ON THE LINEAR INDEPENDENCE OF
THE SET OF DIRICHLET EXPONENTS

ARTŪRAS DUBICKAS

Abstract

Given \( k \geq 2 \) let \( z_1, \ldots, z_k \) be transcendental numbers such that \( z_1, \ldots, z_{k-1} \) are algebraically independent over \( \mathbb{Q} \) and \( z_k \notin \mathbb{Q}(z_1, \ldots, z_{k-1}) \), but \( z_k = (az_i + c)/b \) for some \( i \in \{1, \ldots, k-1\} \) and some \( a, b \in \mathbb{N}, \ c \in \mathbb{Z} \) satisfying \( \gcd(a, b) = 1 \). We prove that then there exists a nonnegative integer \( q \) such that the set of so-called Dirichlet exponents \( \log(n + a_j) \), where \( n \) runs through the set of all nonnegative integers for \( j = 1, \ldots, k-1 \) and \( n = q, q + 1, q + 2, \ldots \) for \( j = k \), is linearly independent over \( \mathbb{Q} \). As an application we obtain a joint universality theorem for corresponding Hurwitz zeta functions \( \zeta(s, a_1), \ldots, \zeta(s, a_k) \) in the strip \( \{ s \in \mathbb{C} : 1/2 < \Re(s) < 1 \} \). In our approach we follow a recent result of Mishou who analyzed the case \( k = 2 \).

1. Introduction

For any given complex number \( \alpha \notin \{0, -1, -2, -3, \ldots\} \) we consider the set

\[ \mathcal{D}(\alpha) := \{ \log \alpha, \log(1 + \alpha), \log(2 + \alpha), \ldots \}, \]

where \( \log \) stands for the principal branch of the natural logarithm. The set \( \mathcal{D}(\alpha) \) is known as the set of Dirichlet exponents of the Hurwitz zeta function

\[ \zeta(s, \alpha) := \sum_{n=0}^{\infty} \frac{1}{(n + \alpha)^s} = \sum_{n=0}^{\infty} e^{-s \log(n + \alpha)}, \]

where \( \alpha \) is a real number in the interval \((0, 1)\). More generally, for each integer \( q \geq 0 \) let us denote

\[ \mathcal{D}_q(\alpha) := \{ \log(q + \alpha), \log(q + 1 + \alpha), \log(q + 2 + \alpha), \ldots \}, \]

so that \( \mathcal{D}_0(\alpha) = \mathcal{D}(\alpha) \).

Recall that a (finite or infinite) set of complex numbers \( V \) is linearly dependent over \( \mathbb{Q} \) if there exist some \( m \in \mathbb{N} \), distinct \( v_1, \ldots, v_m \in V \) and nonzero
$r_1, \ldots, r_m \in \mathbb{Q}$ such that $\sum_{j=1}^m r_jv_j = 0$ and linearly independent otherwise. Obviously, if $x$ is a transcendental number then the set $D(x)$ is linearly independent over $\mathbb{Q}$. The set $D(x)$ for algebraic $x$ have been studied by Cassels [3] (see also [4] and [5]). The question if there is an algebraic number $x$ for which the set $D(x)$ is linearly independent over $\mathbb{Q}$ is still open (see [4], [5] and also [7], [12]). A finite set of distinct complex numbers $v_1, \ldots, v_m$ is algebraically dependent over $\mathbb{Q}$ if there is a nonzero polynomial $P(z_1, \ldots, z_m) \in \mathbb{Q}[z_1, \ldots, z_m]$ such that $P(v_1, \ldots, v_m) = 0$ and algebraically independent otherwise.

The main result of this note is the following:

**Theorem 1.** Let $k \geq 2$ be an integer and let $x_1, \ldots, x_{k-1}, x_k$ be some transcendental numbers. Suppose that the numbers $x_1, \ldots, x_{k-1}$ are algebraically independent over $\mathbb{Q}$ and $x_k \in \mathbb{Q}(x_1, \ldots, x_{k-1})$, and suppose for each $i = 1, \ldots, k-1$ we have $x_k \neq (ax_i + c)/b$ for $a, b, c \in \mathbb{N}$, $c \in \mathbb{Z}$ satisfying $\text{gcd}(a, b) = 1$. Then there is an integer $q \geq 0$ such that set of Dirichlet exponents

$$D(x_1) \cup \cdots \cup D(x_{k-1}) \cup D_q(x_k)$$

is linearly independent over $\mathbb{Q}$.

Following the result of Nesterenko [15], the numbers $\pi$ and $e^\pi$ are algebraically independent over $\mathbb{Q}$, so Theorem 1 can be applied to the numbers

$$x_1 := \pi = 3.14159 \ldots, \quad x_2 := e^\pi = 23.14069 \ldots,$$

$$x_3 := x_1^2 + x_2 = \pi^2 + e^\pi = 33.01029 \ldots.$$

Note that the condition $x_k \neq (ax_i + c)/b$ for integers $a > 0$, $b > 0$ and $c$ satisfying $\text{gcd}(a, b) = 1$ cannot be removed from Theorem 1. Indeed, if $x_k = (ax_i + c)/b$ with some $i \in \{1, \ldots, k-1\}$ and $a$, $b$, $c$ as above then there exists $d \in \mathbb{N}$ for which $u := (bd + c)/a$ is a positive integer. Thus for each $N \in \mathbb{N}$ we have the identity

$$\frac{x_k + d + aN}{x_k + d + a(N-1)} = \frac{ax_i + c + b(d + aN)}{ax_i + c + b(d + aN - a)} = \frac{x_i + u + bN}{x_i + u + b(N-1)}.$$

Consequently, the four logarithms $\log(x_k + d + aN)$, $\log(x_k + d + a(N-1))$, $\log(x_i + u + bN)$, $\log(x_i + u + b(N-1))$ are linearly dependent over $\mathbb{Q}$, and hence the set $D_q(x_i) \cup D_q(x_k)$ is linearly dependent over $\mathbb{Q}$ for any $q \in \mathbb{N}$.

As an application of Theorem 1 we shall prove the following joint universality theorem for Hurwitz zeta functions. (Throughout, $\mu(A)$ stands for the Lebesgue measure of the set $A \subseteq \mathbb{R}$.)

**Theorem 2.** Let $x_1, x_2, \ldots, x_k$, $k \geq 2$, be real transcendental numbers in the interval $(0,1)$ such that for some integers $q_1, q_2, \ldots, q_k \geq 0$ the set of Dirichlet exponents

$$D_{q_1}(x_1) \cup D_{q_2}(x_2) \cup \cdots \cup D_{q_k}(x_k)$$


is linearly independent over $\mathbb{Q}$. For each $j$ in the range $1 \leq j \leq k$ let $K_j$ be a compact subset of the strip \( \{ s \in \mathbb{C} : 1/2 < \Re(s) < 1 \} \) with connected complement and let $f_j(s)$ be a continuous function on $K_j$ which is analytic in the interior of $K_j$. Then for any $\varepsilon > 0$ we have

\[
\liminf_{T \to \infty} \frac{1}{T} \mu \left\{ \tau \in [0, T] : \max_{1 \leq j \leq k} \max_{s \in K_j} |\zeta(s + i\tau, x_j) - f_j(s)| < \varepsilon \right\} > 0.
\]

The subject of “universality” for Dirichlet $L$-functions started with the paper of Voronin [16], where he proved that for every positive number $\varepsilon$ and every continuous non-vanishing function $f(s)$ in the disc $|s| \leq r$, where $0 < r < 1/4$, which is analytic in $|s| < r$ there exists a number $\tau = \tau(\varepsilon)$ for which $\max_{|s| \leq r} |\zeta(s + 3/4 + i\tau) - f(s)| < \varepsilon$. So certain shifts of zeta function are arbitrarily close to every analytic function. Later, this result have been extended to other $L$-functions and it was shown that the set of those $\tau$ for which the shift of the $L$-function by $i\tau$ approximates $f(s)$ has positive density; see, e.g., [8], [9] for some references on this. In particular, for the Hurwitz zeta function $\zeta(s, \alpha)$ it was shown that if $\alpha \in (0, 1/2) \cup (1/2, 1)$ is either rational or transcendental number then for any function $f(s)$ which is continuous in a compact set $K \subset \{ s \in \mathbb{C} : 1/2 < \Re(s) < 1 \}$ with connected complement and analytic in the interior of $K$ we have

\[
\liminf_{T \to \infty} \frac{1}{T} \mu \left\{ \tau \in [0, T] : \max_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} > 0
\]

for any given $\varepsilon > 0$ (see [1], [6]).

Later, certain joint universality theorems when instead of one function $f$ we have several analytic functions $f_1, \ldots, f_k$ and approximate them with some shifts of $\zeta(s, x_j)$, $j = 1, \ldots, k$, were obtained in [2], [11], etc. In particular, the joint universality theorem which asserts the conclusion (1) of Theorem 1 under assumption that all $k$ transcendental numbers $x_1, \ldots, x_k$ are algebraically independent follows from the results of Nakamura in [14]. Laurinčikas proved the same statement under weaker assumption that the set of Dirichlet exponents $D(x_1) \cup \cdots \cup D(x_k)$ is linearly independent over $\mathbb{Q}$ (see [10]). This corresponds to the case $q_1 = \cdots = q_k = 0$ in Theorem 2.

The above mentioned result of Nakamura covers the case when the transcendence degree $\text{trdeg}(\mathbb{Q}(x_1, \ldots, x_k)/\mathbb{Q})$ of the field extension $\mathbb{Q}(x_1, \ldots, x_k)/\mathbb{Q}$ (i.e., the largest cardinality of an algebraically independent subset of $\mathbb{Q}(x_1, \ldots, x_k)$ over $\mathbb{Q}$) is equal to $k$. On the other hand, when $k \geq 3$ and $2 \leq r \leq k - 1$ the next simple example

\[
x_1 := \alpha, \quad x_2 := \alpha/r, \quad x_3 := (\alpha + 1)/r, \ldots, x_{r+1} := (\alpha + r - 1)/r,
\]

where $\alpha$ and $x_j$, $j = r + 1, \ldots, k$, are algebraically independent transcendental numbers in the interval $(0, 1)$ (so that $\text{trdeg}(\mathbb{Q}(x_1, \ldots, x_k)/\mathbb{Q}) = k - r \leq k - 2$), shows that the first $r + 1$ Hurwitz zeta functions are linearly dependent

\[
r^j \zeta(s, x_1) = \zeta(s, x_2) + \cdots + \zeta(s, x_{r+1}).
\]
Therefore, no joint universality theorem holds for these $k$ Hurwitz zeta functions $\zeta(s, x_j)$, $j = 1, \ldots, k$. Theorem 2 deals with the remaining case when the transcendence degree of the field extension $Q(x_1, \ldots, x_k)/Q$ is equal to $k - 1$. The case $k = 2$ was recently analyzed by Mishou [13]. We will follow his approach. It seems likely that the conclusion (1) is true for any distinct transcendental numbers $x_1, \ldots, x_k \in (0, 1)$ for which $\text{trdeg}(Q(x_1, \ldots, x_k)/Q) = k - 1$.

2. Proof of Theorem 1

Recall that if $P(z_1, \ldots, z_m) = \sum_{i} p_i z_1^{i_1} \cdots z_m^{i_m} \in \mathbb{C}[z_1, \ldots, z_m]$, where $i = (i_1, \ldots, i_m)$ and $p_i \in \mathbb{C}\setminus\{0\}$, is a nonzero polynomial then its leading coefficient is the coefficient $p_i$ for $z_1^{i_1} \cdots z_m^{i_m}$ such that the vector $j = (j_1, \ldots, j_m)$ is the largest lexicographically among all vectors $i = (i_1, \ldots, i_m)$ with maximal sum $i_1 + \cdots + i_m = \deg P$. For instance, the leading coefficient of the polynomial $P(z_1, z_2) = z_1^4 + 2z_1z_2^4 + 3z_2^5 - z_1z_2$ is equal to 2.

**Lemma 3.** Suppose that for $m \in \mathbb{N}$ two nonzero polynomials with integer coefficients $P(z_1, \ldots, z_m)$ with positive leading coefficient and $Q(z_1, \ldots, z_m)$, not both constants, are relatively prime. Then there exist infinitely many positive integers $t$ for which

\[(2) \quad P(z_1, \ldots, z_m) + tQ(z_1, \ldots, z_m) = A \prod_{i \in I} (z_i + a_{ij}),
\]

where $I$ is a nonempty subset of the set $\{1, \ldots, m\}$, $A$ is a nonzero integer and $a_{ij} \in \mathbb{N} \cup \{0\}$ (where $a_{ij}$ are not necessarily distinct), if and only if there are $i \in \{1, \ldots, m\}$, $a, b \in \mathbb{N}$, $c \in \mathbb{Z}$, $\gcd(a, b) = 1$ for which $P(z_1, \ldots, z_m) = az_i + c$ and $Q(z_1, \ldots, z_m) = b$.

**Proof.** For $m = 1$ the lemma was proved by Mishou in [13]. Our proof is different from that given in [13] and works for any $m \in \mathbb{N}$.

The lemma is trivial in one direction. If $P(z_1, \ldots, z_m) = az_i + c$ and $Q(z_1, \ldots, z_m) = b$ with $a$, $b$, $c$ as above then there are infinitely many $t \in \mathbb{N}$ for which $c + bt \geq 0$ and $a | (c + bt)$. For each of those $t$ the representation (2) for the polynomial

\[P(z_1, \ldots, z_m) + tQ(z_1, \ldots, z_m) = az_i + c + bt = a(z_i + (c + bt)/a)
\]

holds with $A = a$, $I = \{i\}$ and $\prod_{i \in I} (z_i + a_{ij}) = z_i + (c + bt)/a$.

Assume now that $P, Q \in \mathbb{Z}[z_1, \ldots, z_m]$, not both constants, are relatively prime, and the leading coefficient of $P$ is positive. Assume that there exist infinitely many positive integers $t$ for which (2) holds with $A = A(t) \in \mathbb{Z}\setminus\{0\}$ and $a_{ij} = a_{ij}(t) \in \mathbb{N} \cup \{0\}$. It is clear that the coefficients of the polynomial

\[(3) \quad R_t(z_1, \ldots, z_m) := A(t) \prod_{i \in I} (z_i + a_{ij}(t))
\]
on the right hand side of (2) all have the form \( ut + v \) with some integers \( u, v \) lying in a finite set \( V \). By the condition of the lemma, the nonzero coefficients of \( R_t/A(t) \) are all positive. So if two nonzero coefficients, say \( r_1(t) \) for \( z_1^t \cdots z_m^t \) and \( r_2(t) \) for \( z_1^{t_1} \cdots z_m^{t_m} \), of the polynomial \( R_t \) are unbounded then \( r_1(t) = ut + v \) and \( r_2(t) = u't + v' \) with some integers \( u, u' \neq 0 \). It follows that the modulus of their quotient \( |r_1(t)/r_2(t)| \) is bounded in terms of \( t \). The fact that the quotient of two unbounded coefficients of \( R_t \) must be bounded will be used below several times.

Now we shall prove that all the zeros \(-a_j(t)\) of the polynomial \( R_t \) given in (3) are unbounded in terms of \( t \). For a contradiction assume that \( a_j(t) \) for some fixed pair \( i, j \) is bounded and assume without restriction of generality that \( i = m \). Then \( 0 \leq a_{mj}(t) \leq K \) for certain \( K \in \mathbb{N} \). Since \( a_{mj}(t) \) can only take \( K + 1 \) values, we must have \( a_{mj}(t) = a^* \) for some fixed \( a^* \in \{0, 1, \ldots, K\} \) and infinitely many \( t \in \mathbb{N} \). Thus the factor \( z_m + a^* \) occurs in all those polynomials \( R_t = P + tQ \) defined in (2) corresponding to those \( t \). Then the polynomial

\[
R_t(z_1, \ldots, z_{m-1}, -a^*) = P(z_1, \ldots, z_{m-1}, -a^*) + tQ(z_1, \ldots, z_{m-1}, -a^*)
\]

is zero identically. Thus \( Q(z_1, \ldots, z_{m-1}, -a^*) \) must be the zero polynomial. It follows that \( P(z_1, \ldots, z_{m-1}, -a^*) \) is also the zero polynomial. Hence \( Q(z_1, \ldots, z_m) \) and \( P(z_1, \ldots, z_m) \) are both divisible by the same factor \( z_m + a^* \), a contradiction. This proves that all the zeros \(-a_j(t)\) of \( R_t \) in (3) are unbounded, i.e. \( a_j(t) \to \infty \) as \( t \to \infty \). Since \( A(t) \in \mathbb{Z}\setminus\{0\} \), in view of (3) it follows that all the nonzero coefficients of \( R_t \) are also unbounded except possibly for the leading coefficient \( A(t) \).

Next, if the leading coefficient \( A(t) \) is unbounded then \( A(t) \) and \( A(t) \prod_{i \in I} \prod_j a_j(t) \) are two unbounded coefficients of \( R_t \), which is impossible, because their quotient \( \prod_{i \in I} \prod_j a_j(t) \) tends to infinity as \( t \to \infty \). (Recall that, by the fact established above, the quotient of two unbounded coefficients of \( R_t \) must be bounded.) So \( A(t) \) is bounded. Hence the leading coefficient \( A(t) \) of \( R_t = P + tQ \) must be that of \( P \). This yields \( A(t) = a \), where \( a > 0 \) is the leading coefficient of \( P \).

Suppose next that for infinitely many \( t \in \mathbb{N} \) the product

\[
R_t(z_1, \ldots, z_m) = a \prod_{i \in I} \prod_j (z_i + a_j(t))
\]

contains exactly \( r \geq 2 \) not necessarily distinct factors with the same \( i \), say \( z_i + a_{i1}(t), \ldots, z_i + a_{ir}(t) \). Put \( B = B(t) \) for the constant term of the polynomial \( R_t(z_1, \ldots, z_r)/\prod_{j=1}^r(z_i + a_{ij}(t)) \). Then both \( B(t) \prod_{j=1}^r a_j(t) \) and \( B(t) \sum_{j=1}^r a_j(t) \) are the coefficients of the polynomial \( R_t \) corresponding to its constant term and the term for \( z_i^{r-1} \), respectively. They are both unbounded, so their quotient \( \prod_{j=1}^r a_j(t)/\sum_{j=1}^r a_j(t) \) must be bounded. This is not the case, because all \( a_j(t) \) are unbounded, so the product of \( r \geq 2 \) terms \( \prod_{j=1}^r a_j(t) \) divided by their sum \( \sum_{j=1}^r a_j(t) \) tends to infinity as \( t \to \infty \).
The only remaining possibility is that \( R_i(z_1, \ldots, z_m) = a \prod_{i \in I} (z_i + a_i(t)) \) for infinitely many \( t \in \mathbb{N} \). In case \( |I| \geq 2 \) we see that the constant coefficient of \( R_i \) is equal to \( a \prod_{i \in I} a_i(t) \) and the coefficient for \( z_i \), where \( l \in I \), is equal to \( a \prod_{i \in I \setminus \{l\}} a_i(t) \). They both are unbounded, because \( |I| \geq 2 \). But their quotient \( a_i(t) \) is also unbounded, a contradiction.

It follows that \( |I| = 1 \) and thus \( R_i(z_1, \ldots, z_m) = a(z_i + a_i(t)) \) for some \( i \in \{1, \ldots, m\} \) and infinitely many \( t \in \mathbb{N} \). From \( P + tQ = R_i = az_i + aa_i(t) \) we conclude that \( P(z_1, \ldots, z_m) = az_i + c \), where \( a \in \mathbb{N} \), \( c \in \mathbb{Z} \) and \( Q(z_1, \ldots, z_m) = b \neq 0 \). Then

\[
P + tQ = az_i + c + tb = a(z_i + (c + tb)/a)
\]

has the required form only when \( b > 0 \) and \( a \) divides \( c + bt \) for infinitely many \( t \in \mathbb{N} \). From the equality \( at_1 - bt = c \), where \( t_1 \in \mathbb{Z} \), we see that such positive integers \( t \) exist if and only if \( \gcd(a,b) \) divides \( c \). However, if \( \gcd(a,b) > 1 \) and \( \gcd(a,b) \) divides \( c \) then the polynomials \( P = az_i + c \) and \( Q = b \) are divisible by \( \gcd(a,b) > 1 \), and so they are not relatively prime. Consequently, we must have \( \gcd(a,b) = 1 \). Hence \( P(z_1, \ldots, z_m) = az_i + c \) for some \( i \in \{1, \ldots, m\} \) and \( Q(z_1, \ldots, z_m) = b \) with \( a, b \in \mathbb{N} \), \( c \in \mathbb{Z} \) and \( \gcd(a,b) = 1 \), as claimed in the statement of the lemma.

Now we can give the proof of Theorem 1. Assume that the set

\[
\mathcal{D}(x_1) \cup \cdots \cup \mathcal{D}(x_{k-1}) \cup \mathcal{D}_q(x_k)
\]

is linearly dependent over \( Q \). Since the sets \( \mathcal{D}(x_1) \cup \cdots \cup \mathcal{D}(x_{k-1}) \) and \( \mathcal{D}_q(x_k) \) are both linearly independent over \( Q \), writing \( x_k = P(x_1, \ldots, x_{k-1})/Q(x_1, \ldots, x_{k-1}) \) with two relatively prime polynomials \( P, Q \) in \( \mathbb{Z}[z_1, \ldots, z_{k-1}] \) we must have

\[
\prod_{i \in I} \prod_j (z_i + n_j)^{u_{ij}} = \prod_j (P(x_1, \ldots, x_{k-1})/Q(x_1, \ldots, x_{k-1}) + n_j)^{u_{ij}}
\]

for some \( I \subseteq \{1, \ldots, k-1\} \), \( n_{ij}, n_j \in \mathbb{N} \cup \{0\} \), \( n_j \geq q \) and \( u_{ij}, u_j \in \mathbb{Z} \setminus \{0\} \). Of course, \( P \) and \( Q \) are not both constants, because \( x_k \) is transcendental. Also, without restriction of generality, by multiplying both \( P \) and \( Q \) by \(-1\) if necessary, we may assume that the leading coefficient of \( P \) is positive.

Since the numbers \( x_1, \ldots, x_{k-1} \) are algebraically independent, the equality (4) must be the identity, namely,

\[
\prod_{i \in I} \prod_j (z_i + n_j)^{u_{ij}} = \prod_j (P(z_1, \ldots, z_{k-1})/Q(z_1, \ldots, z_{k-1}) + n_j)^{u_{ij}}.
\]

Note that the polynomials \( P + n_jQ \) and \( P + n_lQ \) with \( n_j \neq n_l \) can have only constant common factor, since \( P \) and \( Q \) are relatively prime. Hence selecting any \( n_j \geq q \) on the right hand side of (5) we see that the corresponding polynomial \( P(z_1, \ldots, z_{k-1}) + n_j Q(z_1, \ldots, z_{k-1}) \) must be a constant multiplied by certain product \( \prod_{i \in I} (z_i + n_{i_l})^{v_{i_l}} \), where \( I_1 \subseteq I \), \( n_{i_l} \in \mathbb{N} \cup \{0\} \) and \( v_{i_l} \in \mathbb{N} \). However, by Lemma 3, this is impossible for \( q \) large enough whenever \( (P, Q) \neq (az_i + c, b) \).
with $a$, $b$, $c$ as in Lemma 3. This completes the proof of Theorem 1, since the condition of the theorem and that of the lemma which exclude the case $P(z_1, \ldots, z_{k-1}) = az_j + c$, $Q(z_1, \ldots, z_{k-1}) = b$, where $i \in \{1, \ldots, k-1\}$, $a, b \in \mathbb{N}$, $c \in \mathbb{Z}$ and $\gcd(a, b) = 1$, are the same.

3. Proof of Theorem 2

Assume that the set of Dirichlet exponents
\[ \mathcal{D}_q(x_1) \cup \cdots \cup \mathcal{D}_q(x_k) \]
is linearly independent over $\mathbb{Q}$. Evidently, its subset
\[ \mathcal{D}_q(x_1) \cup \cdots \cup \mathcal{D}_q(x_k) , \]
where $q := \max_{1 \leq j \leq k} q_j$, is linearly independent over $\mathbb{Q}$ too. Take a maximal subset $M_1$ of the finite set $\bigcup_{j=1}^{k} (\mathcal{D}(x_j) \setminus \mathcal{D}_q(x_j))$ for which the set
\[ \mathcal{D}_1 := M_1 \cup \mathcal{D}_q(x_1) \cup \cdots \cup \mathcal{D}_q(x_k) \]
is linearly independent over $\mathbb{Q}$. This means that each of the $qk - |M_1|$ remaining logarithms $\log(n + x_j) \not\in \mathcal{D}_1$, where $0 \leq n \leq q - 1$ and $1 \leq j \leq k$, is a linear combination with rational coefficients of some elements of $\mathcal{D}_1$. (Of course, the choice of the set $M_1$ is not necessarily unique.)

Fix an integer $m \geq q$ such that each of the logarithms $\log(n + x_j) \not\in \mathcal{D}_1$ is expressible in the form
\[ \log(n + x_j) = \sum_{i=1}^{k} \sum_{r=0}^{m-1} c_{j,n,r,i} \log(i + x_r) \]
with $c_{j,n,r,i} \in \mathbb{Q}$. (Some of the coefficients $c_{j,n,r,i}$ can be zeros.) Therefore, by increasing $q$ to $m$ if necessary and adding more logarithms to the set $M_1$ we may assume that each $\log(n + x_j)$ which is not in the set
\[ \mathcal{D} := M \cup \mathcal{D}_m(x_1) \cup \cdots \cup \mathcal{D}_m(x_k) , \]
where
\[ M := M_1 \cup \{ \log(q + x_1), \ldots, \log(m - 1 + x_1) \} \cup \cdots \]
\[ \cup \{ \log(q + x_k), \ldots, \log(m - 1 + x_k) \} , \]
is a linear combination of at most $km$ logarithms of the set $M$. Obviously, there exists a positive integer $\ell$ such that for each $\log(n + x_j) \not\in \mathcal{D}$ we have the representation
\[ \ell \log(n + x_j) = \sum_{\log(i + x_j) \in M} c_{i,r} \log(i + x_r) \]
with $c_{i,r} \in \mathbb{Z}$. 

Let $K_j$ be the sets and let $f_j(s)$ be the functions described in Theorem 2. Fix $\varepsilon > 0$. Let $K$ be a simply connected compact subset of the strip $\{ s \in \mathbb{C} : 1/2 < \Re(s) < 1 \}$ such that the union $\bigcup_{j=1}^{k} K_j$ is included in the interior of $K$. By Mergelyan’s theorem (see Lemma 5 in [13]), there exist polynomials with complex coefficients $p_j(s)$, $j = 1, \ldots, k$, such that

$$\max_{1 \leq j \leq k} \max_{s \in K_j} |f_j(s) - p_j(s)| < \varepsilon. \quad (7)$$

By Gonek’s lemma (see Lemma 7, (29) and (30) in [13]), there is a large positive integer $v > m$ such that for each sufficiently large integer $t$ and each $j = 1, \ldots, k$ we have

$$\max_{s \in K} \left| p_j(s) - \sum_{0 \leq n < v} \frac{1}{(n + \alpha_j)^{\gamma}} - \sum_{v \leq n \leq t} \frac{\exp(2\pi i\theta_{n,j})}{(n + \alpha_j)^{\gamma}} \right| < \varepsilon$$

with some $\theta_{n,j} \in \mathbb{R}$. Selecting $\theta_{n,j} = 0$ for $n = m, \ldots, v$, we can rewrite the above inequality in the form

$$\max_{s \in K} \left| p_j(s) - \sum_{0 \leq n < m} \frac{1}{(n + \alpha_j)^{\gamma}} - \sum_{m \leq n \leq t} \frac{\exp(2\pi i\theta_{n,j})}{(n + \alpha_j)^{\gamma}} \right| < \varepsilon. \quad (8)$$

For $\delta > 0$ let $B_T(\delta)$ be a set of those $t \in [T, 2T]$ for which

$$\|-(\tau/2\pi) \log(n + \alpha_j) - \theta_{n,j}\| \leq \delta \quad \text{when } m \leq n \leq t, 1 \leq j \leq k$$

and

$$\|-(\tau/2\pi) \log(n + \alpha_j)\| \leq \delta \quad \text{when } \log(n + \alpha_j) \in M.$$

Observe that in view of (6) the second inequality implies that for each sufficiently small $\delta$ there is a positive constant $c_0$ (which depends on $\ell$, $M$ and the coefficients $c_{t,r}$ in $qm - |M|$ equalities (6)) such that

$$\|-(\tau/2\pi) \log(n + \alpha_j)\| \leq c_0 \delta \quad \text{for each } n = 0, 1, \ldots, m - 1 \text{ and each } j = 1, \ldots, k. \quad (9)$$

Since the logarithms involved in the definition of $B_T(\delta)$ are linearly independent over $\mathbb{Q}$, by Kronecker’s theorem (see Lemma 6 in [13]), the Lebesgue measure of the set $B_T(\delta)$ satisfies

$$\mu(B_T(\delta)) \sim \varepsilon_1 T \quad \text{as } T \to \infty, \quad \text{where } \varepsilon_1 := (2\delta)^{k(t-m+1)+|M|}. \quad (10)$$

For each $j = 1, \ldots, k$ and each $t \in B_T(\delta)$ we have

$$\max_{s \in K} \left| \sum_{m \leq n \leq t} \frac{\exp(2\pi i\theta_{n,j})}{(n + \alpha_j)^{\gamma}} - \sum_{m \leq n \leq t} \frac{1}{(n + \alpha_j)^{\gamma+t}} \right| < \varepsilon$$

whenever $\delta$ is small enough. Similarly, by (9), we obtain

$$\max_{s \in K} \left| \sum_{0 \leq n < m} \frac{1}{(n + \alpha_j)^{\gamma}} - \sum_{0 \leq n < m} \frac{1}{(n + \alpha_j)^{\gamma+t}} \right| < \varepsilon.$$
when $\delta$ is small enough. Combined with (8) this gives

$$\max_{1 \leq j \leq k} \max_{s \in K} \left| p_j(s) - \frac{1}{\sum_{0 \leq n \leq t} (n + \alpha_j)^{\tau+it}} \right| < 3\epsilon. \tag{11}$$

The next two inequalities are standard and can be obtained by considering the second moments of the involved functions. Firstly, for any pair of positive numbers $\epsilon_1$, $\epsilon_2$ and a set

$$A_T(\epsilon, z) := \left\{ \tau \in [T, 2T] : \max_{1 \leq j \leq k} \max_{s \in K} \left| \zeta(s + i\tau, \alpha_j) - \frac{1}{\sum_{0 \leq n \leq z} (n + \alpha_j)^{\tau+it}} \right| < \epsilon \right\} \tag{12}$$

we have

$$\lim_{T \to \infty} \inf \frac{\mu(A_T(\epsilon, z))}{T} > 1 - \epsilon_2 \tag{13}$$

for each sufficiently large $z$ (see Lemma 9 in [13]). Secondly, let $C_T(\delta)$ be a subset of $B_T(\delta)$ for which the inequality

$$\max_{1 \leq j \leq k} \max_{s \in K} \left| \sum_{0 \leq n \leq z} (n + \alpha_j)^{\tau+it} \right| < \epsilon \tag{14}$$

holds uniformly for $z > t$. Then (see Lemma 11 in [13] and (10)) for each sufficiently large $t$ we have

$$\lim_{T \to \infty} \inf \frac{\mu(C_T(\delta))}{T} > \frac{1}{2} \lim_{T \to \infty} \frac{\mu(B_T(\delta))}{T} = \frac{\epsilon_1}{2}.$$

Hence selecting $\epsilon_2 = \epsilon_1/4$ in (13) we obtain

$$\lim_{T \to \infty} \inf \frac{\mu(A_T(\epsilon, z) \cap C_T(\delta))}{T} > \frac{\epsilon_1}{4}$$

for each sufficiently large $z$. Finally, for $\tau \in A_T(\epsilon, z) \cap C_T(\delta)$ combining (7), (11), (12), (14) we find that

$$\max_{1 \leq j \leq k} \max_{s \in K} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < 7\epsilon.$$

This completes the proof of Theorem 2.

\section*{References}


ON THE LINEAR INDEPENDENCE OF THE SET OF DIRICHLET EXPO


Artūras Dubickas
DEPARTMENT OF MATHEMATICS AND INFORMATICS
VILNIUS UNIVERSITY
NAUGARDUKO 24, VILNIUS LT-03225
LITHUANIA
E-mail: arturas.dubickas@mif.vu.lt