omega_1(\phi^{-4}), \\ \frac{5}{4} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^{-8n} P_{n-1}^{(0,1)}(3/2) &= \frac{1}{20 + 6\sqrt{10}} = \frac{1}{(\xi_{1,1,2}\xi_{1,2,1}\xi_{2,1,1})^{-2} + 1} = (-1)^3 \Omega_1(-\phi^{-6}). \end{aligned}$$

MAPLE re-confirms this, too. In general, we will see in Subsection 11.2, and this can be calculated directly from (10), that

$$(-1)^k \Omega\left((-1)^k \phi^{-2k}\right) = \frac{1}{\psi_k}, \quad \psi_k = 2F_{k+1}^2 - 2(-1)^k + 2F_{k+1} \sqrt{F_{k+1}^2 - (-1)^k}. \quad (13)$$

Here  $F_k$  is the standard Fibonacci sequence with the (standard) seed values  $F_1 = F_2 = 1$ .

$\ell = 3, k = 1$  Next, though we do not have explicit values for  $W_2(n)$  in a closed form (but see (37) and Section 10), but numerical values can be calculated easily to a very high precision. So, in particular, the first identity of (ii) for  $\ell = 3$ , minding the values in Table 5 and further calculations,

give the correct anticipated value, and this shows that our method indeed works! Thus, MAPLE does indeed confirm that

$$\sum_{n=1}^{\infty} (-1)^{n+1} W_2(n) \phi^{-2n} = 0.0839748528310781_+ = \frac{2}{13 + 3\sqrt{13}} = \frac{1}{\xi_3^{-2} + 1}.$$

$\ell = 4, k = 2$ . A still more interesting example occurs in the second identity of (ii) in case  $\ell = 4$ . Numerically, we have:

$$A = \sum_{n=1}^{\infty} W_2(n) \phi^{-4n} = 0.0428848639793538_+,$$

$$B = \sum_{n=1}^{\infty} W_1^2(n) \phi^{-4n} = 0.0356498091111648_+,$$

$$C = \frac{2}{21 + 5\sqrt{21}} = \frac{1}{(\xi_{1,3}\xi_{3,1})^{-2} - 1} = 0.0455447255899809_+,$$

$$D = \frac{1}{16 + 12\sqrt{2}} = \frac{1}{(\xi_{2,2}\xi_{2,2})^{-2} - 1} = 0.0303300858899106_+.$$

And thus, we get the correct identity (compare (18) and (19) below for  $k = 2$ )

$$2A + B = 2C + D = 0.1214195370698725_+.$$

As can be anticipated, this identity apparently does not decompose into two identities, since there is no *a priori* reason why  $A$  or  $B$  should be algebraic or other explicit numbers. In fact, we know the formula (46), where  $B$  is expressed in terms of complete elliptic integrals of the first and the third kind. In our case  $w = \phi^{-4}$ . So,  $z = \frac{\phi^2}{3}$ ,  $z\sqrt{w} = \frac{1}{3}$ ,  $zw = \frac{\phi^{-2}}{3}$ , and consequently,

$$4\pi \cdot \left(B + \frac{1}{2}\right) = \frac{4\sqrt{5}}{3} \cdot K\left(\frac{1}{3}\right) + \frac{2}{3} \cdot \Pi\left(\frac{\phi^2}{3}, \frac{1}{3}\right) - \frac{2}{3} \cdot \Pi\left(\frac{\phi^{-2}}{3}, \frac{1}{3}\right).$$

We recall that

$$\Pi(\nu, k) = \int_0^1 \frac{dt}{(1 - \nu t^2) \sqrt{(1 - t^2)(1 - k^2 t^2)}}, \quad K(k) = \Pi(0, k).$$

So, according to the notion of Zagier-Kontsevitch [36], which we mentioned in the end of Subsection 1.1, the number  $B$  is *arithmetic* in the sense that it belongs to the extended ring of periods  $\widehat{\mathcal{P}} = \mathcal{P}[\frac{1}{\pi}]$ , and then so does  $A$ , since algebraic numbers are also periods. In general, in the next subsection we will see that

$$\Omega_2\left((-1)^k \phi^{-2k}\right) \in \widehat{\mathcal{P}} \text{ for } k \in \mathbb{N}.$$

This is exactly what we mean by the item (iii) in Theorem 2. An alternative approach is presented in Section 10.

$\ell = 5, k = 3$ . Further, the last identity of (ii) in case  $\ell = 5$  gives the following. Let us define

$$\begin{aligned} E &= \sum_{n=1}^{\infty} (-1)^{n+1} W_2(n) \phi^{-6n} = 0.0144368544431652_+, \\ F &= \sum_{n=1}^{\infty} (-1)^{n+1} W_1^2(n) \phi^{-6n} = 0.0123983653921924_+, \\ G &= \frac{1}{34 + 8\sqrt{17}} = \frac{1}{(\xi_{1,1,3}\xi_{1,3,1}\xi_{3,1,1})^{-2} + 1} = 0.0149287499273340_+, \\ H &= \frac{2}{85 + 9\sqrt{85}} = \frac{1}{(\xi_{1,2,2}\xi_{2,2,1}\xi_{2,1,2})^{-2} + 1} = 0.0119064699080236_+. \end{aligned}$$

Then

$$E + F = G + H = 0.0268352198353576_+.$$

In fact, the numbers  $G$  and  $H$  are special cases of the numbers which will be investigated in Subsections 11.2 and 12.1, respectively:  $1/G$  is given by the formula (49) ( $N = 3, k = 3$ ), and  $1/H$  - by the formulas (54) and (56) ( $k = 3, r = 2$ ). As noted above,  $E, F \in \widehat{\mathcal{P}}$ .

$\ell = 6, k = 4$ . This is a final example from the ones with  $\ell - k = 2$ . We will see now the sum of three quadratic irrationals. Indeed, let

$$\begin{aligned} I &= \sum_{n=1}^{\infty} W_2(n) \phi^{-8n} = 0.0057706896109551_+, \\ J &= \sum_{n=1}^{\infty} W_1^2(n) \phi^{-8n} = 0.0048993082411535_+, \\ K &= \frac{2}{165 + 13\sqrt{165}} = \frac{1}{(\xi_{1,1,1,3}\xi_{1,1,3,1}\xi_{1,3,1,1}\xi_{3,1,1,1})^{-2} - 1} = 0.0060243137049899_+, \\ L &= \frac{2}{221 + 15\sqrt{221}} = \frac{1}{(\xi_{2,2,1,1}\xi_{2,1,1,2}\xi_{1,1,2,2}\xi_{1,2,2,1})^{-2} - 1} = 0.00450459549723434_+, \\ M &= \frac{1}{96 + 56\sqrt{3}} = \frac{1}{(\xi_{2,1,2,1}\xi_{1,2,1,2}\xi_{2,1,2,1}\xi_{1,2,1,2})^{-2} - 1} = 0.0051814855409225_+. \end{aligned}$$

Again, we used formulas (49), (54) and (56). MAPLE confirms that

$$2I + 3J = 2K + 2L + M = 0.0262393039453711_+.$$

$\ell = 4, k = 1$ , and  $\ell = 5, k = 2$ . Since there are infinitely many identities involving the three functions  $W_1, W_2$ , and  $W_3$ , we will confine to the last three examples, now involving  $W_3$ . First, we will check the validity of the first two identities of (ii) for  $\ell = 4$  and  $\ell = 5$ , respectively. Using the values in Table 6 and further calculations, we thus get the correct value:

$$\sum_{n=1}^{\infty} (-1)^{n+1} W_3(n) \phi^{-2n} = 0.052786404500042_+ = \frac{1}{10 + 4\sqrt{5}} = \frac{1}{\xi_4^{-2} + 1}.$$

Next, let us define

$$\begin{aligned}
N &= \sum_{n=1}^{\infty} W_3(n) \phi^{-4n} = 0.0268901808819348_+, \\
O &= \sum_{n=1}^{\infty} W_1(n) W_2(n) \phi^{-4n} = 0.01983768450229800_+, \\
P &= \frac{1}{(\xi_{1,4} \xi_{4,1})^{-2} - 1} = \frac{1}{16 + 12\sqrt{2}} = 0.0303300858899106_+, \\
Q &= \frac{1}{(\xi_{2,3} \xi_{3,2})^{-2} - 1} = \frac{1}{30 + 8\sqrt{15}} = 0.0163977794943222_+.
\end{aligned}$$

Note that, according to the formula in Subsection 11.1,  $\xi_{2,2} \xi_{2,2} = \xi_{1,4} \xi_{4,1}$ ; and so  $D = P$ . Then we get the correct identity:

$$N + O = P + Q = 0.0467278653842328_+.$$

$\ell = 6, k = 3$ . This is the final example. Let

$$\begin{aligned}
R &= \sum_{n=1}^{\infty} (-1)^{n+1} W_3(n) \phi^{-6n} = 0.0090670451347138_+, \\
S &= \sum_{n=1}^{\infty} (-1)^{n+1} W_1(n) W_2(n) \phi^{-6n} = 0.0069673221927849_+, \\
T &= \sum_{n=1}^{\infty} (-1)^{n+1} W_1^3(n) \phi^{-6n} = 0.0059673995757538_+, \\
U &= \frac{1}{52 + 10\sqrt{26}} = \frac{1}{(\xi_{1,1,4} \xi_{1,4,1} \xi_{4,1,1})^{-2} + 1} = 0.0097096621545399_+, \\
V &= \frac{1}{74 + 12\sqrt{37}} = \frac{1}{(\xi_{1,2,3} \xi_{2,3,1} \xi_{3,1,2})^{-2} + 1} = 0.0068030380839281_+, \\
W &= \frac{1}{74 + 12\sqrt{37}} = \frac{1}{(\xi_{1,3,2} \xi_{3,2,1} \xi_{2,1,3})^{-2} + 1} = 0.0068030380839281_+, \\
X &= \frac{1}{100 + 70\sqrt{2}} = \frac{1}{(\xi_{2,2,2} \xi_{2,2,2} \xi_{2,2,2})^{-2} + 1} = 0.0050252531694167_+.
\end{aligned}$$

Quadratic irrationals can be calculated directly, but it is most convenient to use formula (24),  $s = 1$ . The correct identity, confirmed numerically, is

$$3R + 6S + T = 3U + 3V + 3W + X = 0.0074972468136605_+.$$

**3.2. Overlapping values.** One interesting question which arises while inspecting the above examples and Theorem 2 is as follows.

Some values of  $\Omega_{\mathbf{a}}(w)$  at  $w = (-1)^k \phi^{-2k}$  are left out. Namely, only values for  $k \geq s$  occur in Theorem 2, where  $\mathbf{a} = \{\ell_1, \dots, \ell_s\}$ . Hence, we pose the following problem, which we intend to investigate in the second part [1].

**Problem 1.** *Describe the arithmetic meaning of the values*

$$\Omega_{\ell_1, \ell_2, \dots, \ell_s} \left( (-1)^k \phi^{-2k} \right), \quad 1 \leq k < s.$$

For example, let

$$\begin{aligned} Y &= \sum_{n=1}^{\infty} (-1)^{n+1} W_1^3(n) \phi^{-2n} = 0.0375599269123177_+, \\ Z &= \sum_{n=1}^{\infty} (-1)^{n+1} W_1(n) W_2(n) \phi^{-2n} = 0.0426901777719431_+. \end{aligned}$$

However, MAPLE package `IntegerRelations` and the PSLQ algorithm does not seem to find that the number  $aY + bZ$  for small  $a, b \in \mathbb{Z}$  might be an algebraic number of degree 4; this includes the sum of two quadratic irrationals. It might happen that some combination of  $R$  and  $S$  is the sum of three quadratic irrationals, like in the example  $\{\ell = 6, k = 4\}$  above. However, even for the sum of two quadratic irrationals, the coefficients of the 4th degree polynomial are in the range of 10.000, hence much more powerful numerical calculations are needed to detect any algebraicity.

**3.3. Arithmetic of decomposition formulas.** The main Theorem 2 is derived while investigating the  $g$ -coefficients of a function  $U(z)$  which satisfies the functional equation (4) and the regularity property (3); see Subsection 4.4 and Sections 6 and 8. In fact, there exists the second way to get information on the functions  $W_\ell(n)$ , and it is based on the trace formulas of D. Mayer. This second method of approach towards the whole problem of determining arithmetic and structure of the eigenvalues reduces to investigation of high powers of the operator  $\mathcal{L}_\omega$  (see (11) and Section 8) and then at inspecting the small powers of  $\omega$ .

For example, take the  $k^{\text{th}}$  power of  $\mathcal{L}_\omega$ . First, we have [41, 43]:

$$\text{Tr}(\mathcal{L}_\omega^k) = \sum_{i_1, i_2, \dots, i_k=1}^{\infty} \omega^{i_1+i_2+\dots+i_k-k} \left[ \prod_{s=1}^k \xi_{i_s, i_{s+1}, \dots, i_k, i_1, \dots, i_{s-1}}^{-2} - (-1)^k \right]^{-1}. \quad (14)$$

On the other hand, by our result (43),

$$\begin{aligned} \text{Tr}(\mathcal{L}_\omega^k) &= \sum_{n=1}^{\infty} \lambda_n^k(\omega) = \sum_{n=1}^{\infty} (-1)^{nk+k} \phi^{-2nk} \left( \sum_{\ell=0}^{\infty} \omega^\ell \cdot W_\ell(n) \right)^k \\ &= \sum_{\ell=0}^{\infty} \omega^\ell \sum_{n=1}^{\infty} (-1)^{nk+k} \phi^{-2nk} \sum_{j_1+\dots+j_k=\ell} W_{j_1}(n) \cdots W_{j_k}(n). \end{aligned} \quad (15)$$

Now we proceed with comparing the corresponding coefficients at  $\omega^\ell$ . First, the coefficients at  $\omega^0$  of (14) is equal to  $(\phi^{2k} - (-1)^k)^{-1}$ , while the coefficient at  $\omega^0$  of (15) is equal to

$$\sum_{n=1}^{\infty} (-1)^{nk+k} \phi^{-2nk} = \frac{1}{\phi^{2k} - (-1)^k},$$

so both values do match.

Now look at the first non-trivial case, namely, the power  $\omega^1$ . This way we find the function which interpolates a class of quadratic irrationals as follows. Let

$$\underbrace{\xi_{2,1,\dots,1}}_k \cdot \underbrace{\xi_{1,2,\dots,1}}_k \cdots \underbrace{\xi_{1,1,\dots,2}}_k = \Gamma_{2,k}.$$

(For the notation “ $\Gamma_{2,k}$ ”, see Subsection 11.2). Then (see (10))

$$(-1)^k \Omega_1 \left( (-1)^k \phi^{-2k} \right) = \frac{1}{\Gamma_{2,k}^{-2} - (-1)^k}. \quad (16)$$

In the formula (13), we denoted  $\psi_k = \Gamma_{2,k}^{-2} - (-1)^k$ . The identity (16) corresponds to the case when the sum of  $k$  summands is equal to  $k + 1$ .

**3.4. Refined decomposition.** Now, look at  $\omega^2$  of the expansions (14) and (15). This corresponds to the combinatoric case when the sum of  $k$  summands is equal to  $k + 2$ . So, this can happen either in the case  $3 + 1 + \cdots + 1$  ( $k$  unities), or one of the cases  $2 + 2 + 1 + \cdots + 1$ ,  $2 + 1 + 2 + 1 + \cdots + 1$  ( $k - 1$  unities), and so on. Thus, the decomposition of trace formulas gives in this case the following identity:

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{nk+k} \phi^{-2nk} \sum_{j_1+\dots+j_k=2} W_{j_1}(n) \cdots W_{j_k}(n) \\ &= \sum_{i_1+i_2+\dots+i_k=k+2} \left[ \prod_{s=1}^k \xi_{i_s, i_{s+1}, \dots, i_k, i_1, \dots, i_{s-1}}^{-2} - (-1)^k \right]^{-1}, \text{ for any } k \in \mathbb{N}. \end{aligned} \quad (17)$$

So, the left hand side, minding the identity  $W_0(n) = 1$ , is equal to

$$\begin{aligned} & (-1)^k k \sum_{n=1}^{\infty} (-1)^{nk} \phi^{-2nk} W_2(n) + (-1)^k \binom{k}{2} \sum_{n=1}^{\infty} (-1)^{nk} \phi^{-2nk} W_1^2(n) \\ &= (-1)^k k \Omega_2 \left( (-1)^k \phi^{-2k} \right) + (-1)^k \binom{k}{2} \Omega_{1,1} \left( (-1)^k \phi^{-2k} \right). \end{aligned} \quad (18)$$

On the other hand, using the notation of Subsections 11.2 and 12.1, we get that the r.h.s. of (17) is equal to

$$\frac{(-1)^k k}{\Theta_3 \left( (-1)^k \phi^{-2k} \right)} + \frac{1}{2} \sum_{r=2}^k \frac{(-1)^k k}{\Pi_r \left( (-1)^k \phi^{-2k} \right)}, \quad (19)$$

where the first summand corresponds to the case  $3 + 1 + \cdots + 1$  and all its cyclic permutations, and the second sum - to the cases  $2 + 2 + 1 + \cdots + 1$  ( $\Pi_2$ ),  $2 + 1 + 2 + 1 + \cdots + 1$  ( $\Pi_3$ ), and up to  $2 + 1 + \cdots + 1 + 2$  ( $\Pi_k$ ). The factor “ $\frac{1}{2}$ ” arises from the fact that any of the two “2”s can be cyclically permute to start the sum. Obviously,  $\Pi_r(w) = \Pi_{k-r+2}(w)$ . Since the Taylor coefficients of  $\Omega_{1,1}(w)$  are equal to  $W_1^2(n) = O(n^{-1})$ , we can also calculate the asymptotics of the Taylor coefficients (in  $w = (-1)^k \phi^{-2k}$ ) of the function (19), and this gives the exact asymptotics of the function  $W_2(n)$ .



Moreover, the formula (18) further decomposes as follows.

We see from (18) that the identity which equates it to (19) can be written in the form

$$kA\left((-1)^k\phi^{-2k}\right) + k^2B\left((-1)^k\phi^{-2k}\right) = kC\left((-1)^k\phi^{-2k}\right) + k^2D\left((-1)^k\phi^{-2k}\right), \quad k \in \mathbb{N},$$

where  $A(w), B(w), C(w), D(w)$  are analytic functions for  $|w| < 1$ . This implies

$$k(A - C)\left((-1)^k\phi^{-2k}\right) = k^2(D - B)\left((-1)^k\phi^{-2k}\right).$$

Now the assumption that  $A$  is not identical to  $C$ , after investigating the first non-zero Taylor coefficient of  $A(w) - C(w)$ , leads to a contradiction. Thus,  $A \equiv C$  and  $B \equiv D$ .

This is the general outline how our second way to calculate the functions  $W_\ell(n)$  works, and the first few steps are presented in Subsections 11.2 and 12.1. This method is the main topic of the forthcoming paper [1]. As mentioned, the first way by which we prove all our main results is presented in Section 6.

#### 4. TOOLS AND PRELIMINARY RESULTS

**4.1. Jacobi polynomials.** For  $\alpha, \beta \in \mathbb{Z}$ ,  $m \in \mathbb{N}_0$ , the classical Jacobi polynomials are given as follows ([49], §4.6, Formula 4.6.1 in Russian translation of the book).

$$(x-1)^\alpha(x+1)^\beta P_m^{(\alpha,\beta)}(x) = \frac{1}{2^{m+1}\pi i} \oint_{\mathcal{C}} \frac{(w-1)^{m+\alpha}(w+1)^{m+\beta}}{(w-x)^{m+1}} dw; \quad (20)$$

here a small contour  $\mathcal{C}$  winds  $w = x$  once in the positive direction. For arbitrary  $x$  outside the closed interval  $[-1, 1]$ , one has an asymptotic formula ([49], Theorem 8.21.7)

$$P_m^{(\alpha,\beta)}(x) \sim \frac{\left((x+1)^{1/2} + (x-1)^{1/2}\right)^{\alpha+\beta}}{(x-1)^{\alpha/2}(x+1)^{\beta/2}(x^2-1)^{1/4}} \cdot \frac{1}{\sqrt{2\pi m}} \cdot \left(x + (x^2-1)^{1/2}\right)^{m+1/2}, \quad \text{as } m \rightarrow \infty. \quad (21)$$

We will now need the case  $m = \mathbf{X} - 1$ ,  $(\alpha, \beta) = (0, 1)$  (see Theorem 2). Moreover, one can extract the second asymptotic term. In this particular case this reads as

$$\frac{5}{4} P_{\mathbf{X}-1}^{(0,1)}(3/2) \sim \frac{\phi^{2\mathbf{X}}}{\sqrt{\mathbf{X}}} \cdot \frac{5^{1/4}}{2\sqrt{\pi}} + O\left(\frac{\phi^{2\mathbf{X}}}{\mathbf{X}^{3/2}}\right), \quad \text{as } \mathbf{X} \rightarrow \infty$$

(see Proposition 4). Here and in the sequel one can think of  $\mathbf{X} = n$ , which is exactly the index of an eigenvalue  $\lambda_n$  - notations  $n$  or  $\mathbf{X}$  are reserved for this purpose throughout this paper. Though at some places we choose to use an unspecified variable  $\mathbf{X}$  to denote that it is a function, a generic version of  $n$ .

The generating function for Jacobi polynomials is crucial in our investigations. For sufficiently small  $w$ , we have ([49], formula (4.4.5)):

$$\begin{aligned}\Delta^{(\alpha,\beta)}(w) &= \sum_{m=0}^{\infty} P_m^{(\alpha,\beta)}(x)w^m \\ &= 2^{\alpha+\beta}(1-2xw+w^2)^{-\frac{1}{2}} \\ &\quad \times \left[1-w+(1-2xw+w^2)^{\frac{1}{2}}\right]^{-\alpha} \left[1+w+(1-2xw+w^2)^{\frac{1}{2}}\right]^{-\beta}.\end{aligned}\quad (22)$$

In particular, when  $\alpha = 0$ ,  $\beta = 1$ , this gives

$$\sum_{m=0}^{\infty} P_m^{(0,1)}(x)w^m = \frac{1+w}{(1-2xw+w^2)^{1/2}(x+1)w} - \frac{1}{(x+1)w}.$$

We will frequently refer to these identities. Now we can explain why precisely Jacobi polynomials  $P_m^{(0,1)}(x)$  appear in the investigations of the operator  $\mathcal{L}$ .

**4.2. Mayer-Ruelle operator.** For the same class of functions  $\mathbf{V}$  and for a complex number  $s$ ,  $\Re(s) > \frac{1}{2}$ , one defines *the Mayer-Ruelle transfer operator* by [37, 39, 44, 52]

$$L_s[f(t)](z) = \sum_{m=1}^{\infty} \frac{1}{(z+m)^{2s}} f\left(\frac{1}{z+m}\right).$$

Thus,  $L_1 = \mathcal{L}$ , and the operator  $L_s$  is extended to all complex numbers  $s$  by an analytic continuation. As the fundamental contribution, it was proved by D. Mayer that [37, 44]

$$\det(1 - L_s^2) = \prod_{n=1}^{\infty} \left(1 - \beta_n^2(s)\right) = Z(s),$$

where on the left we have the Fredholm determinant of the operator defined as the product involving eigenvalues  $\beta_n(s)$  of the operator  $L_s$  (as given in the middle), and on the right - the Selberg zeta function for the full modular group [32, 39]. So, in this notation,  $\beta_n(1) = \lambda_n$ . Note that

$$-\frac{1}{2} \frac{d}{ds} \beta_1(s) \Big|_{s=1} = \frac{\pi^2}{12 \log 2}$$

is the *Lévy constant*. We avoided to use the notation  $\lambda_n(s)$  since this is reserved for the eigenvalues of the operator  $\mathcal{L}_\omega$ ; see the formula (43) and around. From the work of Lewis and Zagier [39] we know that  $\beta_n(s) = -1$  for a certain  $n \in \mathbb{N}$  if and only if  $s = \frac{1}{2} + it$  is a spectral parameter for the *hyperbolic Laplace-Beltrami operator* corresponding to an odd Maass wave form (then  $\frac{1}{4} + t^2$  is an eigenvalue of this operator), and  $\beta_n(s) = 1$  for a certain  $n \in \mathbb{N}$  if this spectral parameter corresponds to an even Maass wave form, or  $2s$  is a non-trivial zero of the Riemann zeta function. We remind that the hyperbolic Laplace-Beltrami operator acts on the functions defined on  $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathfrak{h}$ , and is given by

$$\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right);$$

here  $\mathfrak{h}$  is the upper half-plane. The formula (14), in case  $\omega = 1$  and when extended to the operator  $L_s$ , can be written as

$$\mathrm{Tr}(L_s^k) = \sum_{n=1}^{\infty} \beta_n^k(s) = \sum_{|h|=k} \frac{\tau(h)^{-2s}}{1 - (-1)^k \tau(h)^{-2}}.$$

Here the summation is over all strictly periodic quadratic irrationals  $0 < h < 1$  of period  $k$ , and

$$\tau(h) = \frac{Q_k + P_{k-1} + \sqrt{(Q_k + P_{k-1})^2 - 4(-1)^k}}{2}; \quad (23)$$

the notation  $\frac{P_s}{Q_s}$  stands for the  $s$ th convergent to  $h$ . Note that we can write the generic summand for  $\mathrm{Tr}(L_s^k)$  as

$$\frac{\tau(h)^{-2s+2}}{\tau(h)^2 - (-1)^k} = \frac{2^{2s-1}(Q_k + P_{k-1} + \sqrt{D})^{-2s+2}}{D + (Q_k + P_{k-1})\sqrt{D}}, \quad D = (Q_k + P_{k-1})^2 - 4(-1)^k. \quad (24)$$

Now, look at (22), where we put  $\alpha = 0$ . If we imagine for a moment that  $1 + w \doteq Q_k + P_{k-1}$  and  $D \doteq (1 - 2xw + w^2)$  - this equivalence will become clear in Subsections 11.2 and 12.1 - we get that the above, including the constant factor, is exactly the generating function for Jacobi polynomials  $P_m^{(0,2s-1)}(x)$ . Whence we derive an important observation:

*The operator  $L_s$  is governed by Jacobi polynomials  $P_m^{(0,2s-1)}(x)$  the same way as  $\mathcal{L}$  is governed by  $P_m^{(0,1)}(x)$  in this paper.*

The constants  $x$  and  $\beta$  in our first approach are always equal to  $\frac{3}{2}$ , 1, respectively, while  $\alpha$  can attain any integral value. Not so for the second approach described in Subsection 3.3, Sections 11 and 12. See, for example, the formula (58), where  $x$  can attain many different rational values, while the pair  $(\alpha, \beta) = (0, 1)$  is always fixed. This general complex  $s$  case is a central topic of the third part of our study [2].

**4.3. Classification of non-trivial zeros of  $\zeta(s)$ .** As a small aside and as an illustration of deep relation between cusp forms for  $\mathrm{PSL}_2(\mathbb{Z})$ , non-trivial zeros of the Riemann zeta function  $\zeta(s)$ , and eigenvalues of  $\mathcal{L}$ , we pose two related problems.

We re-iterate that  $Z(s)$  vanishes at  $s = \frac{\rho}{2}$ , where  $\rho$  is a non-trivial zero of  $\zeta(s)$ . An important consequence of the results in the current paper is the fact that labelling eigenvalues  $\lambda_n$  with an integer  $n$  is canonical. This corresponds to polynomials of degree  $(n + 1)$ , or, rather, rational functions  $u_0(n, z)$  with denominator of degree  $(n + 1)$ ; see (8). This ordering, as is implied by Theorem 1, corresponds exactly to ordering  $\lambda_n$  according to their absolute magnitude. In particular, for each nontrivial zero  $\rho$  of the Riemann zeta function there exist an integer  $N = t(\rho)$  such that

$$\lambda_N\left(\frac{\rho}{2}\right) = 1.$$

For example, if we order non-trivial zeros in  $\Re(s) > 0$  according to the magnitude of an imaginary part, numerical calculations show that

$$t(\rho_1) = 1, \quad t(\rho_2) = 2, \quad t(\rho_3) = 1, \quad t(\rho_4) = 3, \quad t(\rho_5) = 1, \quad t(\rho_6) = 3.$$

In particular, we pose

**Problem 2.** *Given an integer  $N \in \mathbb{N}$ . What one can be said about the set  $t^{-1}(N)$ ? Is it infinite? How the conjectural distribution of non-trivial Riemann zeros change if we limit ourselves to the set  $t^{-1}(N)$ ? How can one compute the set  $t^{-1}(1)$  effectively?*

Of course, this gives also classification of non-trivial zeros of  $Z(s)$ .

The question about trivial zeros of  $Z(s)$  seems also of big interest. Let  $k \in \mathbb{N}$ . It is known that the order of vanishing at  $s = 1 - k$  of  $Z(s)$  equals the dimension of the corresponding space of cusp forms  $M_{2k}$  for  $\mathrm{PSL}_2(\mathbb{Z})$  [15]. This dimension is

$$\dim_{\mathbb{C}} M_{2k} = \begin{cases} \lfloor \frac{k}{6} \rfloor, & \text{if } k \equiv 1 \pmod{6}, \\ \lfloor \frac{k}{6} \rfloor + 1, & \text{if } k \not\equiv 1 \pmod{6}. \end{cases}$$

In particular,

**Problem 3.** *Describe how the number  $\dim_{\mathbb{C}} M_{2k}$  distributes among different factors of*

$$Z(1 - k) = \prod_{n=1}^{\infty} \left(1 - \lambda_n^2(1 - k)\right).$$

These questions will be addressed in [2].

Another classification of Riemann zeros was proposed in [23, 24] by Garunkštis & Steuding, and essentially it is the following.

If  $\zeta(s, \alpha)$  is the Hurwitz zeta function, then

$$\zeta(s, 1) = \zeta(s), \quad \zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s).$$

Then the authors consider the trajectory of any particular non-trivial zero of  $\zeta(s, \alpha)$  on the critical line, as  $\alpha$  ranges from 1 to  $\frac{1}{2}$ . If the zero ends up on the critical line, then the zero is called *stable*. If it end up at the zero of  $2^s - 1 = 0$ , it is called *unstable*. For example, out of the first 10 zeros with  $\Im(s) > 0$ , the zeros 1, 3, 6, 9 are unstable, the rest are stable.

**4.4. g-coefficients of an analytic function.** We will now introduce some special coefficients of a holomorphic function.

**Proposition 3.** *Every function  $f(z)$  which is analytic in the half-plane  $\Re(z) > -\frac{1}{2}$  can be expanded in the following way:*

$$f(z) = \sum_{j=1}^{\infty} a_j \frac{(z - \phi^{-1})^{j-1}}{(z + \phi)^{j+1}},$$

where  $|a_j| < C(f, \epsilon) \cdot (1 + \epsilon)^j$  for every  $\epsilon > 0$ . We call  $a_j$  the  $j$ th golden coefficient, or  $g$ -coefficient, of the analytic function  $f(z)$ .

*Proof.* For  $j, \ell \in \mathbb{N}$ , note a simple identity

$$\oint_{\mathcal{C}} \frac{(z - \phi^{-1})^{\ell-1}}{(z + \phi)^{\ell+1}} \cdot \frac{(z + \phi)^j}{(z - \phi^{-1})^j} dz = \begin{cases} \frac{2\pi i}{2\phi-1}, & \text{if } j = \ell, \\ 0, & \text{if } j \neq \ell; \end{cases} \quad (25)$$

here small a contour  $\mathcal{C}$  in the half-plane  $\Re(z) > -\frac{1}{2}$  winds the point  $\phi^{-1}$  in the positive direction. So, we define  $j$ th  $g$ -coefficient of the function  $f(z)$  by the formula

$$a_j = \frac{2\phi - 1}{2\pi i} \oint_{\mathcal{C}} f(z) \cdot \frac{(z + \phi)^j}{(z - \phi^{-1})^j} dz, \quad j \geq 1.$$

Let

$$x = \frac{z - \phi^{-1}}{z + \phi}, \quad z = \frac{x\phi + \phi^{-1}}{-x + 1}.$$

The  $z$ -half plane  $\Re(z) > -\frac{1}{2}$  is mapped in the  $x$ -plane to the disc  $|x| < 1$ . Given  $f$ , analytic in  $\Re(z) > -\frac{1}{2}$ . Let

$$\mathcal{U}_f(x) = (z + \phi)^2 f(z) = \left(\frac{2\phi - 1}{x - 1}\right)^2 f\left(\frac{x\phi + \phi^{-1}}{-x + 1}\right). \quad (26)$$

Note that  $(2\phi - 1)^2 = 5$ . Then  $g(x)$  is analytic inside  $|x| < 1$  and  $a_j$  is its Taylor coefficient at  $x^{j-1}$ :

$$\mathcal{U}_f(x) = \sum_{j=1}^{\infty} a_j x^{j-1}.$$

The bound on  $a_j$  is then a standard consequence of the Cauchy's formula.  $\square$

**We will see in Section 8 that this single trick, the transform  $f \mapsto \mathcal{U}_f$ , is the main ingredient into the solution of the whole problem.**

To avoid typographic complications, when  $f$  is clear, we will denote  $\mathcal{U}_f$  by  $g$ . For example, the  $g$ -coefficients of the dominant eigenfunction of  $\mathcal{L}$ , namely,  $f(z) = \frac{1}{z+1}$ , are given by

$$a_j = \frac{2\phi - 1}{2\pi i} \int_{\mathcal{C}} \frac{1}{z + 1} \cdot \frac{(z + \phi)^j}{(z - \phi^{-1})^j} dz.$$

Expand the contour to a large circle. The residue of the function under the integral at  $z = -1$  is equal to  $(-1)^j \phi^{-2j}$ . Make a substitution  $z = \frac{1}{w} - 1$ . We are left to calculate the residue at  $w = 0$ . This gives, for  $j \geq 1$ ,

$$a_j = (2\phi - 1) \cdot (1 - (-1)^j \phi^{-2j}) \sim 2\phi - 1, \text{ as } j \rightarrow \infty.$$

In this particular case we have

$$g(x) = \frac{5}{(1-x)(x\phi^{-1} + \phi)} = (2\phi - 1) \left( \frac{1}{1-x} + \frac{1}{x + \phi^2} \right).$$

5. THE COEFFICIENTS  $K(j, \ell)$ 

Now we explore the array of algebraic numbers  $K(j, \ell) \in \mathbb{Q}(\sqrt{5})$ ,  $\phi^{j+\ell} \cdot K(j, \ell) \in \mathbb{Q}_+$ , which, in fact, is our first method (as described in the beginning of Subsection 3.3) to approach the problem. This array governs the whole collection of the eigenvalues  $\lambda_n$ .

**5.1. Combinatorics and asymptotics.** We will need the following crucial result.

**Proposition 4.** *Let  $\ell \in \mathbb{N}$ . The  $j$ th  $g$ -coefficient of the function*

$$\frac{(z + 1 - \phi^{-1})^{\ell-1}}{(z + 1 + \phi)^{\ell+1}}$$

is given by

$$K(j, \ell) = 5 \cdot 2^{j-\ell-2} \cdot \phi^{-\ell-j} \cdot P_{j-1}^{(\ell-j, 1)}(3/2) = \frac{\phi^{-\ell-j}}{2^{j+1}\pi i} \oint \frac{(w-1)^{\ell-1}(w+1)^j}{\left(w - \frac{3}{2}\right)^j} dw; \quad (27)$$

the contour winds  $\frac{3}{2}$  in the positive direction.

Next, we have the first and the second symmetry properties

$$\ell K(j, \ell) = j K(\ell, j), \quad j, \ell \geq 1, \quad (28)$$

$$2\phi K(j, \ell) - K(j-1, \ell) = 2\phi K(\ell, j) - K(\ell-1, j), \quad j, \ell \geq 2. \quad (29)$$

We have the ‘‘averaging’’ recurrence

$$2\phi K(j, \ell) = K(j-1, \ell) + 2\phi^{-1} K(j-1, \ell-1) + K(j, \ell-1), \quad \ell, j \geq 2. \quad (30)$$

These coefficients are positive, and  $\sum_{j=1}^{\infty} K(j, \ell) \equiv 1$  for every  $\ell \in \mathbb{N}$ . Further, for fixed  $\ell$ , for  $j \leq \ell$   $K(j, \ell)$  monotonically increases, achieves its maximum at  $j = \ell$ , and then monotonically decreases.

We have

$$K(\ell, \ell) \sim \frac{\sqrt[4]{5}}{2\sqrt{\pi}\sqrt{\ell}}.$$

Next,  $K(j, \ell)$  is of fast decay when  $|j - \ell|$  increases:

$$K(j, \ell) = \frac{\sqrt[4]{5}}{2\sqrt{\pi}} \cdot \left( \frac{1}{\sqrt{\ell}} + \frac{j-\ell}{4\ell^{3/2}} + \frac{B}{\ell^{3/2}} \right) \cdot \exp\left( -\frac{\sqrt{5}(j-\ell)^2}{2(j+\ell)} \right),$$

which holds uniformly for  $|j - \ell| < \ell^{2/3}$ , and where  $|B| < 2$ .

*Proof.* By the above remark,

$$K(j, \ell) = \frac{2\phi - 1}{2\pi i} \oint_{\mathcal{C}} \frac{(z + \phi)^j}{(z - \phi^{-1})^j} \cdot \frac{(z + 1 - \phi^{-1})^{\ell-1}}{(z + 1 + \phi)^{\ell+1}} dz.$$

here  $\mathcal{C}$  is a small circle around  $\phi^{-1}$  in the positive direction. Let  $a, b, c, d$  be distinct real numbers.

We will generally explore the integral

$$\mathcal{I} = \oint \frac{(z+a)^j(z+b)^{\ell-1}}{(z+c)^j(z+d)^{\ell-1}} \frac{dz}{(z+d)^2};$$

here the contour winds  $-c$  in the positive direction. Let

$$\frac{z+b}{z+d} = w \Rightarrow \frac{z+a}{z+c} = \frac{w(d-a) + (a-b)}{w(d-c) + (c-b)} = \frac{p}{r} \cdot \frac{w+q}{w+s}.$$

So, after this change to the variable  $w$ , the integral  $\mathcal{I}$  transforms into

$$\mathcal{I} = \frac{p^j}{r^j(d-b)} \oint \frac{w^{\ell-1}(w+q)^j}{(w+s)^j} dw.$$

Further, let us make the next change  $w \mapsto \frac{q}{2}w - \frac{q}{2}$ . This gives

$$\mathcal{I} = \frac{p^j \left(\frac{q}{2}\right)^\ell}{r^j(d-b)} \oint \frac{(w-1)^{\ell-1}(w+1)^j}{\left(w-1 + \frac{2s}{q}\right)^j} dw.$$

In our case,  $a = \phi$ ,  $b = 1 - \phi^{-1}$ ,  $c = -\phi^{-1}$ ,  $d = 1 + \phi$ . So,

$$p = d - a = 1, \quad r = d - c = 2\phi, \quad q = \frac{a-b}{d-a} = 2\phi^{-1}, \quad s = \frac{c-b}{d-c} = -\frac{1}{2\phi}, \quad -1 + \frac{2s}{q} = -\frac{3}{2}.$$

So,

$$\mathcal{I} = \frac{\phi^{-\ell-j}}{2^j(2\phi-1)} \oint \frac{(w-1)^{\ell-1}(w+1)^j}{\left(w - \frac{3}{2}\right)^j} dw.$$

where the contour goes around the point  $w = \frac{3}{2}$  in the positive direction. This integral can be expressed in terms of Jacobi polynomials, as the definition (20) shows. In our case,  $m = j - 1$ ,  $\alpha = \ell - j$ ,  $\beta = 1$ ,  $x = \frac{3}{2}$ , and

$$K(j, \ell) = \frac{2\phi - 1}{2\pi i} \cdot \mathcal{I}.$$

This gives the formula of the proposition.

Next, as was shown before, the  $j$ th golden coefficient of  $f(z)$  is the Taylor coefficient at  $x^{j-1}$  of the function  $g(x)$ ; see (26). So,  $K(j, \ell)$  is the Taylor coefficient at  $x^{j-1}$  of

$$\Delta_\ell(x) = 5 \cdot \frac{(2x\phi^{-1} + 1)^{\ell-1}}{(-x + 2\phi)^{\ell+1}} = \sum_{j=1}^{\infty} K(j, \ell) \cdot x^{j-1}, \quad (31)$$

and so, by a direct calculation,

$$K(j, \ell) = \frac{5\phi^{-\ell-j}}{2\pi i} \oint_{|w|=1} \frac{(2w+1)^{\ell-1}}{(2-w)^{\ell+1}w^j} dw = \frac{5}{\ell} \cdot (2\phi)^{-\ell-j} \cdot \sum_{i=0}^{\min\{\ell-1, j-1\}} \frac{(\ell+j-i-1)!}{(\ell-i-1)!(j-i-1)!} \cdot \frac{4^i}{i!}.$$

This gives the positivity (what we already know) and the first symmetry property (28). In Proposition 5, item iv), we will see how to prove this symmetry property directly using the generating function. Taking in (31)  $x = 1$  gives the needed summation property.

Now we will calculate the bivariate generating function of coefficients  $K(j, \ell)$ . According to (31), we have

$$\begin{aligned}\Theta(x, y) &= \sum_{j, \ell=1}^{\infty} K(j, \ell) x^{j-1} y^{\ell-1} \\ &= \frac{5}{(2\phi - x)^2} \sum_{\ell=1}^{\infty} \frac{(2x\phi^{-1} + 1)^{\ell-1}}{(2\phi - x)^{\ell-1}} y^{\ell-1} \\ &= \frac{5}{(2\phi - x)(2\phi - x - y - 2xy\phi^{-1})}.\end{aligned}\tag{32}$$

The double series converges for  $|x| < 1$ ,  $|y| < 1$ . Explicit expression implies another symmetry property

$$\Theta(x, y) \cdot (2\phi - x) = \Theta(y, x) \cdot (2\phi - y).$$

On the level of coefficients, this reads as the second symmetry property (29).

Further, for  $\ell \geq 2$ , we have

$$(2\phi - x)\Delta_{\ell}(x) = 5 \cdot \frac{(2x\phi^{-1} + 1)^{\ell-1}}{(-x + 2\phi)^{\ell}} = (2x\phi^{-1} + 1)\Delta_{\ell-1}(x).$$

This gives the recurrence (30) with initial values

$$K(1, \ell) = \frac{5}{(2\phi)^{\ell+1}}, \quad K(j, 1) = \frac{5j}{(2\phi)^{j+1}}.$$

We can prove the asymptotic for  $K(j, \ell)$  using Gnedenko's theorem, the discrete version of the Central Limit Theorem, the so called *local limit theorems* [25]. Indeed, let

$$\Xi(x) = \frac{2x\phi^{-1} + 1}{2\phi - x} = \sum_{j=0}^{\infty} p(j)x^j.$$

Then

$$p(j) = \begin{cases} 5(2\phi)^{-j-1} & \text{for } j \geq 1, \\ (2\phi)^{-1} & \text{for } j = 0. \end{cases}$$

Let  $\mathbf{X}$  be a random discrete variable with probabilities

$$P(\mathbf{X} = j) = p(j).$$

This is indeed a random variable since  $\sum_{j=0}^{\infty} p(j) = \Xi(1) = 1$ . Further,

$$\begin{aligned}\mathbb{E}(\mathbf{X}) &= \sum_{j=0}^{\infty} jp(j) = \Xi'(1) = 1, \\ \mathbb{E}(\mathbf{X}^2) &= \sum_{j=0}^{\infty} j^2 p(j) = \frac{d}{dx} \left( \Xi'(x)x \right) \Big|_{x=1} = 1 + \frac{2}{\sqrt{5}}, \\ \mathbb{D}(\mathbf{X}) &= \mathbb{E}(\mathbf{X}^2) - (\mathbb{E}(\mathbf{X}))^2 = \frac{2}{\sqrt{5}}.\end{aligned}$$



Thus, the needed asymptotics is the standard result in probability theory and results in asymptotic expansions. The rigorous proof is presented in Appendix 12.1.

Considering the behaviour of  $K(j, \ell)$  for  $\ell$  fixed, we can strengthen this as follows. Let  $j, \ell \geq 1$ .

- i) Fix  $\ell$ .  $K(j, \ell)$  strictly increases until reaching its maximum value at  $j = \ell$ , and then strictly decreases.
- ii) Fix  $j$ .  $K(j, \ell)$  strictly increases until reaching its maximum value at  $\ell = j - 1$ , and then strictly decreases.
- iii)  $K(\ell - 1, \ell) < K(\ell, \ell) < K(\ell, \ell - 1) < K(\ell - 1, \ell - 1)$ , for  $\ell \geq 2$ .
- iv) We have:

$$1 - \frac{1}{2(\ell - 1)} < \frac{K(\ell, \ell)}{K(\ell - 1, \ell - 1)} < 1 - \frac{1}{2\ell}.$$

If we imagine the axis  $j$  going to the right, and the axis  $\ell$  - downwards (as in Table 3), due to the first symmetry property, in the upper-triangle of this quadrant, the item ii) is stronger than the item i), and in the lower-triangle - the other way round. We are aware that in the proximity of the diagonal the behaviour is more complicated. Meanwhile the item iii) follows from i) and ii) immediately. Now, the item iv) follows from asymptotics. We then prove by induction on  $\min\{j, \ell\}$  all i)-iii) at once using the ‘‘averaging’’ recurrence (30), all the cases  $j = \ell - 1, \ell, \ell + 1$  being set by the asymptotics. For example, the inequality  $K(\ell, \ell - 1) > K(\ell, \ell)$  for large  $\ell$  follows from the fact  $\frac{3}{4} + B > \frac{\sqrt{5}}{4}$ . We only note that the item iv) can be demonstrated to hold true purely algebraically minding the recurrence on  $\mathbf{K}(n)$  of Proposition 5 below. Thus, all the above claims have a purely combinatoric-algebraic proof.  $\square$

**5.2. Infinite integer matrix.** In order to understand the arithmetic structure of  $K(j, \ell)$  much deeper, let us define

$$\mathbf{K}(j, \ell) = \frac{(2\phi)^{j+\ell}}{5} \cdot K(j, \ell) \in \mathbb{N}.$$

These integers for  $1 \leq j, \ell \leq 10$  are presented in Table 3. We can immediately rewrite the arithmetic part of Proposition 4.

**Proposition 5.** *The following identities for the integers  $\mathbf{K}(j, \ell)$  hold:*

- i) *The generating functions*

$$\sum_{j=1}^{\infty} \mathbf{K}(j, \ell) x^{j-1} = \frac{(4x + 1)^{\ell-1}}{(1 - x)^{\ell+1}},$$

$$\Theta(x, y) = \sum_{j, \ell=1}^{\infty} \mathbf{K}(j, \ell) x^{j-1} y^{\ell-1} = \frac{1}{(1 - x)(1 - x - y - 4xy)};$$

- ii) *Let  $\mathbf{K}(n) = \mathbf{K}(n, n)$ ; the diagonal of the rational function  $\Theta$  is given by*

$$\Omega(w) = \sum_{n=1}^{\infty} \mathbf{K}(n) w^n = \frac{\Omega_1(4\phi^2 w)}{5} = \frac{1 + 4w - \sqrt{1 - 12w + 16w^2}}{10\sqrt{1 - 12w + 16w^2}};$$

$\ell \setminus j$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	1	7	18	34	55	81	112	148	189	235
3	1	12	58	164	355	656	1092	1688	2469	3460
4	1	17	123	519	1530	3606	7322	13378	22599	35935
5	1	22	213	1224	4830	14556	36302	78968	155079	281410
6	1	27	328	2404	12130	46006	140532	364708	835659	1737385
7	1	32	468	4184	25930	120456	445012	1371848	3666339	8746360
8	1	37	633	6689	49355	273531	1200367	4352263	13505994	36917710
9	1	42	823	10044	86155	557106	2851597	12005328	42920374	133862060
10	1	47	1038	14374	140705	1042431	6122452	29534168	120475854	426019410

TABLE 3. Integer matrix  $\{\mathbf{K}(j, \ell), j, \ell \in \mathbb{N}\}$ . This matrix, coupled with the constant  $\phi$ , contains all the information on eigenvalues of  $\mathcal{L}$ , trace formulas, and decompositions of trace formulas.

iii)  $\mathbf{K}(1, \ell) = 1, \mathbf{K}(j, 1) = j$ ;

iv)  $\ell \mathbf{K}(j, \ell) = j \mathbf{K}(\ell, j)$ ; in terms of  $\Theta$ , this reads as

$$\frac{\partial}{\partial y} \left( y \Theta(x, y) \right) = \frac{1}{(1 - x - y - 4xy)^2},$$

which is symmetric in  $(x, y)$ ;

v)  $\mathbf{K}(j, \ell) - \mathbf{K}(j - 1, \ell) = \mathbf{K}(\ell, j) - \mathbf{K}(\ell - 1, j)$ , for  $j, \ell \geq 2$ ; in terms of  $\Theta$ , this reads as

$$(1 - x) \Theta(x, y) = \frac{1}{1 - x - y - 4xy},$$

which is symmetric in  $(x, y)$ ;

vi)  $\mathbf{K}(j, \ell) = \mathbf{K}(j - 1, \ell) + 4\mathbf{K}(j - 1, \ell - 1) + \mathbf{K}(j, \ell - 1)$ , for  $j, \ell \geq 2$ ;

vii) For  $n \geq 3$ , we have the recurrence

$$(n + 1)\mathbf{K}(n + 1) - (8n + 10)\mathbf{K}(n) - (32n - 72)\mathbf{K}(n - 1) + 64(n - 2)\mathbf{K}(n - 2) = 0.$$

In items i) and ii) we have touched on a surface the wide research topic which investigates *diagonals* of rational functions in positive or zero characteristic.

In general, a diagonal of a rational function is not always algebraic. For example,

$$\text{diag} \left( \frac{1}{1 - x - y - z} \right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} w^n,$$

which is not algebraic. The list of H. A. Schwarz tells when Gauss hypergeometric function  ${}_2F_1$  is algebraic, and theorem of F. Beukers and G. Heckman [11] tells exactly when a higher dimensional one  ${}_nF_{n-1}$  is such. Nevertheless, we have the following result.

**Theorem** (Furstenberg [22, 38]). *Suppose  $a_n \in \mathbb{Q}$ ,  $n \in \mathbb{N}_0$ .*

- 1) If  $f(w) = \sum_{n \geq 0} a_n w^n$  is an algebraic function, that is,  $P(w, f(w)) = 0$  for some  $P \in \mathbb{Q}[x, y]$ , then  $f$  is a diagonal of a rational power series in two variables.
- 2) If  $f(w) = \sum_{n \geq 0} a_n w^n$  is a diagonal of a rational power series, then it is an algebraic series modulo  $p$  for almost all prime  $p$ .
- 3) If  $f(w) = \sum_{n \geq 0} a_n w^n$  is a diagonal of a rational power series in two variables, then it is algebraic;
- 4) If  $\sum_{n \geq 0} a_n w^n, \sum_{n \geq 0} b_n w^n \in \mathbb{F}[[w]]$  are algebraic over a finite field  $\mathbb{F}$ , so is  $\sum_{n \geq 0} a_n b_n w^n$ .

Thus,  $\Omega$  satisfies

$$(5\Omega^2 + \Omega)(1 - 12 + 16w^2) = w.$$

Modulo any prime  $\neq 5$  this gives a quadratic equation. For example, if  $F(w) = \Omega(w) \pmod{2}$ , then  $F^2 = F + w$ . So,

$$F(w) = \sum_{k=1}^{\infty} w^{2^k}.$$

Since

$$\Theta(x, y) = \frac{1}{(1-x)(1-x-y)} \pmod{2},$$

this gives a combinatoric corollary

$$\sum_{j=0}^n \binom{n+j}{j} \equiv \begin{cases} 1 \pmod{2}, & \text{if } n = 2^k - 1, \quad k \geq 0 \\ 0 \pmod{2}, & \text{if } n \neq 2^k - 1, \quad k \geq 0. \end{cases}$$

On the other hand, modulo 5,

$$H(w) = \Omega(w) \pmod{5} = \sum_{k=1}^{\infty} k w^k = \frac{w}{(1-w)^2}.$$

it is well-known that the item 4) of the above Theorem fails for characteristic 0, what we will soon see in Subsection 9.2 in case of the function  $\Omega_{1,1}(w)$ .

These results are just a glimpse into a broad subject, which includes Skolem-Mahler-Lech theorem, Christol's theorem, cellular automata, results of Adamczewski and Bell, and others [38]. It also hints at the possibility of a  $p$ -adic analogue of the results of the current paper.

The sequence  $\{\mathbf{K}(n), n \in \mathbb{N}\}$  is not yet contained in Online Encyclopedia of Integer Sequences [54]. However, it can be expressed in terms of the sequence A084772, which is given by

$$\frac{1}{\sqrt{1-12w+16w^2}} = \sum_{n=0}^{\infty} a(n)w^n,$$

as follows:

$$\mathbf{K}(n) = \frac{a(n) + 4a(n-1)}{10}, \quad n \geq 1.$$

Finally, note that

$$\mathbf{\Omega}'(w) = \frac{1 - 4w}{(1 - 12w + 16w^2)^{3/2}}.$$

This gives the identity

$$(10\mathbf{\Omega}(w) + 1) \cdot (1 - 4w) = \mathbf{\Omega}'(w) \cdot (1 + 4w) \cdot (1 - 12w + 16w^2).$$

Comparing the coefficient at  $w^n$  for  $n \geq 3$  gives the item vii) of Proposition 5.

**5.3.  $\mathbf{K}(j, \ell)$  as a function in  $\ell$ .** We can rewrite the item v) of Proposition 5, if coupled with with the symmetry property iv), as follows:

$$(j - \ell)\mathbf{K}(j, \ell) = j\mathbf{K}(j - 1, \ell) - (\ell - 1)\mathbf{K}(j, \ell - 1).$$

Now, let us use vi) to express  $\mathbf{K}(j, \ell - 1)$ , and substitute this into the above expression. We obtain:

$$(j - 1)\mathbf{K}(j, \ell) = (\ell + j - 1)\mathbf{K}(j - 1, \ell) + 4(\ell - 1)\mathbf{K}(j - 1, \ell - 1). \quad (33)$$

Let us now introduce

$$\mathbf{K}(j, t) = Y_j(t),$$

where  $t$  is an integer, but this time we consider the latter expression as a polynomial in  $t$ . The recurrence (33) in terms of  $Y$  reads as

$$(j - 1)Y_j(t) = (t + j - 1)Y_{j-1}(t) + 4(t - 1)Y_{j-1}(t - 1), \quad Y_1(t) = 1.$$

Thus, the polynomials we obtain are as follows:

$$\begin{aligned} Y_1(t) &= 1, \\ Y_2(t) &= 5t - 3, \\ Y_3(t) &= \frac{25}{2}t^2 - \frac{45}{2}t + 13, \\ Y_4(t) &= \frac{125}{6}t^3 - 75t^2 + \frac{655}{6}t - 51, \\ Y_5(t) &= \frac{625}{24}t^4 - \frac{625}{4}t^3 + \frac{9875}{24}t^2 - \frac{1925}{4}t + 205, \\ Y_6(t) &= \frac{625}{24}t^5 - \frac{1875}{8}t^4 + \frac{23125}{24}t^3 - \frac{16125}{8}t^2 + \frac{25025}{12}t - 819, \\ Y_7(t) &= \frac{3125}{144}t^6 - \frac{4375}{16}t^5 + \frac{231875}{144}t^4 - \frac{83125}{16}t^3 + \frac{84875}{9}t^2 - \frac{35455}{4}t + 3277, \\ Y_8(t) &= \frac{15625}{1008}t^7 - \frac{3125}{12}t^6 + \frac{149375}{72}t^5 - \frac{56875}{6}t^4 + \frac{3769375}{144}t^3 - \frac{512225}{12}t^2 + \frac{3130985}{84}t - 13107. \end{aligned}$$

Now, let

$$\sum_{j=1}^{\infty} Y_j(t)x^{j-1} = \psi(t, x).$$

We know that for  $\ell \in \mathbb{N}$ ,

$$\psi(\ell, x) = \frac{(4x + 1)^{\ell-1}}{(1 - x)^{\ell+1}}.$$

And in general, for  $t \in \mathbb{R}$ ,

$$\psi(t, x) = \frac{(4x + 1)^{t-1}}{(1-x)^{t+1}}.$$

Indeed, the initial condition is  $\psi(t, 0) = 1$ , and we note that the Taylor coefficients of the latter function satisfy the same recurrence, minding the differential equation

$$(1-x) \frac{\partial}{\partial x} \psi(t, x) = (t+1)\psi(t, x) + 4(t-1)\psi(t-1, x).$$

Thus,

$$Y_j(t) = \sum_{s=0}^{j-1} \binom{t-1}{s} 4^s (-1)^{j+s+1} \binom{-t-1}{j-s-1}.$$

We will need these polynomials further, starting from Subsection 6.3.

## 6. THE MAIN RECURRENCE

**6.1. The main recurrence.** As before, let  $\mathbf{X} \in \mathbb{N}$ . Let us define

$$W_0(\mathbf{X}) = 1, \quad u_0(\mathbf{X}, z) = \frac{(z - \phi^{-1})^{\mathbf{X}-1}}{(z + \phi)^{\mathbf{X}+1}},$$

and then the functions  $W_v(\mathbf{X})$ ,  $u_v(\mathbf{X}, z)$  recurrently by

$$\sum_{r=0}^v u_{v-r}(\mathbf{X}, z) W_r(\mathbf{X}) = \frac{(-1)^{\mathbf{X}+1} \phi^{2\mathbf{X}}}{(z+1)^2} u_v\left(\mathbf{X}, \frac{1}{z+1}\right) + \sum_{r=0}^{v-1} u_{v-r-1}(\mathbf{X}, z+1) W_r(\mathbf{X}). \quad (34)$$

This recursion is the core idea of our first method: for the unknown function  $u_v(\mathbf{X}, z)$  we have two terms rather than three, since the last sum is delayed. Also, it turns out that  $W_0$  is the dominant contributor in the asymptotics of  $\lambda_n$ . This recursion works as follows. If we have already determined  $W_r$  and  $u_r$  for  $r \leq v-1$ , the above identity allows to uniquely determine  $g$ -coefficients of  $u_v$ , except for the  $\mathbf{X}$ th coefficient, which can be defined (almost) arbitrarily, as far as a sum over  $\mathbf{X}$ th coefficients of all the functions  $u_\ell$ ,  $\ell \geq 0$ , absolutely converges. In our case, we set this  $\mathbf{X}$ th  $g$ -coefficient for  $\ell \geq 1$  to be equal to 0. This recursion also yields the unique value for  $W_v$ , as we will soon see.

To see clearer the main idea of this first approach, multiply the identity (34) by  $\phi^{-2\mathbf{X}}$ , and sum over  $v \geq 0$ . If we put

$$\Lambda(\mathbf{X}) = \phi^{-2\mathbf{X}} \sum_{\ell=0}^{\infty} W_\ell(\mathbf{X}), \quad U(\mathbf{X}, z) = \sum_{\ell=0}^{\infty} u_\ell(\mathbf{X}, z),$$

we obtain the avatar of the initial functional equation

$$\Lambda(\mathbf{X})U(\mathbf{X}, z) = \Lambda(\mathbf{X})U(\mathbf{X}, z+1) + \frac{(-1)^{\mathbf{X}+1}}{(z+1)^2} U\left(\mathbf{X}, \frac{1}{z+1}\right).$$

We will soon show that

$$\lambda_n = (-1)^{n+1} \Lambda(n).$$

This recurrssion is much more clearly seen if we introduce the  $\mathcal{L}_\omega$ , as given by (11). Then the recurrence is obtained just by comparing the coefficients at  $\omega^v$  of (12). However, we will need the operator only later in Section 8.

**Proposition 6.** *If  $\{a_j, j \in \mathbb{N}\}$  are the  $g$ -coefficients of  $f(z)$ , then the  $g$ -coefficients of*

$$\frac{(-1)^{\mathbf{X}+1}\phi^{2\mathbf{X}}}{(z+1)^2} f\left(\frac{1}{z+1}\right)$$

are given by  $b_j = a_j(-1)^{\mathbf{X}+j}\phi^{2\mathbf{X}-2j}$ . The  $g$ -coefficients of  $f(z+1)$  are given by  $c_j = \sum_{i=1}^{\infty} a_i K(j, i)$ .

*Proof.* Indeed, let

$$f(z) = \sum_{j=1}^{\infty} a_j \frac{(z - \phi^{-1})^{j-1}}{(z + \phi)^{j+1}}.$$

Then

$$\frac{(-1)^{\mathbf{X}+1}\phi^{2\mathbf{X}}}{(z+1)^2} \cdot \frac{(\frac{1}{z+1} - \phi^{-1})^{j-1}}{(\frac{1}{z+1} + \phi)^{j+1}} = (-1)^{\mathbf{X}+j}\phi^{2\mathbf{X}-2j} \frac{(z - \phi^{-1})^{j-1}}{(z + \phi)^{j+1}}.$$

Also,

$$f(z+1) = \sum_{j=1}^{\infty} a_j \frac{(z+1 - \phi^{-1})^{j-1}}{(z+1 + \phi)^{j+1}},$$

and the claim follows from Proposition 4 in Section 5.  $\square$

Let the  $g$ -coefficients of  $u_v(\mathbf{X}, z)$  be given by  $q_v^{(j)}$ ,  $j \geq 1$ ; we omit indication of the dependency on  $\mathbf{X}$ , being aware that  $q_v^{(j)} = q_v^{(j)}(\mathbf{X})$ . Now, let us compare the  $j$ th  $g$ -coefficient of (34). We obtain:

$$\boxed{\sum_{r=0}^v q_{v-r}^{(j)} W_r = q_v^{(j)} (-1)^{\mathbf{X}+j} \phi^{2\mathbf{X}-2j} + \sum_{r=0}^{v-1} \sum_{i=1}^{\infty} q_{v-r-1}^{(i)} K(j, i) W_r, \quad v \geq 0, \quad j \geq 1.} \quad (35)$$

Let  $v = 0$  in (35). This reads as

$$q_0^{(j)} W_0 = q_0^{(j)} (-1)^{\mathbf{X}+j} \phi^{2\mathbf{X}-2j}.$$

We readily obtain that only  $q_0^{(\mathbf{X})}$  is (potentially) non-zero, and this perfectly accords with the fact that  $u_0(\mathbf{X}, z)$  has only one non-zero  $g$ -coefficient, the  $\mathbf{X}$ th one; in our case it is equal to 1.

**6.2.  $v = 1$  and the asymptotic results for  $W_1(n)$ .** When  $v = 1$ , (35) reads as

$$q_1^{(j)} + q_0^{(j)} W_1 = q_1^{(j)} (-1)^{\mathbf{X}+j} \phi^{2\mathbf{X}-2j} + K(j, \mathbf{X}). \quad (36)$$

When  $j = \mathbf{X}$ , we obtain

$$W_1 = K(\mathbf{X}, \mathbf{X}).$$

Note that this will correspond to the second largest contributor to the exact value of  $\lambda_n$ ; it is equal to  $\phi^{-2n}K(n, n)$ , which is of size  $\frac{\phi^{-2n}}{\sqrt{n}}$  - this guarantees the success of our approach! When  $j \neq \mathbf{X}$ , (36) gives

$$q_1^{(j)} = \frac{K(j, \mathbf{X})}{1 - (-1)^{\mathbf{X}+j}\phi^{2\mathbf{X}-2j}}.$$

As we now see, the value of  $q_1^{(\mathbf{X})}$  cannot be extracted from (36), and it can be defined arbitrarily. Indeed, choosing another value for  $q_\ell^{(\mathbf{X})}$ ,  $\ell \geq 1$ , leads to a function  $\tilde{U}(\mathbf{X}, z)$  which is different from  $U(\mathbf{X}, z)$  by a constant factor  $\sum_{\ell \geq 0} q_\ell^{(\mathbf{X})}$ , provided the last series is absolutely convergent. We therefore always choose  $q_\ell^{(\mathbf{X})} = 0$  for  $\ell \geq 1$ . In terms of the variable  $\omega$  this means the following. The equation (12) remains valid even if we multiply it by any function  $t(\omega)$ , analytic for  $|\omega| \leq 1$ . *A posteriori*, the choice  $q_j^{(\mathbf{X})} = 0$  gives the canonical normalization of the function  $U$  which satisfies (4) for the  $n$ th eigenvalue, and this proves item (iv) of Theorem 2.

$n$	$W_1(n)$	$W_1(n)\sqrt{n}$	$n$	$W_1(n)$	$W_1(n)\sqrt{n}$
1	0.4774575140626314	0.47745751406263	14	0.1132518994300057	0.42374980606847
2	0.3191519488288150	0.45134901449152	15	0.1093746232463436	0.42360609433078
3	0.2525179078163117	0.43737384615885	16	0.1058705808693317	0.42348232347732
4	0.2157725901346884	0.43154518026938	17	0.1026834287993105	0.42337462294015
5	0.1917523856954763	0.42877136926285	18	0.0997680672051097	0.42328006119965
6	0.1744105888874710	0.42721694851264	19	0.0970879075478749	0.42319637764101
7	0.1610997341266682	0.42622983277780	20	0.0946129110363578	0.42312180125288
8	0.1504537306202003	0.42554741270543	21	0.0923181531988805	0.42305492505240
9	0.1416824491533456	0.42504734746005	22	0.0901827554098868	0.42299461723883
10	0.1342909272767424	0.42466519929053	23	0.0881890773429909	0.42293995713338
11	0.1279504805135870	0.42436373560926	24	0.0863220981936739	0.42289018820186
12	0.1224328715287222	0.42411990800860	100	0.0422070239152352	0.42207023915235
13	0.1175738863751953	0.42391867598136	1000	0.0133401861665838	0.42185372697076

TABLE 4. The function  $W_1(n)$  for  $1 \leq n \leq 24$ ,  $n = 100, 1000$ .

Proposition 4 tells that

$$W_1(n) \sim \frac{5^{1/4}}{2\sqrt{\pi}\sqrt{n}} = \frac{1}{\sqrt{n}} \cdot 0.4218301030679226_+.$$

The convergence rate is  $n^{-3/2}$ . The numerical data is presented in Table 4.

**6.3.  $v = 2$  and asymptotic results for  $W_2(n)$ .** As a next step, let us read the recurrence (35) in case  $v = 2$ . This gives:

$$q_2^{(j)} + q_1^{(j)}W_1 + q_0^{(j)}W_2 = q_2^{(j)}(-1)^{\mathbf{X}+j}\phi^{2\mathbf{X}-2j} + K(j, \mathbf{X})W_1 + \sum_{i=1}^{\infty} q_1^{(i)}K(j, i).$$

When  $j = \mathbf{X}$ , this yields

$$W_2 = K^2(\mathbf{X}, \mathbf{X}) + \sum_{i \neq \mathbf{X}} \frac{K(\mathbf{X}, i)K(i, \mathbf{X})}{1 - (-1)^{i+\mathbf{X}}\phi^{2\mathbf{X}-2i}}. \quad (37)$$

For example, this gives

$$\begin{aligned} W_2(1) &= \frac{25}{16\phi^4} + \sum_{s \neq 1} \frac{25s}{4^{s+1}\phi^4(\phi^{2s-2} - (-1)^{s+1})} \\ &= 0.267632296633129768709622137665599100606923184706474041907380_+, \\ W_2(2) &= \frac{35^2}{2^8\phi^8} + \sum_{s \neq 2} \frac{s(25s - 15)^2}{2^{2s+5}\phi^8(\phi^{2s-4} - (-1)^s)} \\ &= 0.160274183422101231629872112692281197699627281556900277836798_+, \\ W_2(3) &= \frac{290^2}{2^{12}\phi^{12}} + \sum_{s \neq 3} \frac{25 \cdot s(25s^2 - 45s + 26)^2}{3 \cdot 2^{2s+8}\phi^{12}(\phi^{2s-6} - (-1)^{s+1})} \\ &= 0.120655345445838262410173716774122740053340142551806782250004_{+..+} \end{aligned}$$

In general,

$$W_2(\mathbf{X}) = 25 \frac{Y_{\mathbf{X}}(\mathbf{X})^2}{(2\phi)^{4\mathbf{X}}} + \frac{25}{\mathbf{X}} \sum_{s \neq \mathbf{X}} \frac{s(Y_{\mathbf{X}}(s))^2}{2^{2s+2\mathbf{X}}(\phi^{2s+2\mathbf{X}} - (-1)^{s+\mathbf{X}}\phi^{4\mathbf{X}})},$$

where polynomials  $Y_s$  were introduced in Subsection 5.3. In neither in these cases the ISC, the *Inverse Symbolic Calculator* [53] gives no results. These are new constants. Choosing  $j \neq \mathbf{X}$  gives the value for  $q_2^{(j)}$ :

$$\begin{aligned} q_2^{(j)} &= \frac{K(j, \mathbf{X})K(\mathbf{X}, \mathbf{X})}{(1 - (-1)^{j+\mathbf{X}}\phi^{2j-2\mathbf{X}})(1 - (-1)^{j+\mathbf{X}}\phi^{2\mathbf{X}-2j})} \\ &+ \sum_{i \neq \mathbf{X}} \frac{K(j, i)K(i, \mathbf{X})}{(1 - (-1)^{\mathbf{X}+j}\phi^{2\mathbf{X}-2j})(1 - (-1)^{\mathbf{X}+i}\phi^{2\mathbf{X}-2i})}. \end{aligned}$$

Based on the identity (37), we can calculate numerical values for  $W_2(n)$ , and this is summarized in the Table 5.

Now we will derive asymptotic formula for the sequence  $W_2(n)$ .

**Proposition 7.** *We have an asymptotic formula*

$$W_2(n) \sim \frac{5^{1/4}}{4\sqrt{2}\sqrt{\pi}\sqrt{n}} = \frac{1}{\sqrt{n}} \cdot 0.1491394631939741_+.$$

Numerical data is presented in Table 5. However, differently from  $v = 1$  case in Subsection 6.1, this time, and in general for  $v \geq 2$ , the convergenve is only of order  $n^{-1}$ .

*Proof.* Let us separate (37) into four terms:

$$\begin{aligned} W_2 &= K^2(\mathbf{X}, \mathbf{X}) + \sum_{i < \mathbf{X}} \frac{K(\mathbf{X}, i)K(i, \mathbf{X})}{1 - (-1)^{i+\mathbf{X}}\phi^{2\mathbf{X}-2i}} + \sum_{i > \mathbf{X}} \frac{K(\mathbf{X}, i)K(i, \mathbf{X})(-1)^{i+\mathbf{X}}\phi^{2\mathbf{X}-2i}}{1 - (-1)^{i+\mathbf{X}}\phi^{2\mathbf{X}-2i}} \\ &+ \sum_{i > \mathbf{X}} K(\mathbf{X}, i)K(i, \mathbf{X}) = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4. \end{aligned}$$



$n$	$W_2(n)$	$W_2(n)\sqrt{n}$	$n$	$W_2(n)$	$W_2(n)\sqrt{n}$
1	0.2676322966331298	0.26763229663313	14	0.0461769229840162	0.17277822498163
2	0.1602741834221012	0.22666192389381	15	0.0443945360300747	0.17193929870708
3	0.1206553454458383	0.20898118851697	16	0.0427964974407240	0.17118598976290
4	0.0992583510259317	0.19851670205186	17	0.0413534145890492	0.17050449633061
5	0.0857703979434073	0.19178844025867	18	0.0400420336967085	0.16988396135665
6	0.0764132038812498	0.18717335912032	19	0.0388436875599233	0.16931570866817
7	0.0694718541272077	0.18380524913915	20	0.0377431970151391	0.16879270842804
8	0.0640703504588479	0.18121831712981	21	0.0367280773808485	0.16830919472793
9	0.0597172873901818	0.17915186217055	22	0.0357879546763424	0.16786038662576
10	0.0561149803515228	0.17745114876641	23	0.0349141291268267	0.16744228107545
11	0.0530716542113401	0.17601876402250	24	0.0340992439956194	0.16705149680786
12	0.0504575689489552	0.17479014609200	100	0.0157926629640446	0.15792662964044
13	0.0481814350440456	0.17372063457675	1000	0.0048047468687205	0.15193943685718

TABLE 5. The function  $W_2(n)$  for  $1 \leq n \leq 24$ ,  $n = 100, 1000$ .

Then, minding the asymptotics of  $K(i, \ell)$ , we have

$$\mathcal{S}_1 = \frac{B}{\mathbf{X}}, \quad \mathcal{S}_2 = \frac{B}{\mathbf{X}}, \quad \mathcal{S}_3 = \frac{B}{\mathbf{X}},$$

and the main asymptotics comes from  $\mathcal{S}_4$ . □

**6.4.  $v = 3$  and general asymptotic results for  $W_\ell(n)$ .** The previous subsection, the case  $v = 2$ , already shows the main features how our method works. Now, we will work in detail the case  $v = 3$ , and this will clearly show how the case of general  $v$  works. So, let us rewrite the recurrence (35) for  $v = 3$ . This gives

$$\begin{aligned} q_3^{(j)} + q_2^{(j)}W_1 + q_1^{(j)}W_2 + q_0^{(j)}W_3 = \\ q_3^{(j)}(-1)^{\mathbf{X}+j}\phi^{2\mathbf{X}-2j} + K(j, \mathbf{X})W_2 + \sum_{i=1}^{\infty} q_1^{(i)}K(j, i)W_1 + \sum_{i=1}^{\infty} q_2^{(i)}K(j, i). \end{aligned}$$

In particular, when  $j = \mathbf{X}$ , this gives

$$W_3 = K(\mathbf{X}, \mathbf{X})W_2 + \sum_{i=1}^{\infty} q_1^{(i)}K(\mathbf{X}, i)W_1 + \sum_{i=1}^{\infty} q_2^{(i)}K(\mathbf{X}, i).$$

Thus,

$$\begin{aligned} W_3 = K^3(\mathbf{X}, \mathbf{X}) + 2 \sum_{i \neq \mathbf{X}} \frac{K(\mathbf{X}, i)K(i, \mathbf{X})K(\mathbf{X}, \mathbf{X})}{1 - (-1)^{i+\mathbf{X}}\phi^{2\mathbf{X}-2i}} + \\ \sum_{i \neq \mathbf{X}} \frac{K(\mathbf{X}, i)K(i, \mathbf{X})K(\mathbf{X}, \mathbf{X})}{(1 - (-1)^{i+\mathbf{X}}\phi^{2i-2\mathbf{X}})(1 - (-1)^{i+\mathbf{X}}\phi^{2\mathbf{X}-2i})} \\ + \sum_{i, j \neq \mathbf{X}} \frac{K(\mathbf{X}, i)K(i, j)K(j, \mathbf{X})}{(1 - (-1)^{\mathbf{X}+i}\phi^{2\mathbf{X}-2i})(1 - (-1)^{\mathbf{X}+j}\phi^{2\mathbf{X}-2j})}. \end{aligned}$$

When  $j \neq \mathbf{X}$ , we obtain

...

The Table 6 summarizes numerical results.

$n$	$W_3(n)$	$W_3(n)\sqrt{n}$	$n$	$W_3(n)$	$W_3(n)\sqrt{n}$
1	0.1680289608296186	0.16802896082962	14	0.0262634917733556	0.09826898799627
2	0.0994912028536390	0.14070180841243	15	0.0252135206075704	0.09765154541237
3	0.0730151760824401	0.12646599469838	16	0.0242744501237882	0.09709780049516
4	0.0589532295056262	0.11790645901125	17	0.0234283081740980	0.09659738923134
5	0.0503513491297441	0.11258903941293	18	0.0226609289213482	0.09614217904963
6	0.0445129374472323	0.10903398369814	19	0.0219609781880675	0.09572568462311
7	0.0402451220926773	0.10647858454066	20	0.0213192670506264	0.09534266071137
8	0.0369582987959716	0.10453385479900	21	0.0207282581249678	0.09498881188226
9	0.0343299221825138	0.10298976654754	22	0.0201817037603623	0.09466058137767
10	0.0321680798925768	0.10172440041481	23	0.0196743764518228	0.09435499478917
11	0.0303507625625911	0.10066209152128	24	0.0192018649467395	0.09406954245869
12	0.0287962642532822	0.09975318550973	100	0.0087452046646542	0.08745204664654
13	0.0274475597086833	0.09896358391601	500	0.0037560671544927	0.08398821485500

TABLE 6. The function  $W_3(n)$  for  $1 \leq n \leq 24$ ,  $n = 100, 500$ .

**Proposition 8.** *We have an asymptotic formula*

$$W_3(n) \sim \frac{5^{1/4}}{6\sqrt{3}\sqrt{\pi}\sqrt{n}} = \frac{1}{\sqrt{n}} \cdot 0.0811812411861842_+.$$

*Proof.* Similarly as before,

$$W_3(\mathbf{X}) \sim \sum_{i,j > \mathbf{X}} K(\mathbf{X}, i)K(i, j)K(j, \mathbf{X}).$$

□

## 7. ASYMPTOTICS FOR $W_\ell(\mathbf{X})$

Now, based on the above examples, we will derive formula for  $W_\ell(\mathbf{X})$  for any  $\ell$ .

Let us rewrite (35) in two cases,  $j = \mathbf{X}$ , and  $j \neq \mathbf{X}$ , minding that we have chosen  $q_v^{(\mathbf{X})}$  to be zero for  $v \geq 1$ , and only  $q_v^{(\mathbf{X})}$  is nonzero, equal to 1, for  $v = 0$ . This reads as

$$W_v = \sum_{r=0}^{v-2} \sum_{i \neq \mathbf{X}} q_{v-r-1}^{(i)} K(\mathbf{X}, i) W_r + K(\mathbf{X}, \mathbf{X}) W_{v-1}, \quad (38)$$

$$q_v^{(j)} + \sum_{r=1}^{v-1} q_{v-r}^{(j)} W_r = q_v^{(j)} (-1)^{\mathbf{X}+j} \phi^{2\mathbf{X}-2j} + \sum_{r=0}^{v-2} \sum_{i \neq \mathbf{X}} q_{v-r-1}^{(i)} K(j, i) W_r + K(j, \mathbf{X}) W_{r-1}. \quad (39)$$

Again, based on the above examples, we easily see that the closed-form formulas for  $W$  and  $q$  has the following appearances:

$$W_\ell = K(\mathbf{X}, \mathbf{X})^\ell + \sum_{h=1}^{\ell-1} \sum_{i_1, \dots, i_h \neq \mathbf{X}} \mathfrak{w}(\ell, h; i_1, \dots, i_h) \cdot K(\underline{\mathbf{X}}, i_1) K(i_1, i_2) \cdots K(i_h, \mathbf{X}) K(\mathbf{X}, \mathbf{X})^{\ell-h-1},$$

$$q_\ell^{(j)} = \sum_{h=0}^{\ell-1} \sum_{i_1, \dots, i_h \neq \mathbf{X}} \mathfrak{q}(\ell, h, j; i_1, \dots, i_h) \cdot K(\underline{j}, i_1) K(i_1, i_2) \cdots K(i_h, \mathbf{X}) K(\mathbf{X}, \mathbf{X})^{\ell-h-1}.$$

For  $h = 0$ , the product  $K(\underline{j}, i_1) K(i_1, i_2) \cdots K(i_h, \mathbf{X})$  is just  $K(j, \mathbf{X})$ . In fact, coefficients  $\mathfrak{w}(\ell, h; i_1, \dots, i_h)$  and  $\mathfrak{q}(\ell, h, j; i_1, \dots, i_h)$  depend also on  $\mathbf{X}$ , but we omit the indication of this dependency.

Now, let us plug these two expressions into (38).

In a completely analogous manner we get that

$$W_\ell(\mathbf{X}) \sim \sum_{i_1, i_2, \dots, i_{\ell-1} > \mathbf{X}} K(\mathbf{X}, i_1) \cdot K(i_1, i_2) \cdots K(i_{\ell-1}, \mathbf{X}) \sim \frac{5^{1/4}}{2^{(\ell+1)/2} \pi^{\ell/2} \mathbf{X}^{1/2}} \cdot \mathcal{I}_\ell, \quad (40)$$

where  $\mathcal{I}_\ell = \int_{[0, \infty)^{\ell-1}} \exp\left(-Q_\ell(\alpha_1, \alpha_2, \dots, \alpha_{\ell-1})\right) d\alpha_1 \cdots d\alpha_{\ell-1}.$

Here  $Q_\ell$  is the quadratic form

$$Q_\ell(\alpha_1, \alpha_2, \dots, \alpha_{\ell-1}) = \sum_{t=1}^{\ell-1} \alpha_t^2 - \sum_{t=1}^{\ell-2} \alpha_t \alpha_{t+1}.$$

(An empty integral here is equal to 1 by convention, so the asymptotic formula holds for  $\ell = 1$  as well). We know, and it can be double checked, that

$$\mathcal{I}_1 = 1, \quad \mathcal{I}_2 = \frac{\sqrt{\pi}}{2}, \quad \mathcal{I}_3 = \frac{2\pi}{3\sqrt{3}}.$$

For  $\ell \geq 4$  the matters look as if different. In fact, the integral  $\mathcal{I}_\ell$  has a form of the Gaussian integral. We can transform it, with the help of a linear change, into the quadratic form of a diagonal unary form. But then the *solid angle*  $[0, \infty)^{\ell-1}$  transforms into the solid angle spanned by some normal vectors, and thus all we need is the formula for solid angles in the space  $\mathbb{R}^\ell$ . For  $\ell \geq 4$  such closed-form formula does not exist, only the series expansion [47]. Yet, by a lucky chance, the values  $\mathcal{I}_\ell$  can be given a very neat closed form. In our case, we have the following.

$$\begin{aligned} \mathcal{I}_\ell &= \int_{[0, \infty)^{\ell-1}} \exp\left(-\sum_{t=1}^{\ell-1} \alpha_t^2\right) \cdot \exp\left(\sum_{t=1}^{\ell-2} \alpha_t \alpha_{t+1}\right) d\alpha_1 \cdots d\alpha_{\ell-1} \\ &= \int_{[0, \infty)^{\ell-1}} \exp\left(-\sum_{t=1}^{\ell-1} \alpha_t^2\right) \cdot \sum_{k_1, k_2, \dots, k_{\ell-2}=0}^{\infty} \frac{(\alpha_1 \alpha_2)^{k_1}}{k_1!} \frac{(\alpha_2 \alpha_3)^{k_2}}{k_2!} \cdots \frac{(\alpha_{\ell-2} \alpha_{\ell-1})^{k_{\ell-2}}}{k_{\ell-2}!} d\alpha_1 \cdots d\alpha_{\ell-1}. \end{aligned}$$

The integral splits. Since

$$\int_0^\infty \exp(-\alpha^2) \alpha^k \, d\alpha = \frac{1}{2} \Gamma\left(\frac{k+1}{2}\right), \text{ for } k \in \mathbb{N}_0,$$

we thus have

$$\mathcal{I}_\ell = \sum_{k_1, k_2, \dots, k_{\ell-2}=0}^\infty \frac{\Gamma\left(\frac{k_1+1}{2}\right) \Gamma\left(\frac{k_1+k_2+1}{2}\right) \dots \Gamma\left(\frac{k_{\ell-3}+k_{\ell-2}+1}{2}\right) \Gamma\left(\frac{k_{\ell-2}+1}{2}\right)}{2^{\ell-1} k_1! k_2! \dots k_{\ell-2}!}.$$

In particular,  $\mathcal{I}_3 = \mathcal{I}_3(1)$ , where

$$\begin{aligned} \mathcal{I}_3 &= \sum_{a=0}^\infty \frac{\Gamma\left(\frac{a+1}{2}\right)^2}{4 \cdot a!} = \frac{1}{4} \sum_{a=0}^\infty B\left(\frac{a+1}{2}, \frac{a+1}{2}\right) = \frac{1}{4} \sum_{a=0}^\infty \int_0^1 x^{(a-1)/2} (1-x)^{(a-1)/2} \, dx \\ &= \frac{1}{4} \int_0^1 \frac{1}{\sqrt{x(1-x)} - x(1-x)} \, dx = \frac{2\pi}{3\sqrt{3}}, \end{aligned}$$

and

$$\mathcal{I}_4 = \sum_{a,b=0}^\infty \frac{\Gamma\left(\frac{a+1}{2}\right) \Gamma\left(\frac{a+b+1}{2}\right) \Gamma\left(\frac{b+1}{2}\right)}{8 \cdot a! b!} = 1.9687012432153024_+ = \frac{\pi^{3/2}}{2\sqrt{2}}.$$

In general, as MAPLE numerically confirms,

$$\mathcal{I}_\ell = \frac{(2\pi)^{(\ell-1)/2}}{\ell^{3/2}}.$$

Now, and gathering everything together, we thus obtain the following.

**Proposition 9.** *For  $\ell \in \mathbb{N}$ , we have the formula*

$$W_\ell(n) = \frac{5^{1/4}}{2\sqrt{\pi} \cdot \ell^{3/2} \sqrt{n}} + \frac{b(\ell, n)}{n},$$

where the function  $b(\ell, n)$ , for a fixed  $\ell$ , as a function in  $n$ , is bounded from above.

It may seem from the Tables 4, 5, and 6 that  $b(\ell, n) > 0$ . This is not true, as we will soon see! In fact, for every fixed  $n$  there exists  $L = L(n)$  such that  $b(\ell, n) < 0$ , for  $\ell > L(n)$ . Indeed, let us define now

$$T(\ell) = \sup_{n \in \mathbb{N}} |b(\ell, n)|, \quad C(\ell, n) = \frac{b(\ell, n)}{T(\ell)}.$$

Thus,  $|C(\ell, n)| \leq 1$ . With some ‘‘bootstrapping’’, we will get an improvement of Proposition 9. Indeed, by the formulas in the Subsecion 11.1, we have the following:

$$\sum_{i+j=\ell} \frac{1}{(\xi_{i,j} \xi_{j,i})^{-2} - 1} < \frac{c_1}{\ell^2}, \text{ for a certain } c_1 > 0 \text{ and all } \ell \in \mathbb{N}.$$

Let  $c_2 = 5^{1/4} (2\sqrt{\pi})^{-1}$ . Further, minding the fact that  $W_\ell(n) > 0$ , by the formula (9),

$$\phi^{-4n} W_\ell(n) = \phi^{-4n} W_\ell(n) W_0(n) < \frac{c_1}{(\ell+2)^2} < \frac{c_1}{\ell^2} \implies \left| \frac{c_2}{\ell^{3/2} \sqrt{n}} + \frac{T(\ell) C(\ell, n)}{n} \right| < \frac{c_1 \phi^{4n}}{\ell^2}.$$

Imagine now that we consider only  $n \in [1, \dots, N]$ , for some fixed  $N \in \mathbb{N}$ . We see that for  $\ell$  large enough, starting from some  $\ell = L$ , all  $C(\ell, n)$  should be negative, otherwise the above inequality and the formula (9) will be invalidated. Thus,

$$T(\ell) < \frac{1}{|C(\ell, n)|} \cdot \left| \frac{c_1 \phi^{4n} n}{\ell^2} + \frac{c_2 \sqrt{n}}{\ell^{3/2}} \right|, \text{ for every } n \in \mathbb{N}.$$

Setting  $n = 1$ , we obtain the following

$$T(\ell) < \frac{c_3}{\ell^{3/2}},$$

provided that

$$\inf_{\ell \in \mathbb{N}} |C(\ell, 1)| = c_4 > 0. \quad (41)$$

We have the following result, a refinement of (9):

**Proposition 10.** *For  $\ell \in \mathbb{N}$ , we have the formula*

$$W_\ell(n) = \frac{5^{1/4}}{2\sqrt{\pi} \cdot \ell^{3/2} \sqrt{n}} + \frac{B(\ell, n)}{\ell^{3/2} n},$$

where the function  $B(\ell, n)$ , is uniformly bounded.

The property (41) is equivalent too

$$\frac{c_2}{\ell^{3/2}} \sim (-b(\ell, 1)) > c_4 |b(\ell, n)| \text{ for all } \ell, n \in \mathbb{N}.$$

This is equivalent to the inequality

$$W_\ell(n) < \frac{c_5}{\ell^{3/2}}.$$

## 8. GENERALIZED OPERATOR

Now, we need to investigate the following question: how we can be sure that  $(-1)^{n+1} \Lambda(n)$  gives *all* eigenvalues of  $\mathcal{L}$ , that there are no ‘‘sporadic’’ ones? The answer is provided by the trace formulas, and for this purpose we will employ a generalized operator  $\mathcal{L}_\omega : \tilde{\mathbf{V}} \mapsto \tilde{\mathbf{V}}$ ,  $\omega \in \mathbb{C}$ ,  $|\omega| \leq 1$ , which has been already defined by (11). Then  $\mathcal{L}_1[f(t)](z) = \mathcal{L}[f(t)](z)$ . If  $G(\omega, z)$  is the eigenfunction of this operator with the eigenvalue  $\lambda(\omega)$ , then it satisfies the regularity condition (3), and the functional equation (12). In fact, the whole essence of our first method (as said before) can be expressed in the following way. Let us rewrite the functional equation (12) in terms of a function  $g$ , where

$$G(\omega, z) = \frac{1}{(z + \phi)^2} g\left(\omega, \frac{z - \phi^{-1}}{z + \phi}\right), \quad z = \frac{x\phi + \phi^{-1}}{1 - x}.$$

(See Subsection 4.4). We then obtain

$$\lambda(\omega)g(\omega, x) = \phi^{-2}g(\omega, -\phi^{-2}x) + \lambda(\omega) \frac{5\omega}{(2\phi - x)^2} g\left(\omega, \frac{2x\phi^{-1} + 1}{-x + 2\phi}\right). \quad (42)$$

A function  $g(\omega, x)$  is analytic for  $|\omega| \leq 1$ ,  $|x| \leq 1$  (excluding only  $x = 1$  which corresponds to  $z = \infty$ ),  $\sup_{|x| \leq 1} |(1 - x)g(\omega, x)| < +\infty$ . This is just a reformulation of a regularity property for  $G(\omega, z)$ . Note that since  $G(\omega, z)$  is analytic in the cut  $z$ -plane  $\mathbb{C} \setminus (-\infty, -1]$ , then  $g(\omega, x)$  is analytic

in the cut  $x$ -plane  $\mathbb{C} \setminus (-\infty, -\phi^2] \cup [1, \infty)$ .

The form (42) of a functional equation can be tackled immediately: the first term on the right is just a scaling in a variable  $x$  of  $g(\omega, x)$ , so everything is governed by the Taylor coefficients of powers of the argument of the second summand on the right. So, if

$$g(\omega, x) = \sum_{\ell=0}^{\infty} \sum_{i=1}^{\infty} q_{\ell}^{(i)} \omega^{\ell} x^{i-1}, \quad \lambda_n(\omega) = (-1)^{n+1} \phi^{-2n} \sum_{\ell=0}^{\infty} W_{\ell}(n) \omega^{\ell},$$

comparing coefficients at  $\omega^v x^{j-1}$  of the above functional equation we obtain exactly the recurrence (35).

Let us return back to a variable  $z$ . Multiply now the recurrence (34) by  $\omega^v$ , and sum over integers  $v \geq 0$ . We see that, if we put

$$\Lambda(\omega, \mathbf{X}) = \phi^{-2\mathbf{X}} \sum_{\ell=0}^{\infty} \omega^{\ell} \cdot W_{\ell}(\mathbf{X}),$$

then

$$\lambda_n(\omega) = (-1)^{n+1} \Lambda(\omega, n) \tag{43}$$

is an eigenvalue of  $\mathcal{L}_{\omega}$  with the eigenfunction  $U(n, \omega, z)$ , where

$$U(\mathbf{X}, \omega, z) = \sum_{\ell=0}^{\infty} \omega^{\ell} \cdot u_{\ell}(\mathbf{X}, z).$$

This holds since we obtain exactly the identity (12). Similarly as in the case of  $\mathcal{L}_1$ , for fixed  $\omega$ ,  $|\omega| \leq 1$ , the operator  $\mathcal{L}_{\omega}$  is of trace class and is nuclear of order zero [41]. Thus,

$$\mathrm{Tr}(\mathcal{L}_{\omega}) = \sum_{m=1}^{\infty} \frac{\omega^{m-1}}{\xi_m^{-2} + 1}, \quad \mathrm{Tr}(\mathcal{L}_{\omega}^2) = \sum_{i,j=1}^{\infty} \frac{\omega^{i+j-2}}{(\xi_{i,j} \xi_{j,i})^{-2} - 1},$$

and similarly for higher powers, as is given by (14). Now, we know that the eigenvalues of  $\mathcal{L}_{\omega}$  are either analytic functions of  $\omega$  for  $|\omega| \leq 1$ , or at least can be expressed as a Puiseux series, and they are real for real  $\omega$ . Let the set of eigenvalues be the union of  $\{\lambda_n(\omega), n \in \mathbb{N}\}$ , the set which we have constructed via (43), and  $\{\sigma_i(\omega), i \in \mathcal{I}\}$ , where  $\mathcal{I}$  is a finite or a countable set. These are what we called ‘‘sporadic’’ eigenvalues, and it is the set which was left out of the framework of explicit construction in Section 6. We have:

$$\sum_{n=1}^{\infty} \lambda_n^2(\omega) + \sum_{i \in \mathcal{I}} \sigma_i^2(\omega) \equiv \mathrm{Tr}(\mathcal{L}_{\omega}^2), \quad |\omega| \leq 1.$$

In particular, for  $\omega = 0$ , this gives

$$\sum_{i \in \mathcal{I}} \sigma_i^2(0) = \frac{1}{\phi^4 - 1} - \sum_{n=1}^{\infty} \lambda_n^2(0) = 0.$$

Thus, since  $\sigma_\ell(0) \in \mathbb{R}$  are eigenvalues of  $\mathcal{L}_0$ , this implies  $\mathcal{S} = \emptyset$ . Also, comparing the coefficients at the powers of  $\omega$  in the trace formula, we obtain

$$\sum_{n=1}^{\infty} (-1)^{n+1} \phi^{-2n} W_{\ell-1}(n) = \frac{1}{\xi_\ell^{-2} + 1}, \quad \ell \geq 1,$$

and similarly for higher powers, as explained in Subsection 3.3. This gives the first identities of the Theorem 2, part (ii).

## 9. GENERATING FUNCTION FOR THE COEFFICIENTS $W_\ell(\mathbf{X})$

9.1. **The function  $\Omega_1(w)$ .** Let

$$\Omega_1(w) = \sum_{n=1}^{\infty} W_1(n) w^n = \frac{5}{4} \sum_{n=1}^{\infty} \phi^{-2n} P_{n-1}^{(0,1)}\left(\frac{3}{2}\right) w^n = \frac{5}{4} \sum_{n=1}^{\infty} P_{n-1}^{(0,1)}\left(\frac{3}{2}\right) (\phi^{-2}w)^n.$$

Then, according to the formula (22), one has

$$\begin{aligned} \Omega_1(w) &= \frac{5w}{2(1 - 3\phi^{-2}w + \phi^{-4}w^2)^{1/2}(\phi^2 + w) + 2\phi^2(1 - 3\phi^{-2}w + \phi^{-4}w^2)} \\ &= \frac{5w\phi^2}{2((\phi^4 - w)(1 - w))^{1/2}(\phi^2 + w) + 2(\phi^4 - w)(1 - w)} \end{aligned} \quad (44)$$

$$= \frac{\phi^2 + w}{2((\phi^4 - w)(1 - w))^{1/2}} - \frac{1}{2}. \quad (45)$$

Thus way we arrive at identities which were written down and checked numerically in Section 1:

$$-\Omega_1(-\phi^{-2}) = \frac{1}{2\sqrt{2} + 4}, \quad \Omega_1(\phi^{-4}) = \frac{1}{4\sqrt{3} + 6}, \quad -\Omega_1(-\phi^{-6}) = \frac{1}{6\sqrt{10} + 20}.$$

9.2. **The function  $\Omega_{1,1}(w)$ , the first way.** For  $|w| < |t| < 1$ ,  $\Omega_{1,1}(w)$  is just the constant term in  $t$  of the function  $\Omega_1(t)\Omega_1(\frac{w}{t})$ ; so,

$$\Omega_{1,1}(w) = \frac{1}{2\pi i} \oint \frac{\Omega_1(t)\Omega_1(\frac{w}{t})}{t} dt = \frac{25}{16} \sum_{n=1}^{\infty} \phi^{-4n} \left( P_{n-1}^{(0,1)}\left(\frac{3}{2}\right) \right)^2 w^n$$

where the small contour rounds  $t = 0$  once in the positive direction. We have:

$$\Omega_1(t)\Omega_1\left(\frac{w}{t}\right) = \frac{1}{4} \left( \frac{\phi^2 + t}{((\phi^4 - t)(1 - t))^{1/2}} - 1 \right) \cdot \left( \frac{\phi^2 t + w}{((\phi^4 t - w)(t - w))^{1/2}} - 1 \right).$$

Suppose,  $0 < w < 1$  is real. The function  $\Omega_1(t)\Omega_1(w/t)$  is a single valued function in the cut  $t$ -plane  $\mathbb{C} \setminus \{[1, \phi^4] \cup [\phi^{-4}w, w]\}$  with a singular point  $t = 0$ . Consider the contour consisting of the segment  $[1 - iT, 1 + iT]$ , and a semicircle  $C_1$ , given by  $1 + Te^{is}$ ,  $s \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ . For  $|t|$  large and  $w$  fixed, from (44), we have

$$\left| \frac{\Omega_1(t)\Omega_1(\frac{w}{t})}{t} \right| \ll \frac{1}{|t|^3},$$

so the integral over  $C_1$  is  $\ll |t|^{-2}$ . We now take the limit  $T \rightarrow \infty$ . So,  $\Omega_{1,1}(w)$  is equal to the integral of  $(2\pi i)^{-1} \Omega_1(t)\Omega_1(w/t)t^{-1}$  taken over the line  $[1 - i\infty, 1 + i\infty]$ . By the same reasoning, this is also equal, up to the opposite sign, to the integral taken over the contour consisting of the

segment  $[1 + iT, 1 - iT]$  and a semicircle  $C_2$ , given by  $1 + Te^{is}$ ,  $s \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ . Using the standard contour integration techniques, we thus obtain

$$\Omega_{1,1}(w) = \frac{1}{4\pi} \int_1^{\phi^4} \frac{\phi^2 + t}{((\phi^4 - t)(t - 1))^{1/2}} \cdot \left( \frac{\phi^2 t + w}{((\phi^4 t - w)(t - w))^{1/2} t} - \frac{1}{t} \right) dt.$$

This can be double checked with MAPLE to hold true. At the same time we obtain the identity

$$\frac{5}{4} \phi^{-2n} P_{n-1}^{(0,1)}(3/2) = \frac{1}{2\pi} \int_1^{\phi^4} \frac{\phi^2 + t}{((t - 1)(\phi^4 - t))^{1/2} \cdot t^{n+1}} dt, \quad n \geq 1.$$

Thus,

$$\begin{aligned} \Omega_{1,1}(w) &= \frac{1}{4\pi} \int_1^{\phi^4} \frac{\phi^2 + t}{((t - 1)(\phi^4 - t))^{1/2}} \cdot \frac{(\phi^2 t + w)}{((t - w)(\phi^4 t - w))^{1/2} t} dt - \frac{1}{4\pi} \int_1^{\phi^4} \frac{\phi^2 + t}{((t - 1)(\phi^4 - t))^{1/2} t} dt \\ &= \frac{1}{4\pi} \int_1^{\phi^4} \frac{\phi^2 + t}{((t - 1)(\phi^4 - t))^{1/2}} \cdot \frac{\phi^2 t + w}{((t - w)(\phi^4 t - w))^{1/2} t} dt - \frac{1}{2}. \end{aligned}$$

This, after some transformations, can be given the expression (using MAPLE)

$$\begin{aligned} &4\pi \cdot \left( \Omega_{1,1}(w) + \frac{1}{2} \right) \\ &= \frac{2(\phi^2 + 1)(w + \phi^2)}{\phi^4 - w} \cdot K(z\sqrt{w}) + \frac{2\phi^2(1 - w)}{\phi^4 - w} \cdot \Pi(z, \sqrt{w}z) + \frac{2\phi^2(w - 1)}{\phi^4 - w} \cdot \Pi(wz, \sqrt{w}z), \quad (46) \\ &\text{where } z = \frac{(\phi^4 - 1)}{\phi^4 - w}. \end{aligned}$$

According to the formula (2.1,[10]),

$$\sum_{n=0}^{\infty} \left( P_n^{(0,1)}(3/2) \right)^2 w^n = \frac{1}{(1+w)^2} F_4 \left( 1, \frac{3}{2}; 1, 2; \frac{1}{4(w + w^{-1} + 2)}, \frac{25}{4(w + w^{-1} + 2)} \right).$$

**9.3. The function  $\Omega_{1,1}(w)$ , the second way.** We can also present  $\Omega_{1,1}(w)$  by the simpler integral using the function  $\Theta(x, y)$ , defined by (32):

$$\Omega_{1,1}(w) = \frac{1}{2\pi i} \oint \frac{\Theta(t, \frac{w}{t})w}{t} dt, \quad |w| < |t| < 1.$$

Thus,

$$\Omega_{1,1}(w) = \frac{5}{2\pi i} \oint \frac{w}{t(2\phi - t)(2\phi - t - \frac{w}{t} - 2w\phi^{-1})} dt$$



9.4. **The function  $\Omega_2(w)$ .** We already now from Subsection (3.4) that the values of  $\Omega_2(w)$  at  $w = (-1)^k \phi^{-2k}$  belong to  $\widehat{\mathcal{P}}$ .

We will demonstrate this fact in an alternative way. According to the formula (27), we have

$$K(n, i) \cdot K(i, n) = \frac{\phi^{-2n-2i}}{2^{i+n+2}(\pi i)^2} \oint_{x=\frac{3}{2}} \frac{(x-1)^{i-1}(x+1)^n}{\left(x-\frac{3}{2}\right)^n} dx \oint_{u=\frac{3}{2}} \frac{(y-1)^{n-1}(y+1)^i}{\left(y-\frac{3}{2}\right)^i} dy.$$

Thus, we need to show that the function

$$\begin{aligned} & \sum_{\substack{n,i=1 \\ n \neq i}}^{\infty} \frac{\phi^{-2n-2i}}{2^{i+n}(1 - (-1)^{n+i} \phi^{2n-2i})} \cdot \frac{(x-1)^{i-1}(x+1)^n}{\left(x-\frac{3}{2}\right)^n} \cdot \frac{(y-1)^{n-1}(y+1)^i}{\left(y-\frac{3}{2}\right)^i} w^n \\ &= \sum_{\substack{n,i=1 \\ n \neq i}}^{\infty} \frac{\phi^{-2n-2i}}{(1 - (-1)^{n+i} \phi^{2n-2i})} \cdot \frac{(x-1)^{i-1}(x+1)^n}{(2x-3)^n} \cdot \frac{(y-1)^{n-1}(y+1)^i}{(3y-3)^i} w^n \end{aligned}$$

Let  $n = i + s$ ,  $i, s \geq 1$ .

$$\begin{aligned} & \frac{\phi^{-2s-4} w^{s+1} (x+1)^{s+1} (y-1)^s (y+1)}{(1 - (-1)^s \phi^{2s})(2x-3)^{s+1}(2y-3)} \sum_{i=1}^{\infty} \phi^{-4i+4} \cdot \frac{(x^2-1)^{i-1}}{(2x-3)^{i-1}} \cdot \frac{(y^2-1)^{i-1}}{(2y-3)^{i-1}} w^{i-1} \\ &= \frac{\phi^{-2s} w^{s+1} (x+1)^{s+1} (y-1)^s (y+1)}{(1 - (-1)^s \phi^{2s})(2x-3)^s \left( (2x-3)(2y-3) - (x^2-1)(y^2-1) \right)} \end{aligned}$$

## 10. GOLDEN SECTION AND THETA FUNCTIONS

The general Lambert-like series we obtained for  $W_2(n)$  in Subsection 6.3 cannot be evaluated in closed form, but there are exceptions. For example, the 1899 result of Landau claims that ([14], p. 94)

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = \frac{\sqrt{5}}{4} \left( \Theta_3^2(\phi^{-1}) - \Theta_3^2(\phi^{-2}) \right)$$

, where the Jacobi theta function

$$\Theta_3(q) = \sum_{n \in \mathbb{Z}} q^{n^2}.$$

## 11. INTERPOLATION OF PARTIAL TRACE FORMULAS. I

11.1. **Trace of the square of the operator.** To see the structure of trace formulas that will be crucial in Subsections 11.2 and 12.1, we will derive an alternative expression for  $\text{Tr}(\mathcal{L}^2)$ , based on the trace formula (6).

For  $i, j \in \mathbb{N}$ , let  $\xi_{i,j} = p$ . Then

$$\frac{1}{i + \frac{1}{j+p}} = p \Rightarrow \frac{p+j}{ip + ij + 1} = p. \quad (47)$$

Next,  $\xi_{j,i} = \frac{1}{j+p}$ , so

$$(\xi_{i,j} \cdot \xi_{j,i})^{-1} = ip + ij + 1.$$

The important observation here is that in the expression of  $(\xi_{i,j}\xi_{j,i})^{-1}$  we see exactly the denominator which defines the quadratic irrational number  $p$ . This appears to be a general rule, and we will see this in the next subsections - this is also clear from (24) ( $\frac{P_1}{Q_1} = \frac{1}{i}$ ,  $\frac{P_2}{Q_2} = \frac{j}{ij+1}$ ); see [17, 41, 44, 45]. Indeed, the calculations of Subsection 12.1 - see the formula (??) and below - show that (here we have  $k = 2$ )

$$(\xi_{i,j} \cdot \xi_{j,i})^{-2} - 1 = \frac{D}{2} + \frac{ij+2}{2}\sqrt{D}, \quad D = (ij+2)^2 - 4.$$

So, we obtain

$$\begin{aligned} \mathrm{Tr}(\mathcal{L}^2) &= \sum_{i,j=1}^{\infty} \frac{1}{(\xi_{i,j}\xi_{j,i})^{-2} - 1} \\ &= \sum_{i,j=1}^{\infty} \frac{2}{(ij)^2 + 4ij + (ij+2)\sqrt{(ij)^2 + 4ij}} = \sum_{\ell=1}^{\infty} \frac{2\sigma_0(\ell)}{\ell^2 + 4\ell + (\ell+2)\sqrt{\ell^2 + 4\ell}}; \end{aligned}$$

here  $\sigma_0(\ell) = \sum_{j|\ell} 1$ . Let

$$\varpi(x) = \frac{2x^2}{1 + 4x + (1+2x)\sqrt{1+4x}} = \sum_{k=2}^{\infty} (-1)^k \binom{2k-2}{k-2} x^k, \quad |x| < \frac{1}{4}.$$

Thus,

$$\begin{aligned} \mathrm{Tr}(\mathcal{L}^2) &= \sum_{\ell=1}^4 \varpi\left(\frac{1}{\ell}\right) \sigma_0(\ell) + \sum_{\ell=5}^{\infty} \sum_{k=2}^{\infty} (-1)^k \binom{2k-2}{k-2} \frac{\sigma_0(\ell)}{\ell^k} \\ &= \sum_{\ell=1}^4 \varpi\left(\frac{1}{\ell}\right) \sigma_0(\ell) + \sum_{k=2}^{\infty} (-1)^k \binom{2k-2}{k-2} \left( \zeta^2(k) - 1 - \frac{2}{2^k} - \frac{2}{3^k} - \frac{3}{4^k} \right). \end{aligned}$$

This proves the Proposition 1, since

$$\zeta^2(s) = \sum_{\ell=1}^{\infty} \frac{\sigma_0(\ell)}{\ell^s}, \quad \text{for } \Re(s) > 1.$$

**11.2. One non-identity element.** Let

$$\underbrace{\xi_{N,1,\dots,1}}_k \cdot \underbrace{\xi_{1,N,\dots,1}}_k \cdots \underbrace{\xi_{1,1,\dots,N}}_k = \Gamma_{N,k}.$$

Here, as before,  $\xi_{a_1,a_2,\dots,a_k} = [0, \overline{a_1, a_2, \dots, a_k}]$ . We will find the second degree algebraic functions  $\Theta_N(w)$  such that

$$(-1)^k \Theta_N \left( (-1)^k \phi^{-2k} \right) = \Gamma_{N,k}^{-2} - (-1)^k.$$

Of course, we already know that  $\Theta_2(w) = \frac{1}{\Omega_1(w)}$ , see (16). For this purpose we will explicitly calculate  $\Gamma_{N,k}$ . Let

$$\underbrace{\xi_{1,1,\dots,1,N}}_k = p = p(N, k).$$

Let  $F_0 = 0$ ,  $F_1 = 1$  are the initial values for the standard Fibonacci sequence. Binet's formula tells us that

$$F_k = \frac{\phi^k - (-1)^k \phi^{-k}}{\sqrt{5}}.$$

It is the standard fact from the theory of regular continued fractions that  $p$  is positive and satisfies the quadratic equation

$$\frac{F_{k-1}(N+p) + F_{k-2}}{F_k(N+p) + F_{k-1}} = p. \quad (48)$$

Thus,

$$\begin{pmatrix} P_{k-1} & P_k \\ Q_{k-1} & Q_k \end{pmatrix} = \begin{pmatrix} F_{k-1} & NF_{k-1} + F_{k-2} \\ F_k & NF_k + F_{k-1} \end{pmatrix}.$$

Let

$$\psi = \Gamma_{N,k}^{-2} - (-1)^k.$$

Then, according to (24) ( $s = 1$ ), we have

$$\psi = \frac{D + \sqrt{D}(NF_k + 2F_{k-1})}{2}, \quad D = (NF_k + 2F_{k-1})^2 - 4(-1)^k. \quad (49)$$

Note that, from Binet's formula,

$$NF_k + 2F_{k-1} = \frac{1}{\sqrt{5}} \left( N\phi^k - N(-1)^k \phi^{-k} + 2\phi^{k-1} - 2(-1)^{k-1} \phi^{-k+1} \right).$$

Thus,

$$\sqrt{5}\phi^{-k}(NF_k + 2F_{k-1}) = N - N(-1)^k \phi^{-2k} + 2\phi^{-1} - 2(-1)^{k-1} \phi^{-2k+1}.$$

So, if we set

$$T_N(w) = T(w) := (N + 2\phi^{-1}) - w(N - 2\phi),$$

then

$$NF_k + 2F_{k-1} = \frac{1}{\sqrt{5}} \cdot \phi^k \cdot T\left((-1)^k \phi^{-2k}\right).$$

Thus, we get:

**Proposition 11.** *The next identity holds:*

$$\Theta_N(w) = \frac{T(w)^2 - 20w + T(w)\sqrt{T(w)^2 - 20w}}{10w}.$$

In the setting of Subsection 3.3, we obtained that for the investigation of decomposition formulas at  $\omega^1$  we should explore the function  $\Theta_2(w)$ . Thus, when  $N = 2$ , we once again obtain the identity  $\Theta_2^{-1}(w) = \Omega_1(w)$ . Further, for the decomposition formulas at  $\omega^2$ , as is seen from (19), we need to explore  $\Theta_3(w)$ .

For this purpose, we need to calculate the Taylor coefficients of

$$\frac{10w}{T(w)^2 - 20w + T(w)\sqrt{T(w)^2 - 20w}} = \sum_{n=1}^{\infty} L_N(n)w^n.$$

Now, let us define

$$R(w) = \frac{w}{L(w)^2 - w + L(w)\sqrt{L(w)^2 - w}}, \quad L(w) = \alpha + \beta w.$$

Then, according to (22), we have

$$R(w) = \frac{1}{2} \sum_{n=1}^{\infty} w^n P_{n-1}^{(0,1)} \left( \frac{1}{2\alpha\beta} - 1 \right) \alpha^{-n-1} \beta^{n-1}. \quad (50)$$

In the case  $\frac{1}{\Theta_N(w)}$ , we have  $\alpha = \frac{N+2\phi^{-1}}{\sqrt{20}}$ ,  $\beta = -\frac{N-2\phi}{\sqrt{20}}$ ; so,  $\alpha \cdot \beta = -\frac{N^2-2N-4}{20}$ . So,

$$L_N(n) = 5(N + 2\phi^{-1})^{-n-1} (-N + 2\phi)^{n-1} P_{n-1}^{(0,1)} \left( -\frac{10}{N^2 - 2N - 4} - 1 \right) \sim \frac{5^{1/4}}{2\sqrt{N-1}} \cdot \frac{1}{\sqrt{\pi n}}.$$

The asymptotics is the direct consequence of (21). We already know that  $L_2(n) = W_1(n) = K(n, n)$ . Moreover, we arrived at the formula (45) independently. In the special case  $N = 3$  this reads as

$$L_3(n) = 5\phi^{-6n} P_{n-1}^{(0,1)}(9).$$

Note that for  $N = 3$ ,

$$\begin{aligned} T(w) &= (3 + 2\phi^{-1}) - w(3 - 2\phi) = (1 + 2\phi) - w(1 - 2\phi^{-1}) = \phi^3 + \phi^{-3}w, \\ T(w)^2 - 20w &= \phi^6 - 18w + \phi^{-6}w^2 = \phi^{-6}(\phi^{12} - w)(1 - w). \end{aligned}$$

So,

$$\frac{1}{\Theta_3(w)} = \frac{10w\phi^6}{((\phi^{12} - w)(1 - w))^{1/2}(\phi^6 + w) + (\phi^{12} - w)(1 - w)}. \quad (51)$$

## 12. INTERPOLATION OF PARTIAL TRACE FORMULAS. II

**12.1. Two non-identity elements.** To finish our investigations of the function  $W_2(n)$ , let

$$\underbrace{\xi_{\mathbf{2}_1, 1, \dots, 1, \mathbf{2}_r, 1, \dots, 1}}_k \cdot \underbrace{\xi_{\mathbf{1}, \mathbf{2}_2, 1, \dots, 1, \mathbf{2}_{r+1}, 1, \dots, 1}}_k \cdot \underbrace{\xi_{\mathbf{1}, 1, \mathbf{2}_3, 1, \dots, 1, \mathbf{2}_{r+2}, 1, \dots, 1}}_k \cdots \underbrace{\xi_{\mathbf{1}, 1, \dots, 1, \mathbf{2}_{r-1}, 1, \dots, 1, \mathbf{2}_k}}_k = \Delta_{r,k}.$$

(This is cyclic permutation. We only bold-case one number “2” and its position for clarity). In a similar way, we will find the second degree algebraic functions  $\Pi_r(w)$  (see the end of Section 1 for the notation) such that

$$(-1)^k \Pi_r \left( (-1)^k \phi^{-2k} \right) = \Delta_{r,k}^{-2} - (-1)^k.$$

Note that  $\Delta_{r,k} = \Delta_{k-r+2,k}$ , so we should also obtain

$$\Pi_r(w) = \Pi_{k-r+2}(w). \quad (52)$$

Let, for  $2 \leq r \leq k$ ,  $k \geq 2$ ,

$$\underbrace{\xi_1, 1, \dots, 1, 2_{r-1}, 1, \dots, 1, 2_k}_k = p = p(r, k).$$

Again,  $p$  is positive and satisfies the quadratic equation

$$\frac{(F_{k-r+1}F_r + F_{k-r}F_{r-2})(2+p) + (F_{k-r}F_r + F_{k-r-1}F_{r-2})}{(F_{k-r+1}F_{r+1} + F_{k-r}F_{r-1})(2+p) + (F_{k-r}F_{r+1} + F_{k-r-1}F_{r-1})} = p. \quad (53)$$

This can be written

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} p = \frac{(F_{k-r+1}F_r + F_{k-r}F_{r-2})p + (F_{k-r+3}F_r + F_{k-r+2}F_{r-2})}{(F_{k-r+1}F_{r+1} + F_{k-r}F_{r-1})p + (F_{k-r+3}F_{r+1} + F_{k-r+2}F_{r-1})} = p.$$

According to (24), the number

$$\Delta_{r,k}^{-2} - (-1)^k = \varrho = \frac{D}{2} + \frac{a+d}{2} \sqrt{D} \quad (54)$$

$$\text{where } D = (a+d)^2 + 4(bc-ad) = (a+d)^2 - 4(-1)^k, \quad (55)$$

$$\frac{1}{\varrho} = (-1)^k \left( \frac{a+d}{2\sqrt{D}} - \frac{1}{2} \right).$$

We see that in general  $\varrho$  is very close to an integer  $D + (-1)^k$ :

$$\varrho^2 - D\varrho - (-1)^k D = 0 \Rightarrow \frac{1}{D + (-1)^k - \varrho} = \varrho + (-1)^k \Rightarrow \varrho \approx D + (-1)^k - \frac{1}{D + 2(-1)^k}.$$

More precisely,

$$\varrho = D + [\overline{1, D}] \text{ for } k \text{ even,}$$

$$\varrho = D - 2 + [\overline{1, D - 4}] \text{ for } k \text{ odd.}$$

In our case,

$$a + d = F_{k-r+1}F_r + F_{k-r}F_{r-2} + F_{k-r+3}F_{r+1} + F_{k-r+2}F_{r-1}.$$

We recall the definition of Lucas numbers:

$$L_0 = 2, \quad L_1 = 1, \quad L_{s+2} = L_{s+1} + L_s \text{ for } s \geq 0,$$

$$L_j = \phi^j + (-1)^j \phi^{-j}.$$

Thus, from Binet's formula we get the identity

$$F_\alpha F_\beta = \frac{1}{5} L_{\alpha+\beta} - \frac{1}{5} (-1)^\beta L_{\alpha-\beta}.$$

Thus,

$$\begin{aligned} 5(a+d) &= L_{k+1} - (-1)^r L_{k-2r+1} + L_{k-2} - (-1)^r L_{k-2r+2} \\ &+ L_{k+4} - (-1)^{r+1} L_{k-2r+2} + L_{k+1} - (-1)^{r+1} L_{k-2r+3}. \end{aligned}$$

Further, we have:

$$\begin{aligned}
L_{k-2} + 2L_{k+1} + L_{k+4} &= (L_k - L_{k-1}) + 2L_{k+1} + L_{k+4} \\
&= (L_{k+2} - L_{k+1}) - (L_{k+1} + L_k) + 2L_{k+1} + L_{k+4} \\
&= L_k + L_{k+2} + L_{k+4} \\
&= (L_{k+2} - L_{k+1}) + L_{k+2} + (2L_{k+2} + L_{k+1}) \\
&= 4L_{k+2}.
\end{aligned}$$

Thus,

$$5(a + d) = 4L_{k+2} + (-1)^r L_{k-2r+2}. \quad (56)$$

We can double-check this formula in special cases, like  $k = 3$ ,  $r = 3$ ,  $\xi = \xi_{1,2,2}$ , or  $k = 4$ ,  $r = 2$ ,  $\xi = \xi_{2,1,1,2}$ , and this does hold. The number  $k - 2r + 2$  can be also negative, Binet's formula for  $L_k$  remains valid, which implies  $L_{-k} = (-1)^k L_k$ . So,

$$5(a + d) = 4\phi^{k+2} + 4(-1)^k \phi^{-k-2} + (-1)^r \phi^{k-2r+2} + (-1)^{k+r} \phi^{-k+2r-2}$$

Let us put

$$w = (-1)^k \phi^{-2k}, \quad \eta_r = (-1)^r \phi^{-2r}.$$

Thus,

$$5\phi^{-k}(a + d) = \phi^2(4 + \eta_r) + w\phi^{-2}(4 + \eta_r^{-1}) := L_r(w). \quad (57)$$

Then

$$a + d = L_r(w) \cdot \frac{\phi^k}{5}.$$

and we finally obtain

$$\begin{aligned}
\frac{1}{\Pi_r(w)} &= \frac{L_r(w)}{2\sqrt{L_r(w)^2 - 100w}} - \frac{1}{2}, \quad 2 \leq r \leq k, \\
\frac{1}{\Pi_r(\phi^4 w)} &= \frac{L_r(\phi^4 w)}{2\sqrt{L_r(\phi^4 w)^2 - 100\phi^4 w}} - \frac{1}{2} = \frac{S_r(w)}{2\sqrt{S_r(w)^2 - 100w}} - \frac{1}{2},
\end{aligned}$$

where

$$S_r(w) = (4 + \eta_r) + w(4 + \eta_r^{-1}).$$

Let us explore the function

$$\frac{S(w, \eta)}{2\sqrt{S(w, \eta)^2 - 100w\eta^2}} - \frac{1}{2}$$

$$S_r(w, \eta) = \eta^2 + 4\eta + 4w\eta + w.$$

$$S^2 = 16 + \eta^2 + 16w^2 + w^2\eta^{-2} + 8\eta + 32w + 8w\eta^{-1} + 8w\eta + 2w + 8w^2\eta^{-1}.$$

By a direct check,

$$L_r\left((-1)^k \phi^{-2k}\right) = L_{k-r+2}\left((-1)^k \phi^{-2k}\right),$$

so the symmetry property (52) does hold.

Now we are finally in the position to find the exact structure of the functions  $\Omega_{1,1}(w)$  and  $\Omega_2(w)$ , using the identity that equates (18) and (19).

In the special case (57), that is, when  $\alpha_r = 4\phi^2 + (-1)^r \phi^{-2r+2}$ ,  $\beta_r = 4\phi^{-2} + (-1)^r \phi^{2r-2}$ , one has

$$\alpha_r \cdot \beta_r = 17 + 4(-1)^r L_{2r}.$$

Thus, according to (50),

$$\begin{aligned} \sum_{r=2}^k \frac{1}{\Pi_r(w)} &= \frac{1}{2} \sum_{r=2}^k \frac{w}{(L_r(w)/10)^2 - w + (L_r(w)/10) \sqrt{(L_r(w)/10)^2 - w}} \\ &= 25 \sum_{n=1}^{\infty} w^n \sum_{r=2}^k \alpha_r^{-n-1} \beta_r^{n-1} P_{n-1}^{(0,1)} \left( \frac{50}{\alpha_r \beta_r} - 1 \right). \end{aligned} \quad (58)$$

We need to express the latter sum in terms of  $kW(w)$  and  $Y(w)$ ,  $w = (-1)^k \phi^{-2k}$ , which must be possible as explained in Subsection (3.3). Indeed, we need to express

$$\Sigma(u; v) = \sum_{r=2}^k \frac{\beta_r^u}{\alpha_r^v} = \sum_{r=2}^k \frac{(4\phi^{-2} + (-1)^r \phi^{2r-2})^u}{(4\phi^2 + (-1)^r \phi^{-2r+2})^v}, \quad v \geq u + 2 \geq 0.$$

in terms of  $kW(w)$  and  $Y(w)$ . Expanding in terms of powers of  $(-1)^r \phi^{-2r}$  and summing over  $r = 2, \dots, k$ , we see that this indeed can be expressed in terms of  $w$  and  $kw$ . As mentioned, this research (arithmetic of decomposition formulas) is the main topic of the paper [1].

So, we need to sum

$$\sum_{r=2}^k \frac{1}{\varrho(r, k)}.$$

Let  $a(r, k) + d(r, k) = t(r, k)$ ,  $D(r, k) = t(r, k)^2 - 4(-1)^k$ . We have:

$$\frac{1}{\varrho} = \frac{2}{D + t\sqrt{D}} = \frac{\frac{2}{t^2}}{1 - \frac{4(-1)^k}{t^2} + \sqrt{1 - \frac{4(-1)^k}{t^2}}}.$$

Let

$$\frac{4(-1)^k}{t^2} = \eta(r, k).$$

First, we have

$$\frac{\eta}{1 - \eta + \sqrt{1 - \eta}} = \sum_{j=1}^{\infty} \binom{2j}{j} 4^{-j} \eta^j.$$

So,

$$\sum_{r=2}^k \frac{1}{\varrho(r, k)} = \frac{(-1)^k}{2} \sum_{j=1}^{\infty} \binom{2j}{j} (-1)^{kj} \sum_{r=2}^k \frac{1}{t(r, k)^{2j}}.$$

$$\begin{aligned}
\frac{1}{(a+d)^{2j}} &= \frac{5^{2j}}{\left[4\phi^{k+2} + 4(-1)^k\phi^{-k-2} + (-1)^r\phi^{k-2r+2} + (-1)^{k+r}\phi^{-k+2r-2}\right]^{2j}} \\
&= \frac{\left(\frac{5}{4\phi^2}\right)^{2j}\phi^{-2kj}}{\left[1 + (-1)^k\phi^{-2k-4} + \frac{1}{4}(-1)^r\phi^{-2r} + \frac{1}{4}(-1)^{k+r}\phi^{-2k+2r-4}\right]^{2j}} \\
&= \frac{\left(\frac{5}{4\phi^2}\right)^{2j}\phi^{-2kj}}{\left[1 + \phi^{-4}w + \frac{1}{4}(-1)^r\phi^{-2r} + \frac{1}{4}(-1)^r\phi^{2r-4}w\right]^{2j}}, \\
&= \frac{10w\phi^6}{((\phi^{12} - w)(1 - w))^{1/2}(\phi^6 + w) + (\phi^{12} - w)(1 - w)}
\end{aligned}$$

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#### APPENDIX A. ASYMPTOTICS OF JACOBI POLYNOMIALS, IN COLLABORATION WITH NICO TEMME

Consider the integral

$$Q(j, \ell) = \frac{1}{2\pi i} \oint \frac{(w-1)^{\ell-1}(w+1)^j}{(w-\frac{3}{2})^j} dw,$$

where the contour winds  $\frac{3}{2}$  in the positive direction. We will derive its uniform asymptotics uniformly for

$$|j - \ell| < \ell^{2/3}. \quad (59)$$

We write this in the form

$$Q(j, \ell) = \frac{1}{2\pi i} \oint e^{\ell f(w)} \frac{dw}{w-1}, \quad (60)$$

where

$$f(w) = \ln(w-1) + \mu \ln(w+1) - \mu \ln\left(w - \frac{3}{2}\right), \quad \mu = \frac{j}{\ell}.$$

We have

$$f'(w) = \frac{2w^2 - w - 3 - 5\mu w + 5\mu}{(w-1)(w+1)(2w-3)}.$$

The saddle points are

$$w_{\pm}(\mu) = \frac{1}{4}(1 + 5\mu \pm W), \quad W = \sqrt{25 - 30\mu + 25\mu^2}. \quad (61)$$

The argument of the square root  $W$  is positive for all  $\mu \geq 0$ . So, the saddle points are real, and  $w_+$  is monotonously increasing for  $\mu \in (0, \infty)$ , starting at the value  $\mu_+(0) = \frac{3}{2}$ . It turns out after



some analysis that  $w_+$  is the dominant saddle point. It should be remarked that, because  $j$  and  $\ell$  are integers, the integrand in (60) is single-valued, but that the function  $f(w)$  should be described in terms of its branch cuts. Because contributions to the integral from parts of the contour of  $w$  with  $\Re(w) < \frac{3}{2}$  are exponentially small compared with those from the neighbourhood of the saddle point  $w_+$ , we forget about this aspect.

Now, note that

$$f''(w) + (f'(w))^2 = \frac{5(5\mu w - 3w - 5\mu + 7)\mu}{(2w - 3)^2(w + 1)^2(w - 1)}.$$

For the large- $\ell$  asymptotics we need

$$f''(w_+) = \frac{1}{40\mu} W \left( 17 - 30\mu + 25\mu^2 + (3 - 5\mu)W \right).$$

The first order asymptotic result follows from replacing  $f(w)$  in (60) by  $f(w_+) + \frac{1}{2}f''(w_+)(w - w_+)^2$ , which gives when we integrate along the vertical line  $\Re w = w_+$

$$Q(j, \ell) = \frac{e^{\ell f(w_+)}}{(w_+ - 1)\sqrt{2\pi f''(w_+)}} \left( 1 + B\ell^{-1} \right), \text{ as } \ell \rightarrow \infty.$$

Next,

$$\frac{d}{d\mu} f(w_+(\mu), \mu) = \frac{\partial}{\partial w} f(w_+(\mu), \mu) \cdot w'_+ + \frac{\partial}{\partial \mu} f(w_+(\mu), \mu).$$

If  $\ell = j$ , or  $\mu = 1$ , we have

$$\begin{aligned} w_+(1) &= \phi^2, \\ w'_+(\mu)|_{\mu=1} &= \frac{5 + \sqrt{5}}{4}, \\ \frac{\partial}{\partial w} f(w, \mu)|_{\mu=1} &= 0, \\ \frac{\partial}{\partial \mu} f(w, \mu)|_{\mu=1} &= \ln(2\phi), \\ f(w, \mu)|_{\mu=1} &= \ln(2\phi^2). \end{aligned}$$

Thus,

$$\begin{aligned} f(w_+(\mu), \mu)|_{\mu=1} &= \phi^2, \\ \frac{d}{d\mu} f(w_+(\mu), \mu)|_{\mu=1} &= \ln(2\phi). \end{aligned}$$

This the same manner we calculate

$$\frac{d^2}{d\mu^2} f(w_+(\mu), \mu)|_{\mu=1} = -\frac{\sqrt{5}}{4}.$$

$$f(w_+(\mu), \mu) = \ln(2\phi^2) + \ln(2\phi)(\mu - 1) - \frac{\sqrt{5}}{2}(\mu - 1)^2 + B(\mu - 1)^3.$$

Let

$$g(\mu) = f(w_+(\mu), \mu) - \ln(2\phi^2) - \ln(2\phi)(\mu - 1) =$$

$$e^{\ell f(w_+)} = (2\phi^2)^\ell \cdot (2\phi)^{j-\ell} \exp\left(-\frac{(j-\ell)^2}{4\ell}\right) \exp\left(\frac{B(j-\ell)^3}{\ell^2}\right).$$

Further,

$$\frac{d}{d\mu} \frac{1}{w_+(\mu) - 1} = -\frac{w'_+(\mu)}{(w_+(\mu) - 1)^2}.$$

$$\frac{1}{w_+ - 1} = \phi^{-1} + \frac{\sqrt{5}\phi^{-1}}{4}(\mu - 1) + B(\mu - 1)^2.$$

Finally,

$$\frac{1}{\sqrt{f''(w_+)}} = \frac{\sqrt[4]{5}\phi}{\sqrt{2}} + (\mu - 1) + B(\mu - 1)^2.$$

Because

$$f''(w_+) = \frac{1}{\mu} \left(4 - \frac{52}{5}\mu + \mathcal{O}(\mu^2)\right), \quad \mu \rightarrow 0,$$

and

$$f''(w_+) = \frac{1}{\mu^2} \left(\frac{4}{25} + \frac{12}{125}\mu^{-1} + \mathcal{O}(\mu^{-2})\right), \quad \mu \rightarrow \infty,$$

We give the first coefficients of the expansion

$$Q(\ell, j) \sim \frac{(w_+ - 1)^{\ell-1} (w_+ + 1)^j}{(w_+ - \frac{3}{2})^j} \frac{1}{\sqrt{2\pi\ell f''(w_+)}} \sum_{k=0}^{\infty} \frac{a_k(\mu)}{\ell^k}, \quad \ell \rightarrow \infty. \quad (62)$$

First we introduce a transformation

$$f(w) - f(w_+) = \frac{1}{2}s^2,$$

where  $w$  runs along the saddle point contour through  $w_+$  and  $s$  on the imaginary axis. We use this transformation in (60) and obtain

$$Q(\ell, j) = \frac{e^{\ell f''(w_+)}}{2\pi i} \int_{-i\infty}^{i\infty} e^{\frac{1}{2}\ell s^2} g(s) ds, \quad g(s) = \frac{1}{w - 1} \frac{dw}{ds}. \quad (63)$$

We need the coefficients of the expansion

$$w = w_0 + \sum_{k=1}^{\infty} c_k s^k, \quad c_1 = \left. \frac{dw}{ds} \right|_{s=0} = \frac{1}{\sqrt{f''(w_+)}}$$

where we take the positive square root; in this way,  $w$  crosses the saddle point  $w_+$  in the same direction as  $s$  crosses the origin. The next one is

$$c_2 = \frac{(w_+ - 1)(3w_+^2 - 3w_+ - 1 - c_1^2)}{3(2 - 2w_+ + w_+^2)}.$$

With these coefficients we expand  $g(s)$  defined in (63),  $g(s) = \sum_{k=0}^{\infty} g_k s^k$ . Then the coefficients  $a_k(\mu)$  of (62) are given by

$$a_k(\mu) = \frac{(-1)^k 2^k \left(\frac{1}{2}\right)_k g_{2k}}{g_0}, \quad g_0 = \frac{c_1}{w_+ - 1}, \quad k = 0, 1, 2, \dots$$

The first few are  $a_0(\mu) = 1$  and, with  $W$  defined in (61),

$$\begin{aligned} a_1(\mu) &= -\frac{(5\mu^2 + 2\mu + 5)(5\mu^2 - 11\mu + 5)W}{60\mu(5 - 6\mu + 5\mu^2)^2}, \\ a_2(\mu) &= \frac{1}{288\mu^2(5 - 6\mu + 5\mu^2)^3} (125\mu^8 - 450\mu^7 + 685\mu^6 - \\ &\quad 4410\mu^5 + 9972\mu^4 - 4410\mu^3 + 685\mu^2 - 450\mu + 125), \\ a_3(\mu) &= \frac{W}{51840(\mu^3(5 - 6\mu + 5\mu^2))^5} (86875 - 469125\mu + \\ &\quad 1384725\mu^2 - 2389500\mu^3 + 2957775\mu^4 - 12518415\mu^5 + \\ &\quad 23051362\mu^6 - 12518415\mu^7 + 2957775\mu^8 - 2389500\mu^9 + \\ &\quad 1384725\mu^{10} - 469125\mu^{11} + 86875\mu^{12}). \end{aligned}$$

These coefficients are bounded as  $\mu \rightarrow \infty$ , but become undefined as  $\mu \rightarrow 0$ , that is  $j/\ell \rightarrow 0$ . For the expansion in (62) we need the condition  $\mu \geq \mu_0 > 0$ , where  $\mu_0$  is fixed number.

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