THE DIVISORS OF NEWMAN POLYNOMIALS

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Abstract. We prove that every cyclotomic (not necessarily irreducible) polynomial non-vanishing at 1 divides a cyclotomic polynomial with coefficients \{0, 1\}.

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1. Introduction

Polynomials with coefficients in a small set of integers were studied on many occasions and there are many difficult old questions concerning them. A special attention is usually given to the polynomials having coefficients in the set \{0, 1, −1\} or in its subsets \{1, −1\} and \{0, 1\}. The latter polynomials, namely, those with \{0, 1\} coefficients having constant term 1 are called the Newman polynomials.

It is clear that all Newman polynomials of even degree are positive for \(x \leq -\frac{1 + \sqrt{5}}{2}\). Indeed, let \(N(x)\) be a Newman polynomial of even degree \(d\). Then, for \(x \leq -x_0 := -\frac{(1 + \sqrt{5})}{2}\), we have that \(|x| \leq |x|^2 - 1\), and so

\[
N(x) \geq 1 + |x|^d - |x|^d|1| - |x|^d|3| - \cdots - |x|
\]

\[
= 1 + |x|^d - |x|(|x|^d - 1)/(|x|^2 - 1)
\]

\[
\geq 1 + |x|^d - (|x|^d - 1) = 2 > 0.
\]

Similarly, all Newman polynomials of odd degree are negative for \(x \leq -x_0\). Consequently, all real roots of Newman polynomials must lie in the interval \((-x_0, -1/x_0) = (-\frac{(1 + \sqrt{5})}{2}, \frac{1 - \sqrt{5}}{2})\). Odlyzko and Poonen [6] obtained some bounds for the location of complex zeros of Newman polynomials. Konyagin [3] was interested in their irreducibility. Borwein and Mossinghoff [2] studied the order of vanishing of Newman polynomials at 1.

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We are interested in determining the set of divisors (possibly reducible) of the Newman polynomials.

By the above, if $G(x) \in \mathbb{Z}[x]$ has a real root and is a divisor of a Newman polynomial, then this root must belong to the interval $(-1 + \sqrt{5})/2, (1 - \sqrt{5})/2$. Recall that the polynomials

$$P(x) = x^3 - x - 1$$

and

$$L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

define the smallest Pisot number (as proved by Siegel [7]) and the smallest known Salem number (as found by Lehmer [4]). The polynomials $-P(-x) = x^3-x+1$ and $L(-x) = x^{10}-x^9+x^7-x^6+x^5-x^4-x^3-x+1$ have, respectively, one and two roots in the above interval. Note that they are both divisors of certain Newman polynomials. For instance, $-P(-x)$ divides $x^5 + x^4 + 1$, whereas $L(-x)$ divides the Newman polynomial $L(-x)(1+x+x^2+\cdots+x^{10})$.

(Using some computations Mossinghoff [5] found 23 Newman polynomials divisible by $L(-x)^2$. However, it is not known whether there is a Newman polynomial divisible by $L(-x)^3$.)

Our latter observation concerning $L(-x)$ can be generalized as follows. Given any positive integers $d_1 < \cdots < d_m$ with $m$ even, set $G(x) = 1 - x^{d_1} + x^{d_2} - \cdots + x^{d_m}$. Then $G(x)(1 + x + x^2 + \cdots + x^{d_m})$ is a Newman polynomial, so all $G(x)$ of the above form belong to the set of divisors of Newman polynomials.

Cyclotomic polynomials are certainly other good candidates for being divisors of Newman polynomials. Recall that $\Phi(x) \in \mathbb{Z}[x]$ is called a cyclotomic polynomial if it is a monic polynomial having all roots on the unit circle. It turns out that these indeed are divisors of cyclotomic Newman polynomials. (Of course, we have to exclude those vanishing at 1, because Newman polynomials have no positive roots.)

**Theorem.** If $\Phi(x)$ is a cyclotomic polynomial such that $\Phi(1) \neq 0$, then there is a cyclotomic Newman polynomial $N(x)$ such that $\Phi(x)|N(x)$.

Cyclotomic polynomials of even degree with coefficients $\pm1$ were described by Borwein and Choi [1].
2. Proof of the theorem

The proof is a straightforward generalization of [6], where it is shown that there are Newman polynomials with cyclotomic factors of arbitrarily high multiplicity.

Since $\Phi(0) \neq 0$, we can write

$$\Phi(x) = \prod_{s=2}^{\ell} \Phi_s(x)^{u_s},$$

where $u_2, u_3, \ldots, u_\ell$ are certain nonnegative integers. Here,

$$\Phi_s(x) = \prod_{(j,s) = 1} (x - \mu_j^s),$$

where $\mu_j^s = \exp\{2\pi\sqrt{-1}/s\}$ and the product is taken over every positive $j \leq s$ relatively prime to $s$. All $\Phi_s$ are known to be irreducible in $\mathbb{Z}[x]$.

Set $u = \max\{u_2, u_3, \ldots, u_\ell\}$. Consider the set of $u(\ell - 1)$ positive integers $k_1, \ldots, k_{u(\ell-1)}$ satisfying the following conditions: $k_r > u(k_1 + \cdots + k_{r-1})$, for $r > 1$, and $(k_r, \ell!) = 1$, for $r \geq 1$. Set

$$\Lambda_s(x) = \frac{(x^s - 1)}{(x - 1)} = 1 + x + x^2 + \cdots + x^{s-1},$$

and

$$N(x) := \Lambda_2(x^{k_1}) \cdots \Lambda_2(x^{k_u}) \Lambda_3(x^{k_{u+1}}) \cdots \Lambda_3(x^{k_{2u}}) \cdots \times \Lambda_\ell(x^{k_{u(\ell-2)+1}}) \cdots \Lambda_\ell(x^{k_{u(\ell-1)}}).$$

The first condition ensures that going from left to right in every polynomial $\Lambda_s(x^{k^r})$ the gaps between consecutive powers of $x$ are greater than the degree of the product of all polynomials to the left of $\Lambda_s(x^{k^r})$. Hence, $N(x)$ is a Newman polynomial. The second condition ensures that $\Lambda_s(\mu^r_j) = 0$, so that $\mu_j$ is the root of $N(x)$ with multiplicity $\geq u$, implying that $\Phi(x)|N(x)$. The fact that $N(x)$ is cyclotomic (as a product of cyclotomic factors) completes the proof.

References


**Njumeno daugianarių dalikliai**

A. Dubickas

Įrodoma, kad kiekvienas ciklotominis daugianarį, kuris nelygus nuliui taške 1, yra ciklotominio daugianario, kurio koeficientai priklauso aibei \( \{0, 1\} \), daliklis.

*Rankraštis gautas*

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