ON A POLYNOMIAL WITH LARGE NUMBER
OF IRREDUCIBLE FACTORS

A. Dubickas *

Abstract. We give an example of a polynomial which has large number of cyclotomic factors. We thus obtain an inequality between the number of all irreducible factors of a polynomial counted with multiplicities and its degree and norm which is not far from being the best possible. It is also shown that this polynomial is vanishing at 1 with high multiplicity.

1. INTRODUCTION

Let $P$ be a polynomial of degree $n$ with integer coefficients such that $P(0) \neq 0$:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0, \quad a_n, a_0 \neq 0.$$ 

In the sequel we use the following notations:

$$H(P) = \max_{0 \leq i \leq n} |a_i|,$$

$$\nu(P) = \max_{|z| \leq 1} |P(z)|,$$

$$s(P) = \frac{\log \nu(P)}{n}.$$ 

Put also $\Omega_c(P)$ for the number of cyclotomic factors of $P$ counted with multiplicities, and put $\Omega(P)$ for the number of all irreducible factors of $P$ counted with multiplicities. Finally, suppose that the polynomial $P$ is vanishing at 1 with multiplicity $r = r(P)$.

In the course of improving the lower bound for the difference between an algebraic number and 1 due to M. Mignotte and M. Waldschmidt [9] (see also [3]), the author [5] introduced the following polynomial:

$$F(x) = F_k(x) = \prod_{1 \leq v < u \leq k} (x^u - v - 1)^{J_u J_v},$$

(1)

where $k \geq 3$ is an integer and $J_u = [k \sin(\pi u/k)]$ for $u = 1, 2, \ldots, k$. In the present paper we show that the polynomial $F$ has large number of cyclotomic factors and is vanishing at 1 with high multiplicity.

In Section 2 we state Theorem 1 concerning the number of cyclotomic factors of the polynomial $F(x^2)$. Since the number of noncyclotomic factors of a polynomial is small (Theorem 2), we conjecture that the number of all irreducible factors of a polynomial is bounded above by the quantity with a slightly better constant than the one obtained in [9] (see Section 2). In Section 3 we state Theorem 3 which shows that the polynomial $F(x)$ has high multiplicity of 1. Finally, in Section 4 we prove our theorems in the reverse order and the corollary.

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2. THE NUMBER OF IRREDUCIBLE FACTORS OF A POLYNOMIAL

The problem of estimating $\Omega(P)$ has been studied by A. Schinzel [11] and E. Dobrowolski [4]. As it follows from their work it is natural to give separate estimates for $\Omega_c(P)$ and for the number of noncyclotomic factors $\Omega(P) - \Omega_c(P)$.

In 1993, C. Pinner and J. Vaaler [9] strengthened and generalized the results of A. Schinzel and E. Dobrowolski. In particular, they proved that

$$\frac{1}{n}(\Omega(P) - \Omega_c(P)) \leq c_1 s(P) \left( \frac{\log \left( \frac{1}{s(P)} \right)}{\log \log \left( \frac{1}{s(P)} \right)} \right)^3,$$

(2)

$$\frac{1}{n}\Omega_c(P) \leq \left( \sqrt{\frac{\zeta(2)\zeta(3)}{\zeta(6)}} + o(1) \right) \sqrt{s(P) \log \left( \frac{1}{s(P)} \right)},$$

(3)

Here $c_1$ is an absolute and computable constant and $o(1) \to 0$ as $s(P) \to 0$. They also showed that the bound (3) is essentially sharp: the constant $\sqrt{\zeta(2)\zeta(3)/\zeta(6)} = 1.39...$ which occurs in (3) cannot by replaced by a constant smaller than $3\sqrt{3}/4 = 1.29...$. In this paper we prove the following inequality:

**Theorem 1.** Suppose that $Q_k(x) = F_k(x^2)$, where the polynomial $F_k$ is given in (1). Then

$$\frac{\Omega_c(Q_k)}{\deg Q_k} \geq \left( \frac{3\sqrt{2}}{\pi} + o(1) \right) \sqrt{s(Q_k) \log \left( \frac{1}{s(Q_k)} \right)},$$

(4)

where $o(1) \to 0$ as $k \to \infty$. Numerically one has $3\sqrt{2}/\pi = 1.35...$.

**Corollary.** For every integer $n$ there exists a polynomial $P$ of degree $n$ such that

$$\frac{1}{n}\Omega(P) \geq \frac{1}{n}\Omega_c(P) \geq \left( \frac{3\sqrt{2}}{\pi} + o(1) \right) \sqrt{s(P) \log \left( \frac{1}{s(P)} \right)},$$

(5)

where $o(1) \to 0$ as $n \to \infty$, $s(P) \to 0$.

Put now $M(P)$ for the Mahler measure of $P$ and let

$$h(P) = \frac{\log M(P)}{n}.$$

Note that the inequalities

$$M(P) \leq \sqrt{\sum_{i=0}^{n} |a_i|^2} \leq \nu(P)$$

imply that $h(P) \leq s(P)$. The following theorem follows from [4], [6], [9]:

**Theorem 2.** We have

$$\frac{1}{n}(\Omega(P) - \Omega_c(P)) \leq \left( \frac{4}{9} + o(1) \right) h(P) \left( \frac{\log \left( \frac{1}{h(P)} \right)}{\log \log \left( \frac{1}{h(P)} \right)} \right)^3,$$

(6)

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where \( o(1) \rightarrow 0 \) as \( h(P) \rightarrow 0 \).

The inequalities of the type (2), (6) are closely related with Lehmer’s problem. If there exists a positive absolute constant \( \delta \) such that \( \log M(T) \geq \delta \) for all irreducible noncyclotomic polynomials \( T \), then

\[
\frac{1}{n} (\Omega(P) - \Omega_c(P)) \leq \frac{h(P)}{\delta}.
\]  

(7)

We will show below that inequality (7) is sharp.

Notice that the right hand side of (2) is small compared to the right hand side of (3) whenever \( s(P) \rightarrow 0 \). Therefore both quantities \( n^{-1}\Omega_c(P) \) and \( n^{-1}\Omega(P) \) are bounded above by

\[
\left( \sqrt{\frac{\zeta(2)\zeta(3)}{\zeta(6)}} + o(1) \right) \sqrt{s(P) \log \left( \frac{1}{s(P)} \right)},
\]

where \( o(1) \rightarrow 0 \) as \( s(P) \rightarrow 0 \). Bearing in mind (5), it is tempting to conjecture that they are bounded above by

\[
\left( \frac{3\sqrt{2}}{\pi} + o(1) \right) \sqrt{s(P) \log \left( \frac{1}{s(P)} \right)},
\]

where \( o(1) \rightarrow 0 \) as \( s(P) \rightarrow 0 \). Then this bound would be the best possible.

It is interesting to note that the inequality

\[
\frac{1}{n} \log |\alpha - 1| \geq \left( \frac{\pi}{4} + o(1) \right) \sqrt{h(P) \log \left( \frac{1}{h(P)} \right)},
\]

obtained using the polynomial (1) in [5] is of the form similar to (5). Here \( o(1) \rightarrow 0 \) as \( h(P) \rightarrow 0 \) and not \( n \rightarrow \infty \) as it is stated in [5]. The author wishes to express his thanks to Professor D. Masser for pointing out this misstatement in [5].

3. POLYNOMIALS WITH ZEROS OF HIGH MULTIPLICITY AT 1.

In 1933, I. Schur [12] showed that the number of real roots of a polynomial with complex coefficients is bounded from above by \( 2 \sqrt{n \log \left( \frac{L(P)}{\sqrt{|a_0a_n|}} \right)} \), where \( L(P) \) is the length of \( P \). G. Szegö [13] proved that this bound is the best possible. In particular, for the multiplicity of 1 of an integer polynomial we have

\[
r = r(P) \leq 2 \sqrt{n \log \left( nH(P) \right)}.
\]

(8)

On the other hand, applying Siegel’s lemma M. Mignotte [7] proved that there exists a polynomial \( P \) such that

\[
r^2 \geq (2 + o(1)) \frac{n \log H(P)}{\log n},
\]

(9)
Recently F. Amoroso [1] improved (8) and (9) showing that
\[ r \leq 1.21 \sqrt{n \log H(P)}, \]
whenever \( r, n \to \infty, \) \( r/n \to 0, \sqrt{n/\log n/r} \to 0, \) and
\[ r^2 \log(n/r) \geq (4 + o(1)) n \log H(P), \] (10)
where \( o(1) \to 0 \) as \( r, n \to \infty, \) \( r/n \to 0, \) \( \sqrt{n/\log n/r} \to 0. \) Earlier E. Bombieri and J. Vaaler [2] obtained inequality (10) with the constant 2 instead of 4 under the conditions \( r, n \to \infty, \) \( r/n \to 0. \) The proofs of both (9) and (10) are the proofs of existance. In fact, M. Mignotte’s proof is based on pigeon–hole principle. The proof of E. Bombieri and J. Vaaler uses the geometry of numbers. The proof of F. Amoroso is also based on their result. We will show below that the polynomial given by (1) is vanishing at 1 with high multiplicity:

**Theorem 3.** Suppose that the polynomial \( F_k(x) \) of degree \( n_k \) is given in (1).

Then
\[ r(F_k)^2 \log (n_k/r(F_k)) \geq \left( \frac{32}{\pi^2} + o(1) \right) n_k \log \nu(F_k), \] (11)
\[ r(F_k) \geq \left( \frac{8}{\pi} + o(1) \right) n_k \sqrt{s(F_k)/\log \left( 1/s(F_k) \right)}, \] (12)
where both \( o(1) \to 0 \) as \( k \to \infty. \)

Since \( H(P) \leq \nu(P) \leq (n + 1)H(P) \) and \( 32/\pi^2 = 3.24... \) is less than 4, inequality (11) gives slightly worse bound than (10).

4. PROOFS

Let us recall first (see [5]) that for \( k \) tending to infinity the following asymptotic formulas hold:
\[ \deg Q_k = 2 \deg F_k = 2n_k \sim \frac{k^5}{\pi^2}, \]
\[ \log \nu(Q_k) = \log \nu(F_k) \leq C \sim \frac{k^3 \log k}{4}, \]
\[ r(F_k) \sim \frac{2k^4}{\pi^2}. \] (15)

By (13)–(15) we obtain (11) immediately. Since
\[ s(F_k) \leq (1 + o(1)) \frac{\pi^2 \log k}{2k^2}, \]
we also obtain (12) and Theorem 3 follows.

As we noticed earlier, Theorem 2 is essentially due to E. Dobrovolski [4] and R. Louboutin [6]. It should be noted that the argument below occurs already in the paper of A. Schinzel [11] (see also [9] or [10], Theorem 3, which is a more general result). Suppose that $P(x)$ factors into irreducible factors in $\mathbb{Z}[x]$ as

$$P(x) = \prod \Phi_p(x)^{n_p} \prod \Psi_q(x)^{m_q}.$$ 

Here $\Phi_p(x)$ denotes the $p^{\text{th}}$ cyclotomic polynomial, $n_p \geq 0$, $m_q \geq 1$, and $\Psi_q(x)$ are distinct, irreducible, noncyclotomic polynomials. Using the multiplicativity of the Mahler measure we get

$$M(P) = \prod M(\Psi_q)^{m_q}.$$ 

Hence,

$$\sum m_q \log M(\Psi_q) \leq \log M(P) = h(P). \tag{16}$$

From (16) we easily get (7) under assumption of Lehmer’s conjecture $\log M(\Psi_q) \geq \delta$:

$$\frac{1}{n}(\Omega(P) - \Omega_c(P)) = \frac{1}{n} \sum m_q \leq \frac{h(P)}{\delta}.$$ 

On the other hand, this inequality is sharp, e.g., for the polynomials $T(x)^\omega$, where $T$ is an irreducible polynomial with $\log M(T) = \delta$ and $\omega$ is a positive integer. Indeed, then we have $\Omega(T^\omega) = \omega$, $\Omega_c(T^\omega) = 0$, $n = \omega \deg T$, $h(T^\omega) = \delta/\deg T$.

In order to get (6) we use (16) and the trivial bound

$$\sum m_q \deg \Psi_q \leq n. \tag{17}$$

Let us separate the sum of multiplicities into two sums:

$$\Omega(P) - \Omega_c(P) = \sum m_q = \sum_1 m_q + \sum_2 m_q,$$

where the sum $\sum_1$ is taken for those $q$ for which $\deg \Psi_q \geq 1/h(P)$ and the sum $\sum_2$ is taken for $q$ such that $\deg \Psi_q < 1/h(P)$. Obviously from (17) we have

$$\sum_1 m_q \leq nh(P).$$

In order to estimate $\sum_2$, we use the lower bound for the Mahler’s measure due to R. Louboutin [6]:

$$\log M(\Psi_q) \geq \left(\frac{9}{4} + o(1)\right)\left(\frac{\log \log \deg \Psi_q}{\log \deg \Psi_q}\right)^3.$$
Hence, by (16)

\[ \sum_2 m_q \leq \frac{nh(P)}{\min_q \log M(\Psi_q)} \leq \left( \frac{4}{9} + o(1) \right) nh(P) \left( \frac{\log (1/h(P))}{\log \log (1/h(P))} \right)^3. \]

Therefore, we have

\[ \frac{1}{n} (\Omega(P) - \Omega_c(P)) = \frac{1}{n} \left( \sum_1 m_q + \sum_2 m_q \right) \leq h(P) + \left( \frac{4}{9} + o(1) \right) h(P) \left( \frac{\log (1/h(P))}{\log \log (1/h(P))} \right)^3 = \left( \frac{4}{9} + o(1) \right) h(P) \left( \frac{\log (1/h(P))}{\log \log (1/h(P))} \right)^3, \]

where \( o(1) \to 0 \) as \( h(P) \to 0 \). Theorem 2 is proved.

In order to prove Theorem 1 we first compute \( \Omega_c(Q_k) \). Denote by \( \tau(a) \) the number of divisors of \( a \). Now via the representation

\[ x^m - 1 = \prod_{d|m} \Phi_d(x) \]

we see that \( x^m - 1 \) is the product of \( \tau(m) \) cyclotomic factors. Hence from (1) and from the definition \( Q_k(x) = F_k(x^2) \) it follows that

\[ \Omega_c(Q_k) = \sum J_u J_v \tau(2u - 2v) = \sum_{v=1}^{k-1} J_v \sum_{\omega=1}^{k-v} J_{v+\omega} \tau(2\omega). \]

Let us write \( \tau(2\omega) \) as follows:

\[ \tau(2\omega) = \sum_{2g/2\omega} + \sum_{2g+1/2\omega} = \sum_{g/\omega} 1 + \sum_{2g+1/\omega} 1. \]

We further find

\[ \sum_{\omega=1}^{k-v} J_{v+\omega} \sum_{g/\omega} 1 = \sum_{g=1}^{k-v} \left( J_{v+g} + J_{v+2g} + \ldots + J_{v+l_g} \right), \]

where \( l = \lfloor (k - v)/g \rfloor \).

Replacing the inner sum by proper integral we have

\[ \sum_{v=1}^{k-1} J_v \sum_{g=1}^{k-v} (J_{v+g} + \ldots + J_{v+l_g}) \sim k^4 \log k \int_0^1 \sin(\pi x) \int_0^{1-x} \sin(\pi(x+y))dydx = \frac{2}{\pi^2} k^4 \log k. \]
Analogously,
\[
\sum_{v=1}^{k-1} J_v \sum_{\omega=1}^{k-v} J_{v+\omega} \sum_{2g+1|\omega} 1 \sim \frac{1}{\pi^2} k^4 \log k.
\]
This implies that
\[
\Omega_c(Q_k) \sim \frac{3}{\pi^2} k^4 \log k. \tag{18}
\]
Combining this with (13) we see that the left hand side of (4)
\[
\frac{\Omega_c(Q_k)}{\deg Q_k} \sim \frac{3 \log k}{k}
\]
as \(k \to \infty\).

On the other hand, by (13) and (14) we have
\[
s(Q_k) = \frac{\log \nu(Q_k)}{\nu_k} \leq \left( \frac{\pi^2}{4} + o(1) \right) \frac{\log k}{k^2}.
\]
Hence the quantity \(\sqrt{s(Q_k) \log \left(1/s(Q_k)\right)}\) does not exceed
\[
\left( \frac{\pi}{\sqrt{2}} + o(1) \right) \frac{\log k}{k}.
\]
The inequality (4) now follows and Theorem 1 is proved.

In order to prove the Corollary we need more elaborate form of (13). Since the function \(f(x) = \sin(\pi x)\) is smooth, we can write
\[
d_k = \deg Q_k = 2 \deg F_k = 2k^5 \int_0^{1} \sin(\pi x) \int_0^{x} \sin(\pi y)(x-y)dydx + O(k^4) = \tag{19}
\]
\[
= \frac{k^5}{\pi^2} + O(k^4).
\]

We construct the polynomial \(P\) of degree \(n\) as follows. Let us take maximal \(k\) such that \(d_k \leq n\). Put
\[
P(x) = Q_k(x)(x^{d} - 1),
\]
where \(d = n - d_k\). We claim that for this polynomial inequality (5) holds. Indeed, by (14)
\[
\log \nu(P) \leq \log \nu(Q_k) + \log \nu(x^d - 1) \leq C + \log 2 \sim \frac{k^3 \log k}{4}
\]
for large \(k\). Since \(d = n - d_k < d_{k+1} - d_k = O(k^4)\), applying (19) we have \(n = d_k + d = k^5/\pi^2 + O(k^4)\). Thus, \(s(P)\) is bounded from above by \((\pi^2/4 + o(1))k^{-2} \log k\).

Therefore, the right hand side of (5) is bounded above by \((3 + o(1))k^{-1} \log k\).

On the other hand, (18) implies
\[
\frac{1}{n} \Omega_c(P) = \frac{1}{n} \left( \Omega_c(Q_k) + \Omega_c(x^d - 1) \right) \sim \frac{\pi^2}{k^5} \left( \frac{3}{\pi^2} k^4 \log k + \tau(d) \right) \sim 3k^{-1} \log k,
\]
since \(\tau(d) \leq d = O(k^4)\). Inequality (5) is proved.
REFERENCES


arturas.dubickas@maf.vu.lt
Faculty of Mathematics
Vilnius University
Naugarduko 24
Vilnius 2006, Lithuania