On polynomials with a root close to an integer

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Let \( P(x) \) be a polynomial with integer coefficients. Let also two positive integers \( n \) and \( H \) denote its degree and height respectively. Suppose \( P(x) \) has no roots at integers. We are interested in finding how well a root of \( P(x) \) can be approximated by an integer.

A more general problem of estimating the distance \( |\alpha - \beta| \), where \( \alpha \) is a root of \( P(x) \) and \( \beta \) is an algebraic number, was extensively studied. The case when \( \beta \) is a conjugate of \( \alpha \) is known as a root separation problem which was studied by K. Mahler [13], R. Gütting [12], M. Mignotte and M. Payafar [15] and by the author [6]. For applications see, e.g., M. Mignotte, M. Petkovic and M. Trajkovic [16].

The case when \( \beta \) is a fixed algebraic number which is not a conjugate of \( \alpha \) was investigated by M. Mignotte [14] (see also [18]), F. Amoroso [2]. A very important particular case \( \beta = 1 \) was also first studied in [14]. Later the distance between an algebraic number and unity was investigated by M. Mignotte and M. Waldschmidt [17], Y. Bugeaud, M. Mignotte and F. Normandin [5]. F. Amoroso [1] showed that these bounds are not far from being sharp. The best lower and upper bounds for \( |\alpha - 1| \) are due to the author [7], [8]. The lower bound for \( |\alpha - 1| \) is given in terms of degree and the Mahler measure of \( \alpha \). P. Borwein and C. Pinner [4] investigated in detail the case \( H = 1 \). In particular, they describe explicitly the extremal polynomial with the closest real root to 1.

Let us define

\[
\delta(P) = \min |\alpha - r|,
\]

where the minimum is taken over \( \alpha \) such that \( P(\alpha) = 0 \) and \( r \in \mathbb{Z} \). V. G. Alexeev, A. Polupanov and I. Shparlinski [10] considered the minimal distance from differences of roots of a polynomial to the nearest integers \( \min |\alpha_i - \alpha_j - r| \) and discussed its applications.

We start with the following simple argument. Suppose that

\[
P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0 = a_n(x - \alpha_1)\ldots(x - \alpha_n),
\]

where \( a_n \neq 0 \) and \( a_j, j = 0 \).

Suppose also that \( \delta(P) = |\alpha_1| \).

Indeed, all the roots of the polynomial

\[\text{if } |r| > H + 2, \text{ then } \delta(P) > 1.\]

then

\[
1 \leq |P(r)| \leq H\delta(P)
\]

a contradiction.

It turns out that it is possible to optimize \( \delta(P) \) in the class of polynomials of bounded height. Let us

\[
Q(x) = x^n - \tau
\]

Suppose that \( \tau \) is the smallest positive number such that

\[
Q(x) = x^n - \tau
\]

With this notation the following theorem is true.

**Theorem.** Let \( H \) be a positive integer such that \( n > n(H) \) then for \( \delta(P) \) in the class of polynomials of bounded height at most \( H \) and \( \delta(P) = \tau \).

We believe that the theorem is true. However, our proof involves a result of Erdős and P. Turán [9] on the angular structure of the complex plane with \( n \) points bounded by \( 16\sqrt{n \log((n + 1)H)} \). For this reason we need \( n \) to be su

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where \( a_n \neq 0 \) and \( a_j, j = 0, 1, \ldots, n \), are integers of the absolute value at most \( H \). Suppose also that \( \delta(P) = |a_1 - r| > 0 \). We will prove that then

\[
\delta(P) > (2H + 2)^{-n}.
\]

Indeed, all the roots of the polynomial \( P \) lie in the open disc \( |z| < H + 1 \). So that if \( |r| \geq H + 2 \), then \( \delta(P) > 1 \). If however \( |r| \leq H + 1 \) and \( 0 < \delta(P) \leq (2H + 2)^{-n} \), then

\[
1 \leq |P(r)| = |a_n||a_1 - r||a_2 - r| \cdots |a_n - r| \\
\leq H \delta(P)(2H + 2)^{-n-1} \leq H/(2H + 2) < 1,
\]

a contradiction.

It turns out that it is possible not only to strengthen the simple inequality \( \delta(P) > (2H + 2)^{-n} \), but to describe explicitly the extremal polynomials which minimize \( \delta(P) \) in the class of polynomials with integer coefficients of a given degree and of bounded height. Let us put

\[
Q(x) = x^n - H(x^{n-1} + x^{n-2} + \cdots + x + 1).
\]

Suppose that \( \tau \) is the smallest positive root of the equation

\[
x(H + 1 - x)^n = H.
\]

With this notation the following theorem holds.

**Theorem.** Let \( H \) be a positive integer. There exists an effective constant \( n(H) \) such that if \( n \geq n(H) \) then for any polynomial \( P(x) \in \mathbb{Z}[x] \) of degree \( n \) and of height at most \( H \) either \( \delta(P) = 0 \) or \( \delta(P) \geq \tau \). Here the equality holds if and only if \( P(x) = \pm Q(\pm x) \).

We believe that the theorem holds for all \( n \geq 2 \) (i.e. \( n(H) \) can be taken as 2). However, our proof involves a result on uniform distribution of roots in angles. In 1950, P. Erdős and P. Turán [9] proved that the number of roots of \( P(x) \in \mathbb{C}[x] \) in an angle of the complex plane with vertex at the origin minus "expected" number of roots is bounded above by \( 16\sqrt{n} \log(L(P))/\sqrt{|a_0a_n|} \), where \( L(P) \) is the length of \( P \) (see, e.g., [3], [11] for further work on this problem). For \( P(x) \in \mathbb{Z}[x] \) this is less than \( 16\sqrt{n} \log((n + 1)H) \). We need this quantity divided by \( n \) to be small. For this reason we need \( n \) to be sufficiently large.

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