ALGEBRAIC NUMBERS WITH BOUNDED DEGREE AND WEIL HEIGHT

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(Received 12 March 2018; accepted 8 May 2018; first published online 18 July 2018)

Abstract

For a positive integer $d$ and a nonnegative number $\xi$, let $N(d, \xi)$ be the number of $\alpha \in \mathbb{Q}$ of degree at most $d$ and Weil height at most $\xi$. We prove upper and lower bounds on $N(d, \xi)$. For each fixed $\xi > 0$, these imply the asymptotic formula $\log N(d, \xi) \sim \xi^d$ as $d \to \infty$, which was conjectured in a question at Mathoverflow [https://mathoverflow.net/questions/177206].

2010 Mathematics subject classification: primary 11R06; secondary 11R09.

Keywords and phrases: Mahler measure, Weil height, counting function, irreducible polynomial.

1. Introduction

For an algebraic number $\alpha$ of degree $d$ over $\mathbb{Q}$ with conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ and minimal polynomial

$$a_d(x - \alpha_1) \cdots (x - \alpha_d) = a_dx^d + \cdots + a_1x + a_0 \in \mathbb{Z}[x],$$

where $a_d > 0$, we denote by $H(\alpha) := \max_{0 \leq j \leq d} |a_j|$ the height of $\alpha$ and by

$$M(\alpha) := a_d \prod_{i=1}^{d} \max\{1, |\alpha_i|\}$$

the Mahler measure of $\alpha$. For $\alpha \in \mathbb{Q}$, these quantities are related by the inequalities

$$H(\alpha)2^{-d} \leq M(\alpha) \leq H(\alpha) \sqrt{d + 1} \quad (1.1)$$

(see, for instance, [15] and [16, Lemma 3.11]).

Set

$$M(d, T) := \#\{\alpha \in \mathbb{Q} : \deg \alpha = d, \ M(\alpha) \leq T\},$$

This research was funded by the European Social Fund according to the activity Improvement of researchers qualification by implementing world-class R&D projects of Measure no. 09.3.3-LMT-K-712-01-0037.

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where \#A stands for the cardinality of the set A. For \( d \geq 2 \) and

\[
V_d := 2^{d+1}(d+1)^{(d-1)/2} \prod_{i=1}^{[(d-1)/2]} \frac{(2i)^{d-2i}}{(2i+1)^{d+1-2i}},
\]

the asymptotic formula

\[
M(d, T) = \frac{dV_d}{2\zeta(d+1)} T^{d+1} + O(T^d (\log T)^{2/d}) \quad \text{as } T \to \infty
\]

has been established in [2] and [10]. (Throughout, \( \zeta(s) \) is the Riemann zeta-function.) See also [1, 11, 17] and the references therein for some generalisations. In [9], this formula is given with an explicit error term: for any \( d \geq 3 \) and any \( T \geq 1 \),

\[
\left| M(d, T) - \frac{dV_d}{2\zeta(d+1)} T^{d+1} \right| \leq 3.37 \cdot 15.01^d \cdot T^d.
\]

This inequality gives the asymptotic formula for \( M(d, T) \) as \( d \to \infty \) in the range \( \log T \gg d^2 \). (Here and below, the notation \( v \gg w \) means that the inequality \( v \geq cw \) holds with some positive constant \( c \).) By [2, Theorem 4], this asymptotic formula holds in a wider range \( \log T \gg d \log d \), but with slightly larger error term in \( T \). However, for small \( T \), for example, \( T \) fixed at \( T = 2 \), the problem of finding the correct order of \( M(d, T) \) is wide open. See, for instance, the papers [3, 5, 13]. More precisely, from the main result of [5] one can derive \( M(d, 2) \gg c d^5 \) with some absolute constant \( c > 0 \), whereas the best known upper bound is only \( M(d, 2) < 2^{(1+\varepsilon)d} \) for any \( \varepsilon > 0 \) and \( d \geq d(\varepsilon) \) [6]. Another interesting case, \( T = 1 \), corresponds to the constant

\[
C := \limsup_{d \to \infty} \frac{\log M(d, 1)}{\log d},
\]  

(1.2)

which has been studied by Erdős [7] and Pomerance [14]. This constant can be expressed as the number of solutions of the equation \( \phi(n) = d \) for \( n \in \mathbb{N} \) (when \( d \) is fixed), where \( \phi \) is Euler's totient function, and bounds can be found using tools from analytic number theory. Erdős and Pomerance showed that \( 0.55 < C \leq 1 \) and Erdős conjectured that \( C = 1 \) [8].

In the upper bound direction, for \( d \) sufficiently large and any \( T \geq 1 \), we showed in [6] that the number of integer polynomials of degree \( d \) and with positive leading coefficient, nonzero constant coefficient and Mahler measure at most \( T \) is bounded above by \( T^{d(1+16 \log \log d/\log d)\varepsilon^{3.58} \sqrt{d}} \) for \( d \) large enough. Furthermore, the factor \( e^{3.58 \sqrt{d}} \) can be removed for \( T \geq 1.32 \). The roots of any such polynomial, irreducible over \( \mathbb{Q} \) and whose coefficients are relatively prime, give \( d \) algebraic numbers of degree \( d \) and Mahler measure at most \( T \). Hence, the main result of [6] yields the inequality

\[
M(d, T) < d T^{d(1+16 \log \log d/\log d)}
\]  

(1.3)

for each \( T \geq 1.32 \) and each sufficiently large integer \( d \), say \( d \geq d_0 \).

In this paper, we consider the related quantity

\[
N(d, \xi) := \# \{ \alpha \in \overline{\mathbb{Q}} : \deg \alpha \leq d, \ h(\alpha) \leq \xi \},
\]
where
\[ h(\alpha) := \frac{\log M(\alpha)}{\deg \alpha} \]
is the Weil height of \( \alpha \). Using [2, Theorem 4] and following the approach of [10], for \( \xi \gg \log d \), one can derive the asymptotic formula
\[ N(d, \xi) \sim \frac{dV_d e^{\xi(d+1)}}{2\xi(d+1)} \quad \text{as } d \to \infty. \tag{1.4} \]
In [12], the problem of finding the asymptotic formula for \( N(d, 1) \) (noting that \( \xi = 1 \) is much less than \( \log d \)), or, less ambitiously, for \( \log N(d, 1) \) as \( d \to \infty \), has been raised. From the discussion in [12] and also from (1.4), one can conjecture that the expected formula is
\[ \log N(d, 1) \sim d^2 \quad \text{as } d \to \infty. \tag{1.5} \]
In this note, we prove the following theorem, which implies (1.5).

**Theorem 1.1.** For each \( \xi \geq 2d^{-1} \log d \) and each sufficiently large \( d \),
\[ -\frac{d \log d}{2} < \log N(d, \xi) - \xi d^2 < \frac{17\xi d^2 \log \log d}{\log d}. \]

It is clear that Theorem 1.1 yields the asymptotic formula
\[ \log N(d, \xi) \sim \xi d^2 \quad \text{as } d \to \infty \quad \text{and} \quad \frac{\xi d}{\log d} \to \infty. \]
Of course, equation (1.4) immediately implies this asymptotic formula, but only in the range \( \xi \gg \log d \). We also remark that, for \( 0 \leq \xi \leq d^{-1} (\log d)^{-3} \), by combining a Dobrowolski-type bound with the above mentioned results [7, 8, 14], one gets
\[ \log N(d_k, \xi) \sim C \log d_k \quad \text{as } d_k \to \infty, \]
where \( C \) is the constant defined in (1.2) and \( (d_k)_{k=1}^\infty \) is some increasing sequence of positive integers.

In fact, the lower bound on \( \log N(d, \xi) - \xi d^2 \) as claimed in Theorem 1.1 will be proved for \( d \geq 1.784 \cdot 10^8 \). In principle, some explicit constant \( D_0 \) such that the upper bound of Theorem 1.1 for \( \log N(d, \xi) - \xi d^2 \) is true for each \( d \geq D_0 \) can also be given. However, it depends on the corresponding bound \( d \geq d_0 \) in (1.3), which was not calculated in [6], so we will not give it here.

For \( \log M(d, T) \), by applying the same arguments, we get the following bounds.

**Theorem 1.2.** For each \( T \geq 38d^{3/2} (\log d)^2 \) and each sufficiently large \( d \),
\[ -\frac{d \log d}{2} < \log M(d, T) - d \log T < \frac{17d \log T \log \log d}{\log d}. \]
We will prove the lower bound on \( \log M(d, T) - d \log T \) for each \( d \geq 6 \). Note that Theorem 1.2 implies the asymptotic formula
\[ \log M(d, T) \sim d \log T \quad \text{as } d \to \infty \quad \text{and} \quad \frac{\log T}{\log d} \to \infty. \]
In the next section, we give some auxiliary results and combine them into Lemma 2.3. Then, in Section 3, we give the proofs of the theorems.
2. Auxiliary results

Let \( d, H \geq 2 \) be two integers. Consider the set \( P(d, H) \) of integer polynomials defined by

\[
P(d, H) := \left\{ a_dx^d + \cdots + a_1x + a_0 \in \mathbb{Z}[x] : a_d > 0, a_0 \neq 0, \max_{0 \leq j \leq d} |a_j| \leq H \right\}.
\]

In [4, Theorem 1], we showed that the number of integer polynomials reducible over \( \mathbb{Q} \) and of degree \( d \) and height at most \( H \) is less than

\[
2H(2H+1)^{d-1} + 2dH(2H+1)^{d-1}(\log(2H))^2.
\]

Here, the first term corresponds to the polynomials whose free term is zero. Since the polynomials with \( a_d < 0 \) are also counted in the above formula, we can remove the factor 2 from the second term and restate this result as shown in the following lemma.

**Lemma 2.1.** For any integers \( d, H \geq 2 \), the number of polynomials in \( P(d, H) \) reducible over \( \mathbb{Q} \) is less than

\[
dH(2H+1)^{d-1}(\log(2H))^2.
\]

Of course, the coefficients of a polynomial irreducible over \( \mathbb{Q} \) are not necessarily coprime (for instance, the coefficients of \( 2x^2 - 6x + 2 \) are all divisible by 2). For this reason, we also need the following result.

**Lemma 2.2.** For any integers \( d \geq 6 \) and \( H \geq 6d \), the set \( P(d, H) \) contains at least

\[
\frac{2^d H^{d+1}}{\zeta(d+1)} - d2^{d+2}H^d
\]

polynomials \( a_dx^d + \cdots + a_1x + a_0 \) satisfying \( \gcd(a_d, \ldots, a_1, a_0) = 1 \).

**Proof.** Let \( g \) be an integer in the range \( 1 \leq g \leq H \). Suppose there are \( N_g(H) \) polynomials in \( P(d, H) \) satisfying \( \gcd(a_d, \ldots, a_1, a_0) = g \). Our aim is to estimate \( N_1(H) \) from below. Clearly,

\[
#P(d, H) = 2H^2(2H+1)^{d-1},
\]

since there are \( H \) possibilities for \( a_d \), \( 2H \) possibilities for \( a_0 \), and \( 2H + 1 \) possibilities for each \( a_j \), where \( j = 1, \ldots, d - 1 \). Consequently,

\[
N_1(H) + N_2(H) + \cdots + N_H(H) = 2H^2(2H+1)^{d-1}.
\]

Observe that \( N_g(H) = N_1([H/g]) \) for \( g = 1, \ldots, H \). Hence,

\[
\sum_{g=1}^H N_1([H/g]) = 2H^2(2H+1)^{d-1}.
\]

Now, by the Möbius inversion formula,

\[
N_1(H) = \sum_{g=1}^H \mu(g)2[H/g]^2(2[H/g] + 1)^{d-1}. \tag{2.1}
\]
Split the sum on the right-hand side of (2.1) into two sums \(N_1(H) = S_1 + S_2\), where \(S_1\) is taken over \(g\) in the interval \(1 \leq g \leq \lfloor H/d \rfloor\) and \(S_2\) is over \([H/d] + 1 \leq g \leq H\). Since \(H/g \leq d\), we find that
\[
|S_2| \leq (H - \lfloor H/d \rfloor)2(H/g)^2(2H/g + 1)^{d-1} < 2d^2(2d + 1)^{d-1}H.
\]
So, in view of
\[
2d^2(2d + 1)^{d-1} < 2d^2(13d/6)^{d-1} \leq 2d^2(13H/36)^{d-1} < 0.5H^{d-1},
\]
we conclude that
\[
|S_2| < 0.5H^d.
\]

To evaluate the sum
\[
S_1 := \sum_{g=1}^{\lfloor H/d \rfloor} \mu(g)2[H/g]^2(2\lfloor H/g \rfloor + 1)^{d-1},
\] (2.2)
we first show that the difference between \(2[H/g]^2(2\lfloor H/g \rfloor + 1)^{d-1}\) and \(2^d(H/g)^{d+1}\) is small, and then investigate
\[
S_0 := \sum_{g=1}^{\lfloor H/d \rfloor} \mu(g)2^d(H/g)^{d+1}.
\] (2.3)

Indeed, both numbers, \(2[H/g]^2(2\lfloor H/g \rfloor + 1)^{d-1}\) and \(2^d(H/g)^{d+1}\), belong to the interval
\[
(2(y - 1)^2(2y - 1)^{d-1}, 2y^2(2y + 1)^{d-1}],
\]
where \(y := H/g \geq 2\). Thus, the difference between them does not exceed the length of the interval, namely,
\[
2y^2(2y + 1)^{d-1} - 2(y - 1)^2(2y - 1)^{d-1} < \frac{(2y + 1)^{d+1} - (2y - 2)^{d+1}}{2}.
\]
By the mean value theorem, the latter difference equals \(1.5(d + 1)y_0^d\) for some \(y_0\) in the interval \([2y - 2, 2y + 1]\). Consequently,
\[
|2[H/g]^2(2\lfloor H/g \rfloor + 1)^{d-1} - 2^d(H/g)^{d+1}| < 1.5(d + 1)(2H/g + 1)^d.
\]
Combining this with (2.2) and (2.3), we derive
\[
|S_1 - S_0| \leq 1.5(d + 1) \sum_{g=1}^{\lfloor H/d \rfloor} (2H/g + 1)^d.
\]

The first term in the above sum is \((2H + 1)^d\). The quotient of the \(g\)th term and the first term is
\[
\frac{(2H/g + 1)^d}{(2H + 1)^d} = \left(\frac{2H + g}{2H + 1}\right)^d \cdot \frac{1}{g^d} \leq \frac{(2H + H/d)^d}{(2H + 1)^d} \cdot \frac{1}{g^d} < \left(1 + \frac{1}{2d}\right)^d \cdot \frac{1}{g^d} < \frac{1.65}{g^d}.
\]
It follows that
\[ |S_1 - S_0| < 1.5(d + 1) \frac{1.65}{\zeta(d)} (2H + 1)^d < \frac{2.5(d + 1)}{\zeta(d)} (2H + 1)^d. \]

Therefore, applying the inequality
\[ \left(1 + \frac{1}{2H}\right)^d \leq \left(1 + \frac{1}{12d}\right)^d < 1.09, \]
we conclude that
\[ |S_1 - S_0| < \frac{(3d + 3)(2H)^d}{\zeta(d)} < 3.5d2^d H^d. \] (2.5)

Next, since the Dirichlet series that generates the Möbius function is the inverse of the Riemann zeta function, from (2.3) we find that
\[ \frac{S_0}{2^d H^{d+1}} = \sum_{g=1}^{[H/d]} \mu(g) g^{d+1} = \frac{1}{\zeta(d + 1)} - \sum_{g=[H/d]+1}^{\infty} \frac{\mu(g)}{g^{d+1}}. \]

This leads to
\[
\left| S_0 - \frac{2^d H^{d+1}}{\zeta(d + 1)} \right| \leq 2^d H^{d+1} \sum_{g=[H/d]+1}^{\infty} \frac{1}{g^{d+1}} < \frac{2^d H^{d+1}}{d(H/d - 1)} \leq \frac{2^d H^{d+1}}{d(5H/6d)} = 2.4^d d^{d-1} H \\
\leq 2.4^d (H/6)^{d-1} H < 0.1H^d.
\]

Combining this with (2.1)–(2.3) and (2.5), we deduce that
\[
\left| N_1(H) - \frac{2^d H^{d+1}}{\zeta(d + 1)} \right| = \left| S_2 + S_1 - S_0 + S_0 - \frac{2^d H^{d+1}}{\zeta(d + 1)} \right| \\
\leq |S_2| + |S_1 - S_0| + |S_0 - \frac{2^d H^{d+1}}{\zeta(d + 1)}| \\
< 0.5H^d + 3.5d2^d H^d + 0.1H^d < 2d^{d+2} H^d.
\]

This yields the required lower bound on \( N_1(H) \) and proves the lemma. \( \square \)

From Lemmas 2.1 and 2.2 we will derive the following lemma.

**Lemma 2.3.** For any \( d \geq 6 \) and any \( H \geq 37d(\log d)^2 \) there are at least
\[ d2^{d-1} H^{d+1} \] (2.6)

algebraic numbers of degree \( d \) and height at most \( H \).

**Proof.** Lemmas 2.1 and 2.2 imply that, for \( d \geq 6 \) and \( H \geq 6d \),
\[ I(d, H) > \frac{2^d H^{d+1}}{\zeta(d + 1)} - d2^{d+2} H^d - dH(2H + 1)^{d-1} (\log(2H))^2, \]
where \( I(d, H) \) is the number of irreducible polynomials in \( \mathbb{Z}[x] \) lying in the set \( P(d, H) \).
By (2.4), we have $(2H + 1)^d < 1.09 \cdot 2^d H^d$. It follows that

$$dH(2H + 1)^{d-1} < \frac{d}{2}(2H + 1)^d < d2^d H^d,$$

and hence

$$d2^{d+2} H^d + dH(2H + 1)^{d-1} (\log(2H))^2 < d2^d H^d (4 + (\log(2H))^2).$$

Therefore,

$$I(d, H) > 2^d H^d (H \zeta(d + 1) - 4d - d(\log(2H))^2) > 2^d H^d (0.98H - 4d - d(\log(2H))^2).$$

Note that the function

$$u(x) := \frac{0.24x}{4 + (\log x)^2} - d$$

is increasing in $x > 0$. Furthermore, one can easily verify that, for each $d \geq 6$,

$$u(74d(\log d)^2) = d\left(\frac{17.76(\log d)^2}{4 + (\log(74d(\log d)^2))^2} - 1\right) > 0.$$ 

Hence, $u(x) > 0$ for $x \geq 74d(\log d)^2$. Now, assuming that

$$H \geq 37d(\log d)^2$$

and $d \geq 6$, from $u(2H) > 0$ we deduce that

$$0.98H - 4d - d(\log(2H))^2 > 0.5H.$$ 

Therefore,

$$I(d, H) > 2^d H^d \cdot 0.5H = 2^{d-1} H^{d+1}.$$ 

This implies (2.6), since each of these polynomials (with positive leading coefficients) gives $d$ algebraic numbers of degree $d$ and height at most $H$. \hfill \square

3. Proofs of the theorems

Proof of Theorem 1.1. We will apply Lemma 2.3 with

$$H := \lfloor e^{\xi d}(d + 1)^{-1/2} \rfloor$$

and $d$ so large that $H \geq 37d(\log d)^2$. (Recall that $\xi \geq 2^{-d^2} \log d$, so the inequality $H \geq 37d(\log d)^2$ holds for $d \geq 1.784 \cdot 10^8$.) Then, by (1.1) and (2.6), each of those $d2^{d-1} H^{d+1}$ algebraic numbers $\alpha$ has degree $d$ and Weil height

$$h(\alpha) = \frac{\log M(\alpha)}{d} \leq \frac{\log(H(\alpha) \sqrt{d + 1})}{d} \leq \frac{\log e^{\xi d}}{d} = \frac{\xi d}{d} = \xi.$$
Hence, for all \(d \geq 1.784 \cdot 10^8\) and \(\xi \geq 2d^{-1} \log d\),

\[
N(d, \xi) \geq d^{2d-1} [e^{\xi d}(d+1)^{-1/2}]^{d+1} > d^{2d-1} \left( \frac{e^{\xi d} - \sqrt{d+1}}{\sqrt{d+1}} \right)^{d+1}
\]

This implies the required lower bound on \(\log N(d, \xi)\).

For the upper bound, we first observe that, by (1.1), each \(\alpha \in \overline{\mathbb{Q}}\) of degree \(d\) whose Mahler measure is bounded by \(T\), satisfies

\[
H(\alpha) \leq 2d M(\alpha) \leq 2^d T.
\]

Thus,

\[
M(d, T) \leq (2^{d+1} T + 1)^{d+1} < (2^{d+2} T)^{d+1} = 2^{(d+1)(d+2)} T^{d+1}.
\]

Next, observe that each \(\alpha\) of degree at most \(d\) and Weil height at most \(\xi\) satisfies \(M(\alpha) \leq e^{\xi \deg \alpha} \leq e^{\xi d}\). Now, using (1.3) with \(T = e^{\xi d}\) for \(j\) in the range \(d_0 \leq j \leq d\), where \(d_0\) is so large that (1.3) is true for \(d \geq d_0\), and (3.1) for \(j < d_0\), we deduce that

\[
N(d, \xi) \leq \sum_{j=0}^{d_0-1} M(j, e^{\xi d}) = \sum_{j=0}^{d_0-1} M(j, e^{\xi d}) + \sum_{j=d_0}^{d-1} M(j, e^{\xi d})
\]

for \(d\) large enough. This proves the required upper bound.

**Proof of Theorem 1.2.** By (1.3), we find that

\[
M(d, T) < T^{d(1+17 \log \log d/d\log d)}
\]

for \(T \geq 1.32\) and \(d\) large enough. This implies the claimed upper bound.

To prove the lower bound, apply Lemma 2.3 with

\[
H := \lfloor T(d+1)^{-1/2} \rfloor,
\]

where \(T \geq 38d^{3/2}(\log d)^2\) and \(d \geq 6\). Then, by (1.1) and (2.6), each of those \(\geq d^{2d-1} H^{d+1}\) algebraic numbers has degree \(d\) and Mahler measure at most \(T\). Consequently, using the bounds \(T - \sqrt{d+1} > T/2\) and \(d \geq 6\), we deduce that

\[
M(d, T) \geq d^{2d-1} [T(d + 1)^{-1/2}]^{d+1} > d^{2d-1} \left( \frac{T - \sqrt{d+1}}{\sqrt{d+1}} \right)^{d+1} \frac{d^{d+1}}{d T^{d+1}}
\]

\[
> d^{2d-1} (d+1)^{-d+1/2} \left( \frac{T}{2} \right)^{d+1} = \frac{2d^d T^d}{4 \sqrt{d+1} (d+1)^{d+1}}
\]

which gives the claimed lower bound.
References


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