A lower bound for the quantity $\|(3/2)^k\|$  

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It is well known that for $k \neq 4$ the inequality $\|(3/2)^k\| > (3/4)^k$ implies that $g(k) = 2^k + [(3/2)^k] - 2$ (where $g(k)$ is the well-known constant in Waring’s problem). Here $\| a \| = \min | a - M |$. In [1] Mahler showed that for any $a, q \in \mathbb{N}$, $a > q \geq 2$, $(a, q) = 1$, and for any $\varepsilon > 0$ there exists $k_0$ such that for $k \in \mathbb{N}$, $k > k_0$ the inequality $\|(a/q)^k\| > (1-\varepsilon)^k$ holds. However the constant $k_0$ in Mahler’s paper is not effective. An effective sharpening of the trivial inequality $\|(a/q)^k\| > (1/q)^k$ was obtained by Baker and Coates in [2]. They showed that there exist effective numbers $k_0, \eta, \theta < \eta < 1$, such that $\|(a/q)^k\| > (1/q)^k$ when $k \in \mathbb{N}, k > k_0$. For the case $a = 3, q = 2$ one obtains $\eta \approx 10^{-64}$.

In [3] Beukers strengthened this result by showing that $\|(3/2)^k\| > (0.5396)^k$ when $k \in \mathbb{N}, k > k_0$.

The theorem stated below strengthens Beukers’ result.

**Theorem.** There exists an effective $k_0$ such that for $k \in \mathbb{N}$, $k > k_0$, the following inequality holds: $\|(3/2)^k\| > (0.5769)^k$.

Let $a, b \in \mathbb{N}$,

$$z^b H(a, b, z) = (1 - z)^{a+b} - \sum_{r=0}^{b-1} C_{a+b}^r (-z)^r.$$

It is shown in [3] that there exists a polynomial $P_n(z) \in \mathbb{Z}[z]$, $\deg P_n \leq n$, such that

$$P_n(z) = H(a, b, z) Q_n(z) + (-1)^{n+b} z^{2n+1} E_n(z),$$

where

$$Q_n(z) = \sum_{r=0}^{n} C_{2n+b-r}^{n} C_{a-n+r-1}^r = \frac{(a+b+n)!}{(a-n-1)! (b+n)!} \int_0^1 (1-t)^{n+b} t^{a-n-r} (1-t+zt)^n dt,$$

$$E_n(z) = (a+b+n)! \int_0^1 (1-t)^{n+b} t^n (1-tz)^{a-n-1} dt.$$

Suppose now that $\theta = 1.0723, a = 2\theta, b = 6m, n = m$ or $m+1, m = 100000, t_0 \in \mathbb{N}$. We note that

$$\frac{3}{2} e^{\theta m} = 2^{30m} \sum_{r=0}^{29m} C_{29m-r}^r \left( \frac{1}{3} \right)^r = 2^{30m} \sum_{r=0}^{29m} C_{29m-r}^r \left( \frac{1}{3} \right)^r.$$

Consequently, by representing $k$ in the form $k = 60m+\delta$, where $\delta \in \mathbb{Z}, 0 < \delta < 6 \times 10^723$, we have

$$\|(3/2)^k\| = \|(3/2)^{\delta m+\delta}\| = \min_{M \in \mathbb{Z}} \max \left( \left(3/2\right)^{\delta} H \left(2\theta m, 6m, -\frac{1}{8}\right) - M \right).$$

Thus

$$\left| \left( \left(3/2\right)^{\delta} P_n \left( -\frac{1}{8} \right) - \delta Q_n \left( -\frac{1}{8} \right) \right) - \delta E_n \left( -\frac{1}{8} \right) \right| \leq \left| \left(3/2\right)^{\delta} H \left(2\theta m, 6m, -\frac{1}{8}\right) \right| - M.$$

**Lemma.** Let $\varepsilon > 0$, $e, f, g, h \in \mathbb{Z}, \theta \in \mathbb{Q}, 0 < \theta < 2$. Then there exists an effective $m_0$ such that for $m > m_0$, $m, \theta \in \mathbb{N}$, there is a natural number $\Delta_m$ that divides the numbers $C_{(2+\theta)m+\delta}^{\delta} C_{(2-\theta)m+\delta}^{\delta}$ for all $r \in \mathbb{Z}$, for which both binomial coefficients make sense and satisfy the inequality

$$\Delta_m > \exp \left( m \cdot \sum_{t \in \mathbb{N}} \left( \frac{1+\theta}{1+3\theta} \right) - \max \left( \frac{1+\theta}{1+6}, \frac{1+\theta}{1+3\theta} \right) + 1, \frac{1}{1+3\theta} + 1, \frac{2\theta - 1}{1+3\theta} + 1 \right) \cdot e^m.$$
where the summation is carried out over all \( t \equiv \mathbb{N} \) for which

\[
\left\lfloor \frac{1 + \theta}{1 + 36} t \right\rfloor + \left\lfloor \frac{t}{1 + 36} \right\rfloor + \left\lfloor \frac{2\theta - 1}{1 + 36} t \right\rfloor = t - 2.
\]

**Proof.** The proof is similar to that of Lemma 2 in [4]. In our case, a corollary of this lemma is that all the coefficients of the polynomial \( Q_n(z) \), and therefore of \( P_n(z) \) as well, are divisible by a number \( \Delta_{m} > (1.541090834955)^m \), provided that \( m > m_0 \). In [3] it is shown that

\[
P_m\left(-\frac{1}{8}\right)Q_{m-1}\left(-\frac{1}{8}\right) - P_{m-1}\left(-\frac{1}{8}\right)Q_m\left(-\frac{1}{8}\right) \neq 0.
\]

Thus, for example, \( \frac{3}{2} \delta P_m\left(-\frac{1}{8}\right) - M Q_m\left(-\frac{1}{8}\right) \neq 0 \) (the case \( \frac{3}{2} \delta P_{m-1}\left(-\frac{1}{8}\right) - M Q_{m-1}\left(-\frac{1}{8}\right) \neq 0 \) is analyzed similarly) and

\[
\frac{\Delta_m}{2^{8m}} < \| (3/2)^k \| Q_m\left(-\frac{1}{8}\right) + \left(\frac{3}{2}\right)^k 8^{-2m-1} E_m\left(-\frac{1}{8}\right) \|.
\]

In our case for \( m > m_1 \), we have

\[
| Q_m\left(-\frac{1}{8}\right) | < (6.633298492)^m,
\]

\[
| E_m\left(-\frac{1}{8}\right) | < (8 \times 1.5409725967)^m.
\]

It is clear that for \( m > m_2 \)

\[
\frac{\Delta_m}{2^{8m}} - \left(\frac{3}{2}\right)^k 8^{-2m-1} \left| E_m\left(-\frac{1}{8}\right) \right| > \left(\frac{1.54109083495}{8}\right)^m.
\]

Consequently, since \( m = (k - \delta)/6 \delta \) for \( k \in \mathbb{N}, k > k_0 \), we obtain the inequality

\[
\| (3/2)^k \| > \left(\frac{1.54109083495}{8 \times 6.633298492}\right)^m > (0.5769)^k.
\]

**References**


