THE MAXIMAL VALUE OF POLYNOMIALS WITH RESTRICTED COEFFICIENTS

ARTŪRAS DUBICKAS AND JONAS JANKAUSKAS

Abstract. Let $\zeta$ be a fixed complex number. In this paper, we study the quantity $S(\zeta, n) := \max_{f \in \Lambda_n} |f(\zeta)|$, where $\Lambda_n$ is the set of all real polynomials of degree at most $n$ with coefficients in the interval $[0, 1]$. We first show how, in principle, for any given $\zeta \in \mathbb{C}$ and $n \in \mathbb{N}$, the quantity $S(\zeta, n)$ can be calculated. Then we compute the limit $\lim_{n \to \infty} S(\zeta, n)/n$ for every $\zeta \in \mathbb{C}$ of modulus 1. It is equal to $1/\pi$ if $\zeta$ is not a root of unity.

1. Introduction

A nonzero polynomial with 0, 1 coefficients is called a Newman polynomial after [6]. There is a variety of different problems in number theory and analysis related to Newman polynomials. See, for instance, [2], [3], [4], [7], [8].

This paper is motivated by the work of Akiyama, Brunotte, Pethő, and Steiner [1] which, at the first glance, has nothing to do with Newman polynomials. They investigate the sequence of integers satisfying $a_{n+1} = -[\lambda a_n] - a_{n-1}$, $n = 1, 2, \ldots$. It is conjectured in [1] that, for any $a_0, a_1 \in \mathbb{Z}$ and $\lambda \in [-2, 2]$, the sequence $a_n$, $n = 0, 1, 2, \ldots$ is periodic. The nontrivial case is when $\lambda \in (-2, 2) \setminus \{ -1, 0, 1 \}$. This problem seems to be very difficult, especially, when the number $\zeta$, defined by the equality $\zeta + \zeta^{-1} = -\lambda$ (so that $|\zeta| = 1$), is not a root of unity. In fact, the only case when the periodicity of the sequence $a_n$, $n = 0, 1, 2, \ldots$, is proved and published [1] is when $\lambda = (1 + \sqrt{5})/2 = 2 \cos(\pi/5)$, so that $\zeta$ corresponding to $\lambda$ is a root of unity. It seems that similar methods can be applied to some other $\lambda$ of the form $2 \cos(\pi r)$ with $r \in \mathbb{Q}$. However, for $\lambda \neq 2 \cos(\pi r)$, i.e., when $\zeta$ is not a root of unity, the periodicity problem seems to be completely out of reach.

We now explain how this periodicity problem is related to polynomials with coefficients in $[0, 1]$ and, in particular, with Newman polynomials.
recurrence equation as \( a_{j+1} + \lambda a_j + a_{j-1} = \{\lambda a_j\} \). Multiplying each equality by \( \zeta^j \) and adding all obtained equalities for \( j = 1, \ldots, n \), using \( \zeta + \zeta^{-1} = -\lambda \), we get

\[
(a_{n+1} - \zeta a_n)\zeta^n = \sum_{j=1}^{n} \{\lambda a_j\} \zeta^j + (a_1 - \zeta a_0).
\]

Put \( r_n := |a_{n+1} - \zeta a_n| \). Then

\[
|r_n| \leq \left| \sum_{j=1}^{n} (\lambda a_j) \zeta^j \right| + |r_0| = \left| \sum_{j=1}^{n} \{\lambda a_j\} \zeta^{j-1} \right| + |r_0|.
\]

One can show easily (see Proposition 2.4 in [1]) that the periodicity of the sequence \( a_n, n = 0, 1, 2, \ldots \), would follow from the inequality

\[
\limsup_{n \to \infty} \frac{r_n}{n} < \frac{\sqrt{1 - \lambda^2}}{\pi}.
\]

The sum \( \sum_{j=1}^{n} \{\lambda a_j\} \zeta^{j-1} \) is equal to the value at \( \zeta \) of a certain polynomial of degree \( \leq n - 1 \) whose coefficients are all in the interval \([0, 1)\). This suggests the problem of finding the maximum \( S(\zeta, n) \) over all degree \( \leq n - 1 \) polynomials with coefficients in the interval \([0, 1]\) at a fixed point of the unit circle \( \zeta \). We shall prove below that \( \lim_{n \to \infty} S(\zeta, n)/n = 1/\pi \) for every \( \zeta \) of modulus 1 which is not a root of unity, so that \( \limsup_{n \to \infty} |r_n|/n \leq 1/\pi \) which is too weak to solve the above problem of periodicity.

Finally, let us consider the case \( \lambda = 1/2 \). Then \( \zeta = (-1 + i\sqrt{15})/4 \) satisfying \( \zeta + \zeta^{-1} = -1/2 \) is not a root of unity. We claim that the sequence \( a_n, n = 0, 1, 2, \ldots \), defined by \( a_{n+1} = -[a_n/2] - a_{n-1}, n = 1, 2, \ldots \), contains at least four equal elements. Indeed, without loss of generality suppose that the sequence \( |a_n|, n = 0, 1, 2, \ldots \), is unbounded. Then, for any \( N \in \mathbb{N} \), there is an index \( n > N \) such that \( |a_n| \geq |a_j| \) for \( j = 0, 1, \ldots, n - 1 \). The corresponding polynomial \( f(z) := \sum_{j=1}^{n} \{a_j/2\}z^{j-1} \) is a Newman polynomial multiplied by \( 1/2 \). The inequality

\[
|r_n| = |a_{n+1} - \zeta a_n| \leq |f(\zeta)|/2 + |a_1 - \zeta a_0|
\]

combined with the inequality \( |a_{n+1} - \zeta a_n| \geq |3(\zeta a_n)| = |a_n|\sqrt{15}/4 \) implies that

\[
|a_n| \leq 2|f(\zeta)|/\sqrt{15} + 4|a_1 - \zeta a_0|/\sqrt{15}.
\]

Hence, by Theorem 4 below, for any \( \varepsilon > 0 \) and any sufficiently large \( n > n(\varepsilon) \), we have \( |a_n| < (2/(\pi\sqrt{15}) + \varepsilon)n < 0.165n \). The interval \([-0.165n, 0.165n] \) contains at most 0.33\( n \) + 1 < 0.33\( n \) < \( n/3 \) distinct integers. Since \( |a_n| \geq |a_j| \), \( j = 0, 1, \ldots, n - 1 \), it includes all integers \( a_0, a_1, \ldots, a_n \). If none of them is repeated more than three times then the set \( \{a_0, a_1, \ldots, a_n\} \) is of cardinality \( \geq (n + 1)/3 \geq n/3 \), a contradiction.
2. Main results

Let $\Lambda_n$ be the set of real polynomials of degree $\leq n - 1$ whose coefficients all lie in the interval $[0, 1]$. Set

$$ S(\zeta, n) := \max_{f \in \Lambda_n} |f(\zeta)| $$

for any $\zeta \in \mathbb{C}$. It is clear that

$$ S(\zeta, n) = 1 + \zeta + \cdots + \zeta^{n-1} $$

for each nonnegative real number $\zeta$.

We remark first that, for any fixed $\zeta \in \mathbb{C}$, the maximum $S(\zeta, n)$ is attained for some polynomial $f(z) = c_0 + c_1 z + \cdots + c_{n-1} z^{n-1} \in \Lambda_n$. Indeed, treating $f(\zeta)$ as a complex continuous function in $n$ real variables $c_0, \ldots, c_{n-1} \in [0, 1]$, by a standard argument of compactness, we see that its modulus $|f(\zeta)|$ attains its maximum for some fixed values of the coefficients $c_0, \ldots, c_{n-1} \in [0, 1]$. It follows that, for any $\zeta \in \mathbb{C}$, there exist a (not necessarily unique) polynomial $f \in \Lambda_n$ such that $S(\zeta, n) = |f(\zeta)|$.

Below, we sometimes use the vector representation of complex numbers. Let us denote the value $f(\zeta)$ whose modulus $|f(\zeta)|$ is the largest among all $f \in \Lambda_n$ by the vector $s$. As we already said above, the vector $s$ satisfying $|s| = S(\zeta, n)$ is not necessarily unique. We begin with the following simple, but important observation:

**Theorem 1.** Let $\zeta \neq 0$, and let $s = f(\zeta) = \sum_{j=0}^{n-1} c_j \zeta^j$ be one of the vectors of maximal length, where $f \in \Lambda_n$. Then $f$ is a Newman polynomial. Moreover, for each $j = 0, 1, \ldots, n - 1$, we have $c_j = 1$ if the projection of the vector $\zeta^j$ to the vector $s$ is positive, and $c_j = 0$ otherwise.

In particular, if $s$ is one of the extremal vectors, then the line passing through the origin and orthogonal to $s$ contains none of the points $1, \zeta, \ldots, \zeta^{n-1}$. Therefore, Theorem 1 suggests the following practical method for the computation of $S(\zeta, n)$. Suppose that $\zeta \neq 0$. Let $\ell$ be any line passing through the origin but through none of the $n$ points $D_n := \{1, \zeta, \ldots, \zeta^{n-1}\}$. Let us rotate the line $\ell$, say, counterclockwise until it reaches at least one of the points of $D_n$. Then rotate $\ell$ again by an angle so small that no point of $D_n$ lies on $\ell$ and stop. At this, first, stop we calculate the sums $r_1$ and $l_1$ of the numbers from $D_n$ that lie on both sides, say, ‘right hand side’ and ‘left hand side’ of $\ell$. (Note that $r_1 + s_1 = 1 + \zeta + \cdots + \zeta^{n-1}$.) Then rotate $\ell$ until it reaches at least one point of $D_n$ again, slightly pass this point, stop for the second time, and calculate $r_2, l_2$, where $r_2 + l_2 = 1 + \zeta + \cdots + \zeta^{n-1}$, and so on. The last, say, kth stop will be when $\ell$ is rotated by the angle $\pi$, so that it reaches its original position (but changes its direction). It is easy to see that $k \leq n$, where the value $n$ for $k$ is attained when no two points of $D_n$ lie on a line passing through the origin. Theorem 1 implies that

$$ S(\zeta, n) = \max \{ |r_1|, |l_1|, |r_2|, |l_2|, \ldots, |r_k|, |l_k| \}. $$
In particular, if \( \zeta \) is a negative real number, then all of its powers are positive and negative real numbers. Let us start with a line, say, orthogonal to the real axis and begin the process described above. Then there is only one stop, giving 
\[ r_1 = 1 + \zeta^2 + \cdots + \zeta^v, \]
where \( u \leq n - 1 \) is the largest even integer, and 
\[ l_1 = -\zeta - \zeta^3 - \cdots - \zeta^v, \]
where \( v \leq n - 1 \) is the largest odd integer. The formula 
\[ S(\zeta, n) = \max [\{|r_1|, |l_1|\}] \] 
yields the following corollary:

**Corollary 2.** Let \( u \) and \( v \) be the largest even and odd numbers, respectively, satisfying \( u, v \leq n - 1 \). If \( \zeta \) is a negative real number then 
\[ S(\zeta, n) = \max (1 + \zeta^2 + \cdots + \zeta^v, -\zeta(1 + \zeta^2 + \cdots + \zeta^{v-1})). \]

Suppose that \( \zeta \) is a complex number of modulus 1. In the evaluation of 
\[ S(\zeta, n) \] 
there are two different cases depending on whether \( \zeta \) is or is not a root of unity. Let throughout \( \zeta_d := \exp(2\pi i/d) \) be a primitive \( d \)-th root of unity. Let also \( U_d \) be the set of its conjugates over \( \mathbb{Q} \), so that \( |U_d| = \varphi(d) \), where \( \varphi(d) \) stands for the Euler totient function. In the next theorem, we calculate the value \( S(\zeta, md) \) for every \( \zeta \in U_d \) and \( m \in \mathbb{N} \).

**Theorem 3.** Suppose that \( m \in \mathbb{N} \) and \( \zeta \in U_d \), where \( d \geq 2 \). Then 
\[ S(\zeta, md) = m/\sin(\pi/d) \] 
if \( d \) is even and 
\[ S(\zeta, md) = m/(2\sin(\pi/2d)) \] 
if \( d \) is odd.

The main theorem of this paper can be stated as follows:

**Theorem 4.** Let \( \zeta \in \mathbb{C} \) be a complex number of modulus 1. If \( \zeta \in U_d \), where \( d \in \mathbb{N} \), then
\[ \lim_{n \to \infty} S(\zeta, n)/n = \begin{cases} 1, & \text{if } d = 1, \\ 1/(d\sin(\pi/d)) & \text{if } d \text{ is even,} \\ 1/(2d\sin(\pi/2d)) & \text{if } d > 1 \text{ is odd.} \end{cases} \]
If \( \zeta \) is not a root of unity, then 
\[ \lim_{n \to \infty} S(\zeta, n)/n = 1/\pi. \]

In the next section, we shall prove Theorems 1, 3 and 4. Some numerical examples will be given in Section 4.

3. Proofs

**Proof of Theorem 1.** The vector \( \mathbf{s} \) is the sum of the vectors \( \zeta^j \), where \( j = 0, \ldots, n - 1 \), scaled by \( c_j \in [0, 1] \). Clearly, \( |\mathbf{s}| > 0 \). Put \( \mathbf{s}_j := \zeta^j \). If there is an index \( j \in \{0, \ldots, n - 1\} \) such that the projection of \( \mathbf{s}_j = \zeta^j \) to \( \mathbf{s} \) is positive (i.e., the scalar product \( (\mathbf{s}_j, \mathbf{s}) \) is positive) and \( c_j < 1 \) then, by replacing \( c_j \) by 1, we obtain that the vector \( \mathbf{s} - c_j \mathbf{s}_j + \mathbf{s}_j = \mathbf{s} + (1 - c_j)\mathbf{s}_j \) has greater length than \( |\mathbf{s}| \), a contradiction. Similarly, suppose that there is an index \( j \in \{0, \ldots, n - 1\} \) such that the projection of \( \mathbf{s}_j = \zeta^j \) to \( \mathbf{s} \) is negative or zero (i.e., \( (\mathbf{s}_j, \mathbf{s}) \leq 0 \)) and \( c_j > 0 \). Then, by replacing \( c_j \) by 0, we obtain that the vector \( \mathbf{s} - c_j \mathbf{s}_j \) has greater length than \( |\mathbf{s}| \), because 
\[ |\mathbf{s} - c_j \mathbf{s}_j|^2 - |\mathbf{s}|^2 = c_j^2|\mathbf{s}_j|^2 - 2c_j (\mathbf{s}_j, \mathbf{s}) \geq c_j^2|\mathbf{s}_j|^2 > 0, \]
a contradiction again. \( \square \)
The following simple lemma will be used in the proof of Theorem 3 and in numerical examples of Section 4:

**Lemma 5.** Let $\Gamma_d$ be the set of complex roots of $z^d - 1 = 0$, where $d \geq 2$, and let $\ell$ be a line passing through the origin but through none of the points of $\Gamma_d$. Then the sum of all numbers from $\Gamma_d$ that lie on one side of $\ell$ belongs to some axis of symmetry of a regular $d$-gon with vertices in $\Gamma_d$, and the modulus of this sum is equal to $1 / \sin(\pi/d)$ for $d$ even, and to $1 / (2 \sin(\pi/2d))$ for $d$ odd.

**Proof.** Consider a half plane in that side of $\ell$, where exactly $k = [d/2]$ points of $\Gamma_d$ are lying. Take $\zeta_d = \exp(2\pi i/d)$. Let $r$ be the smallest positive integer such that $\zeta_r$ is the first vertex of $\Gamma_d$ in that half plane counterclockwise. Then the points of $\Gamma_d$ in this half plane are the powers $\zeta_d^j$, where $j = r, \ldots, r + k - 1$. Note that all sums $\zeta_d^r + \zeta_d^{r+k-1-j}$, where $j = 0, \ldots, [(k-1)/2]$, lie on the same axis of symmetry of a regular $d$-gon, hence so does their sum $\sum_{j=r}^{r+k-1} \zeta_d^j = \frac{1}{2} \sum_{j=0}^{k-1} (\zeta_d^r + \zeta_d^{r+k-1-j})$ on the same side of $\ell$.

Next, recall that $1 + \zeta_d + \cdots + \zeta_d^{d-1} = 0$. Hence on both sides of $\ell$ we get the sums lying on the same axis of symmetry whose moduli are

$$|1 + \zeta_d + \cdots + \zeta_d^{d/2-1}| = |(\zeta_d^{d/2} - 1)/(\zeta_d - 1)| = \frac{\sin(\pi[d/2]/d)}{\sin(\pi/d)}.$$

This is equal to $\frac{1}{\sin(\pi/d)}$ for $d$ even, and to $\frac{\cos(\pi/2d)}{\sin(\pi/2d)} = \frac{1}{2 \sin(\pi/2d)}$ for $d$ odd. \hfill $\Box$

**Proof of Theorem 3.** Suppose that $\zeta \in U_d$, where $d \geq 2$ is an integer. Since $\zeta^d = 1$, we can write the value $f(\zeta)$ of the polynomial $f \in \Lambda_{md}$ at $z = \zeta$ as

$$f(\zeta) = f_1(\zeta) + \cdots + f_m(\zeta),$$

where $f_1, \ldots, f_m \in \Lambda_d$. Hence $S(\zeta, md) \leq mS(\zeta, d)$. Moreover, if $f_0 \in \Lambda_d$ is a polynomial for which $S(\zeta, d) = |f_0(\zeta)|$ then, by setting $f(z) := f_0(z)(1 + z^d + \cdots + z^{(m-1)d}) \in \Lambda_{md}$, we find that $f(\zeta) = m f_0(\zeta)$. Hence $S(\zeta, md) = mS(\zeta, d)$. It remains to show that $S(\zeta, d) = 1 / \sin(\pi/d)$ if $d$ is even and $S(\zeta, d) = 1 / (2 \sin(\pi/2d))$ if $d > 1$ is odd.

Let $f$ be a Newman polynomial of degree $\leq d - 1$ for which we have $S(\zeta, d) = |f(\zeta)|$. Put $s = f(\zeta)$. By Theorem 1, $s$ is the sum of all numbers $\zeta_j$, where $j \in \{0, \ldots, d - 1\}$, that lie on one side of a line $\ell$ orthogonal to $s$ but not on $\ell$ itself. Moreover, none of the points $\zeta_j$ lies on $\ell$. Since $\zeta \in U_d$, the set $\{\zeta_j : j = 0, \ldots, d - 1\}$ is precisely the set of roots of $z^d - 1$, i.e., $\Gamma_d$. By Lemma 5, $|s| = 1 / \sin(\pi/d)$ for $d$ even and $|s| = 1 / (2 \sin(\pi/2d))$ for $d > 1$ odd. This completes the proof of the theorem. \hfill $\Box$

**Proof of Theorem 4.** The case $\zeta = 1$ is obvious. The maximal sum is $1 + \zeta + \cdots + \zeta^{n-1}$, so $S(1, n) = n$ for every positive integer $n$. Suppose that $\zeta \in U_d$ with $d \geq 2$. Choose an integer $m$ such that $md \leq n < (m+1)d$. Since $S(\zeta, n)$ is a nondecreasing function in $n$, we have $S(\zeta, md) \leq S(\zeta, n) \leq S(\zeta, (m+1)d)$. 

\[
S(\zeta, (m+1)d) \leq S(\zeta, (m+1)d) = \frac{\sin(\pi/(md+1))}{\sin(\pi/2d)} \leq \frac{1}{2 \sin(\pi/2d)} \leq \frac{1}{2 \sin(\pi/(2d)).}
\]
Thus, by Theorem 3, for even \( d \geq 2 \), we have

\[
\frac{1 - d/n}{d \sin(\pi/d)} = \frac{n/d - 1}{n \sin(\pi/d)} < \frac{m}{n \sin(\pi/d)} = \frac{S(\zeta, md)}{n} \leq \frac{S(\zeta, n)}{n} \leq \frac{M}{n}.
\]

It follows that \( \lim_{n \to \infty} S(\zeta, n)/n = 1/(d \sin(\pi/d)) \) for each even \( d \geq 2 \). The proof of the case when \( d > 1 \) is odd is similar: one just uses the ‘odd’ part of Theorem 3 instead of its ‘even’ part.

Finally, suppose that \( \zeta = e^{i\phi} \), where \( 0 < \phi < 2\pi \), is a complex number of modulus 1 which is not a root of unity. Then \( \phi/\pi \not\in \mathbb{Q} \). Suppose that \( s = f(\zeta) = \sum_{j=0}^{n-1} c_j\zeta^j \) is one of the vectors of maximal length. Then, by Theorem 1, \( c_j \in \{0,1\} \) with \( c_j = 1 \) if and only if the projection of \( \zeta^j \) to \( s \) is positive. Let \( \ell \) be the line passing through the origin and orthogonal to \( s = |s|e^{i\tau} \). The line \( \ell \) divides the complex plane into two half planes. Let us divide the open half plane with the point \( e^{i\tau} \) into \( 2M \) equal sectors, where for each \( k \in \{-M, -1, 1, \ldots, M\} \) the \( k \)th sector consists of complex numbers whose arguments belong to the interval \( [\tau + \pi(k-1)/2M, \tau + \pi k/2M) \) for \( k > 0 \) and to the interval \( [\tau + \pi k/2M, \tau + \pi(k+1)/2M) \) for \( k < 0 \). (Since this half plane needs to be open, one exception is that the interval corresponding to \( k = -M \) is open \( [\tau - \pi/2, \tau - \pi(M - 1)/2M) \).)

For any \( j \in \{0,1,\ldots,n-1\} \) the vector \( \zeta^j \) is belongs to the sum \( s \) if and only if it lies in one of the above \( 2M \) sectors. The sum of the vectors \( \zeta^j = \cos(j\phi) + i \sin(j\phi) \) is \( f(\zeta) = s = |s|e^{i\tau} \), hence \( f(\zeta)e^{-i\tau} \) is a real number. Using the fact that the number

\[
f(\zeta)e^{-i\tau} = \sum_{j=0}^{n-1} c_j\zeta^j e^{-i\tau} = \sum_{j=0}^{n-1} c_j(\cos(j\phi - \tau) + i \sin(j\phi - \tau))
\]

is real, we obtain that \( \sum_{j=0}^{n-1} c_j \sin(j\phi - \tau) = 0 \), so

\[
|f(\zeta)| = f(\zeta)e^{-i\tau} = \sum_{j=0}^{n-1} c_j \cos(j\phi - \tau).
\]

Suppose that the sector corresponding to the index \( k \) contains \( n_k \) vectors of the set \( \{1, \ldots, n-1\} \), say, \( \zeta^j \) with \( j \in N_k \), where \( N_k \) is a subset of \( \{0,1,\ldots,n-1\} \) of cardinality \( n_k \). Then \( \sum_{j \in N_k} \cos(j\phi - \tau) \) is at least \( n_k \cos(|k|\pi/2M) \) and at most \( n_k \cos((|k| - 1)\pi/2M) \). It follows that

\[
\sum_{k=1}^{M} (n_k + n_{-k}) \cos(k\pi/2M) \leq |f(\zeta)| \leq \sum_{k=1}^{M} (n_k + n_{-k}) \cos((k - 1)\pi/2M).
\]

By an old result of Weyl \([9]\) (see, e.g., Example 2.1 in \([5]\)), the sequence of fractional parts \( \{m\phi/2\pi\}, m = 0, 1, 2, \ldots \), is uniformly distributed in the
interval \([0, 1)\), because \(\phi/2\pi \notin \mathbb{Q}\). Fix \(\varepsilon > 0\). Then fix any \(M = M(\varepsilon) \in \mathbb{N}\) satisfying
\[
\frac{1}{4M} \left( 1 + \frac{1}{\tan(\pi/4M)} \right) < 1 + \varepsilon \quad \text{and} \quad \frac{1}{4M} \left( -1 + \frac{1}{\tan(\pi/4M)} \right) > 1 - \varepsilon.
\]
Such an \(M\) exists, because \(\lim_{x \to \infty} x \tan(\pi/x) = \pi\). Given \(k \in \{1, \ldots, M\}\), \(\zeta^j\) belongs to the \(k\)th sector if and only if there is an \(l \in \mathbb{Z}\) such that
\[
\tau + \pi(k-1)/2M < j\phi - 2\pi l < \tau + \pi k/2M,
\]
i.e., \((k-1)/4M \leq j\phi/2\pi - \pi/2 < k/4M\). Using uniform distribution of \(\{j\phi/2\pi - \pi/2\}, j = 0, 1, \ldots\), in \([0, 1)\), we deduce that \((1 - \varepsilon)n/4M < n_k < (1 + \varepsilon)n/4M\) for each sufficiently large \(n \in \mathbb{N}\). The same bounds hold for \(k \in \{-M, \ldots, -1\}\). Hence
\[
(1 - \varepsilon) \frac{n}{2M} \sum_{k=1}^{M} \cos(k\pi/2M) \leq |f(\zeta)| \leq (1 + \varepsilon) \frac{n}{2M} \sum_{k=1}^{M} \cos((k-1)\pi/2M).
\]
Setting \(x = \pi/2M\) into the identity
\[
1/2 + \cos(x) + \cdots + \cos((M-1)x) = \frac{\sin((M-1/2)x)}{2\sin(x/2)},
\]
we derive that
\[
\sum_{k=1}^{M} \cos((k-1)\pi/2M) = \frac{1}{2} \left( 1 + \frac{1}{\tan(\pi/4M)} \right)
\]
and
\[
\sum_{k=1}^{M} \cos(k\pi/2M) = \frac{1}{2} \left( -1 + \frac{1}{\tan(\pi/4M)} \right).
\]
Hence
\[
(1 - \varepsilon) \frac{n}{4M} \left( -1 + \frac{1}{\tan(\pi/4M)} \right) \leq |f(\zeta)| \leq (1 + \varepsilon) \frac{n}{4M} \left( 1 + \frac{1}{\tan(\pi/4M)} \right).
\]
By the choice of \(M\), this implies that \((1 - \varepsilon)^2 n/\pi \leq |f(\zeta)| \leq (1 + \varepsilon)^2 n/\pi\). Thus
\[
(1 - \varepsilon)^2/\pi \leq S(\zeta, n)/n = |f(\zeta)/n| \leq (1 + \varepsilon)^2/\pi
\]
for each \(n \geq n(\varepsilon)\). However, \(\varepsilon\) can be arbitrarily small, so \(\lim_{n \to \infty} S(\zeta, n)/n = 1/\pi\), as claimed.

4. Practical computations

Take \(\zeta = \exp(2\pi i/5)\) and \(n = 5\). By Lemma 5, we can take any \(\ell\) which goes through none of the roots of \(z^5 - 1 = 0\). Take \(\ell\) such that 1 and \(\zeta\) are on one of its sides. Then, by Lemma 5, we find that \(|1 + \zeta| = 1/(2\sin(\pi/10)) = (1 + \sqrt{5})/2 = 1.61803\ldots\).

Similarly, taking \(\zeta = \exp(9\pi i/7)\) to be one of the roots of \(z^{14} - 1 = 0\) and \(n = 14\), one can choose \(\ell\) to be the imaginary axis. Then one of the
extremal Newman polynomials will be \( f(z) = 1 + z^3 + z^5 + z^6 + z^8 + z^9 + z^{11} \), because 0, 3, \ldots, 11 are the only powers of \( \zeta \) that are on the right hand side of \( \ell \). Lemma 5 and Theorem 3 gives \( f(\zeta) = 1/\sin(\pi/14) = 4.43935 \ldots \).

Take \( \zeta = i \) and \( n = 5 \). By Theorem 1, there are four possible quadrants for the location of \( \mathbf{s} \). The maximum for \( |f(i)| \) is attained by Newman polynomials \( 1 + z + z^4 \) and \( 1 + z^3 + z^4 \), giving \( \mathbf{s} = 2 \pm i \). Hence \( S(i,5) = \sqrt{5} \). Note that the maximal vectors \( 2 \pm i \) do not lie on an axis of symmetry of the square with vertices \( 1, i, -1, -i \). So Lemma 5 does not hold, because there is one ‘double’ vector \( 1 = i^4 \).

It seems likely that when \( \zeta \) is not a root of unity one cannot expect any simple formulae for \( S(\zeta, n) \). For example, for \( \zeta \) satisfying \( \zeta^2 - \zeta/2 + 1 = 0 \), we calculated the value \( S(\zeta,100) = 31.8928 \ldots \). It is easy to see that \( S(\zeta,100)/100 = 0.31892 \ldots \) is quite close to the limit value \( 1/\pi = 0.31830 \ldots \), given by Theorem 4. The value \( S(\zeta,100) \) is attained by the polynomial \( f(z) = z^{97} + z^{96} + z^{95} + z^{92} + z^{91} + z^{90} + z^{87} + z^{86} + z^{82} + z^{81} + z^{78} + z^{77} + z^{76} + z^{73} + z^{72} + z^{71} + z^{68} + z^{67} + z^{66} + z^{62} + z^{58} + z^{57} + z^{54} + z^{53} + z^{52} + z^{49} + z^{48} + z^{44} + z^{43} + z^{39} + z^{38} + z^{35} + z^{34} + z^{33} + z^{30} + z^{29} + z^{28} + z^{25} + z^{24} + z^{20} + z^{19} + z^{16} + z^{15} + z^{14} + z^{11} + z^{10} + z^9 + z^6 + z^5 + z + 1 \).

Finally, we remark that the results of this paper may be applied to polynomials whose coefficients lie in any real interval \([a,b]\). In this case, if \( \zeta \neq 1 \), the constant factor \( b - a \) will appear on the right hand side of the formulas established by Theorems 3 and 4. Indeed, any polynomial \( f(z) = \sum_{j=0}^{n-1} c_j z^j \) with coefficients \( c_j \in [a,b] \) can be written as

\[
 f(z) = (b - a) g(z) + ah(z),
\]

where \( g(z) = \sum_{j=0}^{n-1} ((c_j - a)/(b - a)) z^j \) is a polynomial with coefficients in \([0,1]\) and \( h(z) = 1 + \cdots + z^{n-1} = (z^n - 1)/(z - 1) \). Now, \( h(\zeta) = 0 \) if \( \zeta \neq 1 \) is an \( n \)th root of unity. Furthermore, \( |h(\zeta)| \) is bounded by an absolute constant depending on \( \zeta \) only if \( |\zeta| \leq 1 \) and \( \zeta \neq 1 \), so that \( |h(\zeta)|/n \to 0 \) as \( n \to \infty \). Taking \( n = d \), Theorem 3 may be applied immediately to \( g(z) \). To obtain a corresponding limit in Theorem 4, one can divide the equality by \( n \), and then let \( n \to \infty \).

**Acknowledgements.** We thank the referee of this paper for pointing out an error in the first version of the paper. This research was supported in part by the Lithuanian State Studies and Science Foundation.

**References**


Artūras Dubickas
Department of Mathematics and Informatics
Vilnius University
Naugarduko 24, Vilnius LT-03225, Lithuania
E-mail address: arturas.dubickas@mif.vu.lt

Jonas Jankauskas
Department of Mathematics and Informatics
Vilnius University
Naugarduko 24, Vilnius LT-03225, Lithuania
E-mail address: jonas.jankauskas@gmail.com