Let \( \alpha \) be an algebraic number of degree \( n \) and let \( q \) be the leading coefficient of its minimal polynomial. We find lower bounds for the product of \( s \) conjugates outside the unit circle in terms of \( n \), \( q \) and \( s \). For a non-reciprocal algebraic number of norm \( \pm 1 \) this lower bound is given in terms of \( q \).

1. Introduction.

Let \( \alpha \) be an algebraic number of degree \( n \geq 2 \) with conjugates \( \alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n \). If
\[
P(x) = q(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) = qx^n + q_{n-1}x^{n-1} + \ldots + q_0
\]
is the minimal polynomial of \( \alpha \), then we say that \( q \) is the denominator of \( \alpha \).

Denote the absolute norm of \( \alpha \) by
\[
N(\alpha) = \alpha_1\alpha_2\ldots\alpha_n = \frac{(-1)^n q_0}{q}.
\]
Let
\[
\Lambda(\alpha) = \prod_{j=1}^{n} \max(1, |\alpha_j|)
\]
be the product of conjugates of \( \alpha \) outside the unit circle.

Lehmer’s conjecture on the lower bound for \( \Lambda(\alpha) \) of an algebraic integer \( \alpha \) (i.e. \( q = 1 \)) was stated in 1933 [4]: there exists an absolute constant \( \delta > 0 \) such that \( \Lambda(\alpha) \geq 1 + \delta \) whenever \( \alpha \) is not a root of unity. The best unconditional result so far follows from work of Dobrowolski, Cantor and Straus, Louboutin [6]. For a non-reciprocal algebraic integer Smyth [12] obtained the best possible result \( \Lambda(\alpha) \geq \theta_1 \), where \( \theta_1 \) is the real root of \( x^3 - x - 1 = 0 \).

In this paper we will consider the case \( q \geq 2 \). If \( |N(\alpha)| > 1 \) , then
\[
\Lambda(\alpha) \geq |N(\alpha)| \geq \frac{q + 1}{q} = 1 + \frac{1}{q}.
\]
This lower bound is the best possible because for any root \( \alpha \) of \( qx^n - q - 1 = 0 \) we have \( \Lambda(\alpha) = 1 + 1/q \).

For an arbitrary non-reciprocal algebraic number with denominator \( q \) the natural generalization of Smyth’s result \( \Lambda(\alpha) \geq \theta_q \), where \( \theta_q \) is the real root of \( qx^3 + (q - 1)x^2 - qx - q = 0 \), was proved to be false (see e.g. [11] for a counterexample). However if \( |N(\alpha)| = 1 \), then the inequality \( \Lambda(\alpha) \geq \theta_q \) is believed to be true. The following estimate is a slight refinement of the results obtained by Pathiaux [10], Notari [9] and Lloyd-Smith [5].
Theorem 1. Let $\alpha$ be a non-reciprocal algebraic number of degree $n \geq 2$ and of denominator $q \geq 2$ satisfying $|N(\alpha)| = 1$. Then

$$\Lambda(\alpha) > \frac{\sqrt{2}}{8q} \left( ((256q^4 - 96q^2 + 25)^{1/2} + 5 + 16q^2)^{1/2} + ((256q^4 - 96q^2 + 25)^{1/2} + 5 - 16q^2)^{1/2} \right).$$

(1)

For example, for $q = 2$ the inequality (1) gives $\Lambda(\alpha) > 1.137\ldots$, but $\theta_2 = 1.142\ldots$.

If $\alpha$ is a reciprocal algebraic number of degree $n = 2$, then either $\Lambda(\alpha) = 1$ or $\Lambda(\alpha) \geq 1 + (1 + \sqrt{1+4q})/q$. For $n \geq 3$ we prove the following result.

Theorem 2. Let $\alpha$ be an algebraic number of degree $n \geq 3$ and of denominator $q \geq 2$. Let $s \geq 1$ be the number of conjugates of $\alpha$ lying strictly outside the unit circle. Then

$$\Lambda(\alpha) > 1 + \frac{2s}{q^{n/s}n^{n/2s+1}}, \text{ if } |N(\alpha)| = 1; \quad (2)$$

$$\Lambda(\alpha) > 1 + \frac{s}{4n(3q^2 \log q)^{n/s}}, \text{ if } |N(\alpha)| < 1. \quad (3)$$

The inequalities (2), (3) are stronger than similar results obtained by Lloyd-Smith [5]. Notice that if $s \geq cn$ for some absolute constant $c > 0$, then the lower bound in the inequality (3) depends only on $q$. Therefore in this case an analogue of Lehmer’s conjecture for an algebraic number with fixed denominator is valid. However for small $s$ the lower bounds (2), (3) apparently are far from being optimal. For example, if $s$ is odd number, then at least one the conjugates outside the unit circle is real. Let $\alpha_1$ be this real conjugate. From recent work on the lower bound for $|\alpha - 1|$ (see [2], [3], [7]) it follows that

$$\Lambda(\alpha) \geq |\alpha_1| > 1 + \exp \left( - c_1 \sqrt{n \log n \log q} \right).$$

The latter inequality is much stronger than the inequalities (2), (3) for small odd $s$.

2. Preliminaries.

We need the following lemmas.

Lemma 1. Let $n \geq 2$ and $z_1, z_2, \ldots, z_n \in \mathbb{C}$. Then

$$\prod_{j \neq k} |z_j \tilde{z}_k - 1| \leq n^n \Lambda^{2(n-1)}, \quad (4)$$

where

$$\Lambda = \prod_{j=1}^n \max \left( 1, |z_j| \right).$$
Proof. Put \( \tau_j = \max (1, |z_j|) \). Then

\[
\prod_{j \neq k} |z_j \bar{z}_k - 1| \leq \max_{\omega} \prod_{j \neq k} |\omega_j \bar{\omega}_k - 1|,
\]

where maximum is taken over \( |\omega_i| \leq \tau_i, 1 \leq i \leq n \). Further, following Alexander [1] we bound

\[
\max_{\omega} \prod_{j \neq k} |\omega_j \bar{\omega}_k - 1| \leq \max \max_{\{\omega\}} \prod_{j \neq k} |\omega_j \bar{\omega}_k - y_j \bar{y}_k|,
\]

where the inner maximum is taken over \( |y_i| \leq \tau_i \). Applying maximum modulus principle we get \( |\omega_i| = |y_i| = \tau_i \) and

\[
\max \max_{\{\omega\}} \prod_{j \neq k} |\omega_j \bar{\omega}_k - y_j \bar{y}_k| = \max_{\{\omega\}} \prod_{j \neq k} |y_j \bar{\omega}_k - y_j \bar{\omega}_k| = \tau_1 \tau_2 \ldots \tau_n (2n-1) \max_{|u_i| \leq 1} \prod_{j \neq k} |u_j - u_k| = (\tau_1 \tau_2 \ldots \tau_n)^{2(n-1)} n^n.
\]

This completes the proof of the lemma.

**Lemma 2.** If \( x_1, x_2, \ldots, x_n \geq 1 \), then

\[
\prod_{j=1}^{n} (x_j - 1) \leq \left( \left( x_1 x_2 \ldots x_n \right)^{1/n} - 1 \right)^n. \tag{5}
\]

If \( 0 < x_1, \ldots, x_n \leq 1 \), then

\[
\prod_{j=1}^{n} (1 - x_j) \leq \left( 1 - \left( x_1 \ldots x_n \right)^{1/n} \right)^n. \tag{6}
\]

**Proof.** Put \( x_j - 1 = y_j^n \). Then the inequality (5) can be written in the following form

\[
(1 + y_1 y_2 \ldots y_n)^n \leq (1 + y_1^n) \ldots (1 + y_n^n).
\]

The last inequality easily follows from the geometric-arithmetic mean inequality. The inequality (6) follows from (5).

3. **Proof of Theorem 1.**

Following Smyth [12] and Notari [9] we consider the ratio

\[
\frac{P(z) \text{sgn } q_0}{z^n P(1/z)} = \frac{\text{sgn } q_0 \prod_{|\alpha_j| < 1} (z - \alpha_j)/(1 - \bar{\alpha}_j z)}{\prod_{|\alpha_j| > 1} (1 - \bar{\alpha}_j z)/(z - \alpha_j)} = \frac{f(z)}{g(z)}
\]

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and obtain

$$1 + b_k z^k + b_l z^l + \ldots = \frac{f(z)}{g(z)} = \frac{c_0 + c_1 z + c_2 z^2 + \ldots}{d_0 + d_1 z + d_2 z^2 + \ldots},$$

where $0 < k < l$, $b_k = u/q$, $b_l = v/q^2$. Here $u$ and $v$ are non-zero integers, $c_0 = d_0 = 1/\Lambda(\alpha) = c$. We also have $c_i = d_i$ for $i < k$ and $c_k - d_k = b_k c$, $c_l - d_l = b_k d_{l-k} + b_l c$. If $l \geq 2k$ or $|u| \geq 2$, then $\Lambda(\alpha) \geq \theta_q$ (see Notari [5]). Let $l < 2k$ and $u = 1$ (the case $u = -1$ can be considered analogously).

First, we notice that $|f(z)| = |g(z)| = 1$ for $|z| = 1$. Applying Parseval’s formula

$$\frac{1}{2\pi} \int_0^{2\pi} |h(e^{i\phi})|^2 d\phi = t_0^2 + t_1^2 + t_2^2 + \ldots,$$

where $h(z) = t_0 + t_1 z + t_2 z^2 + \ldots$ with real $t_0, t_1, t_2, \ldots$ is a holomorphic function in a neighbourhood containing the unit disc, to

$$|1 + j_2 z^{l-k} + j_3 z^k + j_1 z^l|^2 |f(z)|^2$$

and to

$$|-1 - j_2 z^{l-k} + j_3 z^k + j_1 z^l|^2 |g(z)|^2$$

we get

$$c^2 + (j_2 c + c_{l-k})^2 + (c_k + j_2 c_{2k-l} + j_3 c)^2 + (c_l + j_2 c_k + j_3 c_{l-k} + j_1 c)^2$$

$$\leq 1 + j_1^2 + j_2^2 + j_3^2,$$

$$c^2 + (j_2 c + d_{l-k})^2 + (-d_k - j_2 d_{2k-l} + j_3 c)^2 + (-d_l - j_2 d_k + j_3 d_{l-k} + j_1 c)^2$$

$$\leq 1 + j_1^2 + j_2^2 + j_3^2.$$

Since $l < 2k$, we have $c_{l-k} = d_{l-k}$, $c_{2k-l} = d_{2k-l}$. Now, using the inequality $(a^2 + b^2)/2 \geq ((a + b)/2)^2$, we get

$$c^2 + (j_2 c + c_{l-k})^2 + \left(\frac{c}{2q} + j_3 c\right)^2 + \left(\frac{b_2 c}{2q} + \frac{c_{l-k}}{2q} + \frac{j_2 c}{2q} + j_3 c_{l-k} + j_1 c\right)^2$$

$$\leq 1 + j_1^2 + j_2^2 + j_3^2.$$

The matrix of the corresponding quadratic form

$$F = c^2 j_1^2 + (j_2 c + j_4 c_{l-k})^2 + \left(\frac{j_4 c}{2} + j_3 c\right)^2$$

$$+ \left(j_4 \left(\frac{b_2 c}{2} + \frac{c_{l-k}}{2q}\right) + \frac{j_2 c}{2q} + j_3 c_{l-k} + j_1 c\right)^2 - j_4^2 - j_1^2 - j_2^2 - j_3^2$$

4
is
\[
\begin{bmatrix}
c^2 - 1 & ac & cc_{l-k} & cA \\
ac^2 & c^2(1 + a^2) - 1 & acc_{l-k} & cc_{l-k} + acA \\
cc_{l-k} & acc_{l-k} & c^2 + c^2_{l-k} - 1 & ac^2 + c_{l-k}A \\
cA & cc_{l-k} + acA & ac^2 + c_{l-k}A & c^2(1 + a^2) + c^2_{l-k} + A^2 - 1
\end{bmatrix},
\]
where \( a = 1/2q, \ A = b_l c/2 + ac_{l-k} \).

Since the quadratic form \( F \) is non-positive, we have that the diagonal minors satisfy
\[
M_1 = c^2 - 1 \leq 0, \\
M_2 = (1 - c^2)^2 - a^2 c^2 \geq 0, \\
M_3 = -(1-c^2)^3 + (1-c^2)(c^2_{l-k} + a^2 c^2) \leq 0.
\]
This implies
\[
|c_{l-k}| \leq \sqrt{(1-c^2)^2 - a^2 c^2}. \tag{7}
\]

In order to find the determinant of the above matrix we denote the \( i \)-th row of the determinant by \( f_i \) and the \( i \)-th column by \( g_i \). Then we replace step by step
\[
f_2 \rightarrow f_2 - af_1, \\
f_3 \rightarrow f_3 - \frac{c_{l-k}}{c} f_1, \\
f_4 \rightarrow f_4 - \frac{A}{c} f_1, \\
f_4 \rightarrow f_4 - af_3, \\
g_4 \rightarrow g_4 - ag_3, \\
g_1 \rightarrow cg_1, \\
f_1 \rightarrow \frac{f_1}{c}.
\]
This does not change the determinant which now is equal to
\[
\begin{vmatrix}
c^2 - 1 & ac & cc_{l-k} & b_l c/2 \\
ac & c^2 - 1 & 0 & cc_{l-k} \\
c_{l-k} & 0 & c^2 - 1 & a \\
b_l c/2 & cc_{l-k} & a & c^2 + c^2_{l-k} - 1 - a^2
\end{vmatrix}.
\]
After doing some algebra we find that this determinant equals to
\[
M_4 = \left((1-c^2)^2 - a^2 c^2 - c^2_{l-k}\right)^2 - \left(ac_{l-k}(1+c^2) + \frac{b_l c(1-c^2)}{2}\right)^2.
\]
Since \( M_4 \geq 0 \) and \(|b_l| \geq 1/q^2 = 4a^2\) using the inequality (7) we have
\[
(1-c^2)^2 - a^2 c^2 - c^2_{l-k} \geq \frac{|b_l|c(1-c^2)}{2} - a|c_{l-k}|(1+c^2),
\]
\[ |c_{l-k}|^2 - a(1 + c^2)|c_{l-k}| + 2a^2 c(1 - c^2) - (1 - c^2)^2 + a^2 c^2 \leq 0. \] 
\[ (8) \]
If
\[ \sqrt{(1 - c^2)^2 - a^2 c^2} \geq \frac{a(1 + c^2)}{2}, \]
then
\[ (1 - c^2)^2 \geq a^2 c^2 + \frac{a^2(1 + c^2)^2}{4} = \frac{a^2}{4}(1 + 6c^2 + c^4). \]
The last inequality implies either \( 1 - c^2 \geq a \) or \( 1 + 6c^2 + c^4 \leq 4 \). In both cases we can easily get good lower bounds for \( \Lambda = 1/c \) which are better than \( \theta_q \). Hence, without loss of generality we may assume that the inequality opposite to (9) holds. It follows that the minimum in the left hand side of (8) in \( |c_{l-k}| \) is achieved when
\[ |c_{l-k}| = \sqrt{(1 - c^2)^2 - a^2 c^2}. \]
From (8) we now get
\[ \sqrt{(1 - c^2)^2 - a^2 c^2} \geq \frac{2ac(1 - c^2)}{1 + c^2}, \]
\[ (1 - c^4)^2 - (1 + c^2) a^2 c^2 \geq 4a^2 c^2(1 - c^2)^2, \]
\[ c^8 - 5a^2 c^6 + (6a^2 - 2)c^4 - 5a^2 c^2 + 1 \geq 0, \]
\[ \Lambda^8 - \frac{5}{4q^2} \Lambda^6 + \left( \frac{3}{2q^2} - 2 \right) \Lambda^4 - \frac{5}{4q^2} \Lambda^2 + 1 \geq 0. \]
Without loss of generality we may assume that the latter inequality is strict and solving it first in \( \Lambda^2 + \Lambda^{-2} \) we find
\[ \Lambda^2 + \Lambda^{-2} > \frac{5 + \sqrt{256q^4 - 96q^2 + 25}}{8q^2}. \]
Using this, we are finally able to estimate \( \Lambda \) as in the right hand side of the inequality (1). This completes the proof of Theorem 1.

Notice that if in the expression
\[ \frac{P(z) \text{ sgn} q_0}{z^n P(1/z)} = 1 + b_k z^k + b_l z^l + \ldots \]
we have \( l < 2k \), then the inequality (1) holds also for \( q = 1 \). This gives the bound \( \Lambda(\alpha) > 1.32497 \ldots \) which is a refinement of the respective inequality due to Smyth [12].

4. Proof of Theorem 2.

First we prove the inequality (2). Since (1) or \( M_2 \geq 0 \) gives \( \Lambda(\alpha) > 1 + 1/4q \) for a non-reciprocal \( \alpha \), without loss of generality we assume that \( \alpha \) is reciprocal and thus \( n \geq 4 \).
Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be outside the unit circle and $\alpha_n = 1/\bar{\alpha}_1$, $\alpha_{n-1} = 1/\bar{\alpha}_2$, $\ldots$, $\alpha_{n-s+1} = 1/\bar{\alpha}_s$. Consider the discriminant

$$D = q^{2n-2} \prod_{i<j} (\alpha_i - \alpha_j)^2 = q^{2n-2} \begin{vmatrix} 1 & \ldots & 1 \\ \alpha_1 & \ldots & \alpha_n \\ \vdots & \vdots & \vdots \\ \alpha_1^{n-1} & \ldots & \alpha_n^{n-1} \end{vmatrix}^2.$$

Subtracting the $n+1-i$-th column of the determinant from the $i$-th ($i = 1, 2, \ldots, s$) we get

$$D = q^{2n-2}(\alpha_1 - 1/\bar{\alpha}_1)^2 \ldots (\alpha_s - 1/\bar{\alpha}_s)^2 \begin{vmatrix} 0 & \ldots & 1 \\ 1 & \ldots & \alpha_n \\ \vdots & \vdots & \vdots \\ \alpha_1^{n-2} + \ldots + (1/\bar{\alpha}_1)^{n-2} & \ldots & \alpha_n^{n-1} \end{vmatrix}^2.$$ 

Now from Hadamard’s inequality

$$\det \|a_{i,j}\|_{i,j=1,2,\ldots,n} \leq \prod_{j=1}^n \sum_{i=1}^n |a_{ij}|^2,$$

the inequality (5) and the inequality

$$1^2 + 2^2 + \ldots + (n-1)^2 < n^3/3$$

we obtain

$$|D| \leq q^{2n-2} \frac{(\Lambda^2/s - 1)^{2s}}{\Lambda^2} \left(\frac{n^3}{3}\right)^s \Lambda^{2n-4} n^{n-s} = \frac{\Lambda^{2n-6}}{q^2} q^{2n}(\Lambda^2/s - 1)^{2s} n^{n+2s} 3^{-s}.$$

Using the evaluation of discriminant of an algebraic number field

$$|D| > \left(\frac{\pi}{4}\right)^n \left(\frac{n^n}{n!}\right)^2$$

(see e.g. Theorem 2.11 in [8]) we get

$$q^2 \Lambda^{6-2n} \left(\frac{\pi}{4}\right)^n \left(\frac{n^n}{n!}\right)^2 < q^{2n}(\Lambda^2/s - 1)^{2s} n^{n+2s} 3^{-s}.$$

If the inequality

$$q^2 \Lambda^{6-2n} \left(\frac{\pi}{4}\right)^n \left(\frac{n^n}{n!}\right)^2 \geq (2.5)^n$$

holds, then

$$(2.5)^{n/2} \leq q^{n/s}(\Lambda^2/s - 1)^{n^{n/2s+1}} 3^{-1/2},$$

$$\Lambda^2/s \geq 1 + \sqrt{3}(2.5)^{n/2s} q^{-n/s} n^{-n/2s-1}.$$
Since $2 \leq s \leq n/2$ (the case $s = 1$ can be treated directly or by our remark in the introduction) we obtain
\[ \Lambda \geq 1 + \frac{\sqrt{3} \cdot 2.5}{2} s q^{-n/s} n^{-n/2s-1} > 1 + \frac{2s}{q^{n/s} n^{n/2s+1}}. \]

If the opposite inequality
\[ q^2 \Lambda^{6-2n} \left( \frac{\pi}{4} \right)^n \left( \frac{n^n}{n!} \right)^2 < (2.5)^n \]
holds, then
\[ \Lambda^{n-3} > q \left( \frac{\pi}{10} \right)^{n/2} \frac{n^n}{n!}. \]
For $n \geq 4$ we have
\[ \left( \frac{\pi}{10} \right)^{n/2} \frac{n^n}{n!} > 1, \]

hence
\[ \Lambda > q^{1/(n-3)} > q^{1/n} > 1 + \frac{\log q}{n} > 1 + \frac{1}{nq^s} \geq 1 + \frac{2s}{q^{n/s} n^{n/2s+1}} \]
which gives the desired inequality (2).

Now we prove the inequality (3). Consider the resultant of the polynomials
\[ P(x) = qx^n + q_{n-1} x^{n-1} + \ldots + q_0 \]
and $Q(x) = x^n P(1/x)$. Since $|N(\alpha)| < 1$, $\alpha$ is not reciprocal and $|\text{Res}(P, Q)| \geq 1$.

We have
\[ 1 \leq |\text{Res}(P, Q)| = q^n |q_0|^n \prod_{j,k=1}^n |\alpha_j - \frac{1}{\alpha_k}| = q^n |q_0|^n \prod_{j,k=1}^n |\alpha_j - \frac{1}{\alpha_k}| \]
\[ = q^n |q_0|^n \prod_{j,k=1}^n \frac{|\alpha_j \bar{\alpha}_k - 1|}{|\alpha_k|} = q^{2n} \prod_{j,k=1}^n |\alpha_j \bar{\alpha}_k - 1| \]
\[ = q^{2n} \prod_{|\alpha_j| > 1} (|\alpha_j|^2 - 1) \prod_{|\alpha_j| < 1} (1 - |\alpha_j|^2) \prod_{j \neq k} |\alpha_j \bar{\alpha}_k - 1|. \]

From the inequalities (4)-(6) using $|N(\alpha)| \geq 1/q$ we obtain
\[ 1 \leq q^{2n}(\Lambda^{2/s} - 1)^s \left( 1 - \left( \frac{|N(\alpha)|}{\Lambda} \right)^{2/(n-s)} \right)^{n-s} n^n \Lambda^{2n-2} \]
\[ \leq q^{2n}(\Lambda^{2/s} - 1)^s (\Lambda^{2/(n-s)} - q^{-2/(n-s)})^{n-s} n^n \Lambda^{2n-4}. \]
Assume now that $s \leq n - 2$ (the case $s = n - 1$ can be considered analogously). Without loss of generality we may also suppose that $\Lambda \leq 1 + 1/12$. Indeed, if $\Lambda > 1 + 1/12$, then the inequality (3) holds since

$$\frac{1}{12} \geq \frac{s}{4n(3q^2 \log q)^{n/s}}.$$ 

Utilizing Bernoulli’s inequality we have

$$\Lambda^{2/(n-s)} - q^{-2/(n-s)} \leq (1 + 1/12)^{2/(n-s)} - e^{-2 \log q/(n-s)} < \frac{1/6 + 2 \log q}{n - s}.$$ 

Hence

$$1 \leq q^{2n/s} (\Lambda^{2/s} - 1) \left( \frac{1/6 + 2 \log q}{n - s} \right)^{n/s - 1} n^{n/s} \left( 1 + \frac{1}{12} \right)^{2n/s}.$$ 

Here we can treat the case $s = 1$ directly or by our remark in the introduction. If $s \geq 2$, then by Bernoulli’s inequality

$$\Lambda^{2/s} - 1 \leq (\Lambda - 1) \frac{2}{s}.$$ 

Therefore

$$1 \leq q^{2n/s} \frac{2(\Lambda - 1)}{s} \left( \frac{1/6 + 2 \log q}{n - s} \right)^{n/s - 1} n^{n/s} \left( 1 + \frac{1}{12} \right)^{2n/s}$$

and

$$\frac{s \log q}{n} \left( 1 - \frac{s}{n} \right)^{n/s - 1} q^{-2n/s} (2 \log q)^{-n/s} < (\Lambda - 1) \left( 1 + \frac{1}{12} \right)^{2n/s} \left( 1 + \frac{1}{12 \log q} \right)^{n/s}.$$ 

Since

$$\left( 1 + \frac{1}{12} \right)^2 \left( 1 + \frac{1}{12 \log q} \right) \leq \left( 1 + \frac{1}{12} \right)^2 \left( 1 + \frac{1}{12 \log 2} \right) < 1.5$$

and

$$\left( 1 - \frac{s}{n} \right)^{n/s - 1} \geq \frac{1}{e},$$

we obtain

$$\frac{s \log q}{en} q^{-2n/s} (2 \log q)^{-n/s} < (\Lambda - 1)(1.5)^{n/s},$$

$$\Lambda - 1 > \frac{s \log 2}{en} (3q^2 \log q)^{-n/s} > \frac{s}{4n(3q^2 \log q)^{n/s}}.$$ 

This completes the proof of the inequality (3).
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