ADDITIVE BASES OF POSITIVE INTEGERS AND RELATED PROBLEMS

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ABSTRACT. Let $A$ be an infinite set of nonnegative integers and let $k \geq 2$ be an integer. We investigate the relation between the number of representations of an integer $n$ by sums of the form $a_1 + \cdots + a_k$, where $a_1, \ldots, a_k \in A$, and the size of $A$. Some related problems are also considered.

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1. Introduction

Let $A$ be an infinite set of nonnegative integers and let $k \geq 2$ be an integer. We will denote by $R_k(A, n)$ the number of representations of $n$ in the form $n = a_1 + a_2 + \cdots + a_k$, where $a_1, a_2, \ldots, a_k \in A$. Similarly, let $r_k(A, n)$ denote the number of such representations of $n$ by ordered $k$-tuples, namely, $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \leq a_2 \leq \ldots \leq a_k$. Setting

$$f(z) = \sum_{j \in A} z^j,$$

we have

$$f(z)^k = \sum_{n=0}^{\infty} R_k(A, n) z^n. \quad (1)$$

Evidently,

$$r_k(A, n) \leq R_k(A, n) \leq k! r_k(A, n). \quad (2)$$

A set $A$ is called a base of $\mathbb{N}$ of order $k$ if $r_k(A, n) \geq 1$ for each $n \in \mathbb{N}$. By (3), the condition $r_k(A, n) \geq 1$ is equivalent to $R_k(A, n) \geq 1$. An old conjecture of Erdős and Turán [4] asserts that if $R_2(A, n) \geq 1$ for each sufficiently large $n$ then $\limsup_{n \to \infty} R_2(A, n) = \infty$. In other words, it says that for no positive integer $v$ we have $R_2(A, n) \in [1, v]$ for all sufficiently large $n$.

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Although this Erdős-Turán conjecture (also known as one of USD 500 problems in [3]) remains open, there are several nontrivial results concerning it. Firstly, Grekos, Haddad, Helou and Pihko [5] proved that the numbers $R_2(A, n)$, $n \geq 0$, cannot all lie in the interval $[1, 5]$. This was later extended to the interval $[1, 7]$ by Borwein, Choi and Chu [1]. Recently, Sándor [10] showed that if $v = \limsup_{n \to \infty} R_2(A, n)$ then $\liminf_{n \to \infty} R_2(A, n) \leq (\sqrt{v - 1})^2$.

It seems likely that for every integer $k \geq 2$ we must have either $\limsup_{n \to \infty} R_k(A, n) = \infty$ or $\liminf_{n \to \infty} R_k(A, n) = 0$.

Of course, such a result, if proved, would imply the Erdős-Turán conjecture (which is the corresponding statement for $k = 2$). In other words, we ask whether there is an interval $[1, v]$ and an integer $n_0$ such that $R_k(A, n) \in [1, v]$ for each $n \geq n_0$. In view of (3) this problem is equivalent to the question on whether there are some integers $n_0, b > 0$ such that $r_k(A, n) \in [1, b]$ for every $n \geq n_0$.

The following result describes the size of the set $A$ provided that $r_k(A, n)$ is bounded from below or from above. Here and subsequently, $A(n)$ stands for the number of elements of the set $A \cap [0, n]$.

**Theorem 1.** Let $k \geq 2$ be an integer. If $r_k(A, n) \geq a$ for each $n \geq n_0$ then $A(n) \geq (ak!n)^{1/k} - k + 1$ for every sufficiently large $n$. If $r_k(A, n) \leq b$ for each $n \geq n_1$ then $A(n) \leq (bk^2(k - 1)!n)^{1/k} + 1$ for every sufficiently large $n$. Finally, if $r_k(A, n) \leq 1$ for each $n \geq 1$ then

$$A(n) \leq \begin{cases} (((k/2)((k/2)!))^2n)^{1/k} + O(n^{1/(2k)}) & \text{for } k \text{ even}, \\ (((k + 1)/2)!)^2n)^{1/k} + O(n^{1/(2k)}) & \text{for } k \text{ odd}. \end{cases}$$

In particular, Theorem 1 combined with (3) implies that

$$a^{1/k} \leq \liminf_{n \to \infty} A(n)n^{-1/k} \leq \limsup_{n \to \infty} A(n)n^{-1/k} \leq (bk^2(k - 1)!)^{1/k}$$

provided that $R_k(A, n) \in [a, b]$ for each sufficiently large $n$.

In this context, Sándor’s result [10] can be easily (and by the same method as in [10]) generalized as follows:

**Theorem 2.** Let $k \geq 2$ be an integer. If $\limsup_{n \to \infty} R_k(A, n) < \infty$ then

$$\limsup_{n \to \infty} R_k(A, n)^{1/2} - \liminf_{n \to \infty} R_k(A, n)^{1/2} \geq 1.$$

By (1) and (2), Theorem 2 deals with coefficients $b_{n,k}$ of the series

$$\sum_{n=0}^{\infty} b_{n,k}^n z^n = \sum_{n=0}^{\infty} b_{n,k}^n z^n.$$
where \(b_n \in \{0, 1\}\). In Section 4, we will give an example of such series with \(b_n > 0\) for each \(n \geq 0\) such that \(\lim_{n \to \infty} b_{n,k} = \gamma > 0\), so no analogue of Theorem 2 holds if the condition \(b_n \in \{0, 1\}\) is replaced by \(b_n > 0\). We also ask for the smallest value of the quotient of the largest and the smallest coefficients that appear in the \(k\)th power of a polynomial \((\sum_{n=0}^{d} b_n z^n)^k = \sum_{n=0}^{d} b_{n,k} z^n\), where \(b_0, \ldots, b_d \geq 0\) and \(b_0, b_d > 0\). It seems that such a question has not been considered earlier, although it seems quite natural. Moreover, its version with Newman polynomials (those with coefficients in \(\{0, 1\}\)) seems to be naturally related to the Erdős-Turán conjecture.

2. Proof of Theorem 1

Put \(m = A(n)\). The number of ordered \(k\)-tuples \((a_1, \ldots, a_k) \in A^k\), where \(0 \leq a_1 \leq \ldots \leq a_k \leq n\), is equal to \(m(m+1) \ldots (m+k-1)/k!\). Since \(r_k(A, N) \geq a\) for \(N \geq m_0\) and \(a_{m+1} > n\), every integer \(N \in [n_0, n]\) is expressible by the sum of at least \(a\) ordered \(k\)-tuples as above. Hence

\[
m(m+1) \ldots (m+k-1)/k! \geq a(n-n_0+1).
\]

Note that \(m(m+1) \ldots (m+k-1) < (m+k-3/2)^k\). Consequently, \((m+k-3/2)^k > ak!(n-n_0+1)\) giving \(m+k-3/2 > (ak!(n-n_0+1))^{1/k}\). This yields the required inequality \(m > (ak!n)^{1/k} - k + 1\) for \(n\) large enough.

We remark that the above proof follows that of Nathanson [9] for \(k = 2\) and \(a = 1\). It seems, however, that his proof of Theorem 4 gives the inequality \(A(0, x) \geq \sqrt{2x} - 1\) only instead of \(A(0, x) \geq 2\sqrt{x} - 1\) as claimed in [9]. (In the notation of [9], \(A(0, x)\) is the number of elements of the set \(A\) in \([0, x]\). One can only claim that \((k^2 + k)/2 \geq x - n_0\), because \(n > x\) can be represented as \(a + a'\) with \(a' > x\).) A corrected inequality \(A(0, x) \geq \sqrt{2x} - 1\) of [9] is exactly the first part of our Theorem 1 with \(k = 2\) and \(a = 1\). Of course, this implies that \(\liminf_{n \to \infty} A(n)/\sqrt{n} \geq \sqrt{2}\) for each set \(A\) which is an asymptotic basis of \(N\) of order 2. We stress that a slightly better constant for liminf replaced by limsup, namely, \(\limsup_{n \to \infty} A(n)/\sqrt{n} \geq 2\sqrt{2}/\pi\) for each \(A\) which is an asymptotic basis of \(N\) of order 2 follows from Theorem 3.3 in [6].

For the second part, suppose that \(m = A(n)\) and \(r_k(A, N) \leq b\) for \(N \geq n_1\). Put \(B = \max_{0 \leq n \leq n_1 - 1} r_k(A, n)\). As above, the number of ordered \(k\)-tuples \((a_1, \ldots, a_k) \in A^k\), where \(0 \leq a_1 \leq \ldots \leq a_k \leq n\), is equal to \(m(m+1) \ldots (m+k-1)/k!\) and \(a_{m+1} > n\). Note that the sum \(a_1 + \cdots + a_k\) lies in the interval \([0, kn]\). Hence

\[
m^k/k! < m(m+1) \ldots (m+k-1)/k! \leq Bn_1 + b(kn - n_1 + 1).
\]

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This yields \( m = A(n) < (bk^2(k - 1)!n)^{1/k} + 1 \) for \( n \) sufficiently large.

Recall that a set of integers \( E \) is called a \( B_k[1] \) set if all possible sums of \( k \) (not necessarily distinct) elements of \( E \) are distinct. Since \( r_k(A,j) \leq 1 \) for all \( j \geq 0 \), the set \( A \cap [0, n] \) is a \( B_k[1] \) set. However, the largest such set which is a subset of \( \{0, 1, \ldots, n\} \) contains at most \( ((k/2)(k/2))^{n/2} + O(n^{1/(2k)}) \) elements for \( k \) even (see [7], [8]) and at most \( (((k + 1)/2)!n)^{1/k} + O(n^{1/(2k)}) \) elements for \( k \) odd (see [2]). This gives the required bound for \( A(n) \).

### 3. Proof of Theorem 2

The result is obvious if \( \lim \inf_{n \to \infty} R_k(A,n)^{1/2} = 0 \), because \( A \) is infinite and so \( R_k(A,n) \geq 1 \) for infinitely many \( n \)’s.

Assume that \( \lim \inf_{n \to \infty} R_k(A,n)^{1/2} > 0 \), but the required inequality does not hold. Then there exist a positive integer \( n_0 \) and two positive numbers \( u, v \) such that \( 0 < u < v < \infty, v > 1, v - u < 1 \) and \( u \leq R_k(A,n)^{1/2} < v \) for every \( n \geq n_0 \).

The inequality \( v - u < 1 \) implies that \( v^2 < v^2 + u \). Take \( w = (v^2 - v + u^2 + u)/2 \). Then \( 0 < w - u^2 < u \) and \( 0 < u^2 - w < v \), hence \( (u^2 - w)^2 < u^2 \) and \( (v^2 - w)^2 < v^2 \). It follows that there is a positive number \( \epsilon_0 \) such that for each \( y \in [v^2, v^2] \) we have \( (y - w)^2 < (1 - \epsilon_0)y \). Hence, as \( u^2 \leq R_k(A,n) \leq v^2 \) for \( n \geq n_0 \), we obtain

\[
(R_k(A,n) - w)^2 < (1 - \epsilon_0)R_k(A,n).
\]

Let \( r \) be a fixed number satisfying \( 1/2 < r < 1 \). Consider the integral

\[
I = \int_0^1 |f(re^{2\pi it}) - w|^{2r}e^{2\pi i t n}dt = \int_0^1 |f(re^{2\pi it}) - \frac{w}{1-re^{2\pi i t}}|dt.
\]

Subtracting \( w/(1 - z) \) from both sides of (2) and substituting \( z = re^{2\pi it} \) with \( t \in [0, 1] \), we obtain

\[
f(re^{2\pi it}) - w/(1-re^{2\pi it}) = \sum_{n=0}^{\infty} (R_k(A,n) - w)r^n e^{2\pi i t n}.
\]

Hence \( I = \int_0^1 \sum_{n=0}^{\infty} b_n r^n e^{2\pi i t n} |dt| \). Applying the inequality of Cauchy-Schwarz \( \int_0^1 |g(t)|^2 dt \leq \left( \int_0^1 |g(t)|^2 dt \right)^{1/2} \) and the Parseval identity

\[
\int_0^1 \sum_{n=0}^{\infty} b_n e^{2\pi i t n} |dt| = \sum_{n=0}^{\infty} |b_n|^2,
\]

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we find that $I^2 \leq \sum_{n=0}^{\infty} (R_k(A,n) - w)^2 r^{2n}$. Estimating the sum of all terms in the range $0 \leq n \leq n_0 - 1$ by the absolute constant

$$c_1 = n_0 \max_{0 \leq n \leq n_0 - 1} (R_k(A,n) - w)^2$$

and each of the terms $n \geq n_0$ by (5), we obtain the inequality

$$I^2 \leq c_1 + (1 - \varepsilon_0) \sum_{n=n_0}^{\infty} R_k(A,n) r^{2n}.$$

Using (2) with $z = r^2$ we further have $I^2 \leq c_1 + (1 - \varepsilon_0) f(r^2)^k$. This yields

$$I \leq c_2 + (1 - \varepsilon_0/2) f(r^2)^{k/2}.$$

(Here and below, $c_1, c_2, \ldots$ are some positive constants that do not depend on $r$.)

Next, we shall estimate $I$ from below using

$$I \geq \int_0^1 |f(re^{2\pi it})|^k dt - w \int_0^1 \frac{dt}{|1-re^{2\pi it}|}.$$

Firstly, by (1) and (6), we have

$$\int_0^1 |f(re^{2\pi it})|^2 dt = \int_0^1 \left| \sum_{j \in A} r^j e^{2\pi i tj} \right|^2 dt = \sum_{j \in A} r^{2j} = f(r^2).$$

Combining this with the inequality $\int_0^1 |g(t)|^2 dt \leq \left( \int_0^1 |g(t)|^k dt \right)^{2/k}$, where $k \geq 2$ (which follows from Hölder’s inequality), we have

$$\int_0^1 |f(re^{2\pi it})|^k dt \geq \left( \int_0^1 |f(re^{2\pi it})|^2 dt \right)^{k/2} = f(r^2)^{k/2}.$$

Note that $|1-re^{2\pi it}| = ((1-r)^2 + 4r \sin^2(\pi t))^{1/2}$. So $|1-re^{2\pi it}| \geq 1 - r$ in the interval $0 \leq t \leq 1 - r$ and $|1-re^{2\pi it}| \geq 2\sqrt{r}\sin(\pi t) \geq 4\sqrt{r}t > 2t$ in the interval $1 - r \leq t \leq 1/2$, because $\sin x \geq 2x/\pi$ for each $x \in [0,\pi/2]$. It follows that

$$\int_0^1 \frac{dt}{|1-re^{2\pi it}|} = 2 \int_0^{1/2} \frac{dt}{|1-re^{2\pi it}|} \leq 2 \int_0^{1-r} \frac{dt}{1-r} + 2 \int_{1-r}^{1/2} \frac{dt}{2t}$$

$$= 2 + \log \frac{1}{2-2r} < 2 + \log \frac{1}{1-r}.$$

Consequently,

$$I \geq f(r^2)^{k/2} - w \left( 2 + \log \frac{1}{1-r} \right).$$
Combining this inequality with (7), we obtain
\[ f(r^2)^{k/2} - w \left( 2 + \log \frac{1}{1 - r} \right) \leq c_2 + (1 - \varepsilon_0/2)f(r^2)^{k/2}. \]
So
\[ \varepsilon_0 f(r^2)^{k/2} \leq c_3 + 2w \log \frac{1}{1 - r}. \]
Select \( r = 1 - 1/T \) with a large integer \( T \) to be chosen later. Then
\[ \varepsilon_0 f(1 - 2/T)^{k/2} \leq \varepsilon_0 f(r^2)^{k/2} \leq c_3 + 2w \log \frac{1}{1 - r} \leq c_4 \log T. \]
From \( R_k(A, n) \geq u^2 \) for \( n \geq n_0 \) using (4) we deduce that \( A(n) > c_5 n^{1/k} \). Thus \( A(T) > c_5 T^{1/k} \) for each sufficiently large integer \( T \). Hence, by (1),
\[ f(1 - 2/T) = \sum_{j \in A} (1 - 2/T)^j > \sum_{j \in A, j \leq T} (1 - 2/T)^j \]
\[ \geq A(T)(1 - 2/T)^T > c_6 T^{1/k}. \]
It follows that
\[ c_4 \log T \geq \varepsilon_0 f(1 - 2/T)^{k/2} > \varepsilon_0(c_6 T^{1/k})^{k/2} > \varepsilon_0 c_7 T^{1/2}, \]
which is a contradiction for \( T \) large enough.

4. Some related problems

Let us consider the series
\[ f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)^{1-1/k}}. \]
Then
\[ f(z)^k = \sum_{n=0}^{\infty} b_{n,k} z^n, \]
where
\[ b_{n,k} = \sum_{j_1, \ldots, j_k \geq 0, j_1 + \cdots + j_k = n} ((j_1 + 1)(j_2 + 1) \cdots (j_k + 1))^{-1+1/k}. \]
Write \( b_{n,k} \) in the form
\[ b_{n,k} = \frac{1}{n^{k-1}} \sum_{j_1, \ldots, j_k \geq 1} \left( \frac{j_1 + 1}{n} \cdots \frac{j_{k-1} + 1}{n} (1 + \frac{k}{n} - \sum_{l=1}^{k-1} \frac{j_l + 1}{n}) \right)^{-1+1/k}, \]
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where the sum is taken over the indices satisfying the inequalities $j_1, \ldots, j_{k-1} \geq 0$, $j_1 + \cdots + j_{k-1} \leq n$. Put

$$B_{n,k} = \frac{1}{nk-1} \sum_{i_1, \ldots, i_{k-1} \geq 1, i_1 + \cdots + i_{k-1} \leq n-1} \left( \frac{i_1}{n} \cdots \frac{i_{k-1}}{n} \left( 1 - \kappa \sum_{l=1}^{k-1} \frac{i_l}{n} \right) \right)^{-1+1/k}.$$  

For each $k \geq 2$, let $T_k$ be a subset of $\mathbb{R}^{k-1}$ consisting of the points $(\theta_1, \ldots, \theta_{k-1}) \in \mathbb{R}^{k-1}$ satisfying $\theta_1, \ldots, \theta_{k-1} \geq 0$ and $\theta_1 + \cdots + \theta_{k-1} \leq 1$. Then the sum $B_{n,k}$ is a Riemann sum of the integral

$$I_k = \int_{T_k} (x_1 \cdots x_{k-1}(1-x_1 - \cdots - x_{k-1}))^{-1+1/k} dx_1 \cdots dx_{k-1}.$$  

Since $k$ is fixed and $n \to \infty$, it is easily seen that

$$\lim_{n \to \infty} b_{n,k} = \lim_{n \to \infty} B_{n,k} = I_k.$$  

Hence no analogue of Theorem 2 holds for the series with nonnegative coefficients. In particular, for $k = 2$, the integral is expressible by Euler’s beta function. Indeed,

$$I_2 = \int_{T_2} (x_1(1-x_1))^{-1/2} dx_1 = \int_0^1 \frac{dx_1}{\sqrt{x_1(1-x_1)}} = \frac{\Gamma(1/2)^2}{\Gamma(1)} = \pi,$$

so the coefficients $b_{n,2}$ of the series $\sum_{n=0}^{\infty} b_{n,2} z^n = \left( \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n+1}} \right)^2$ tend to $\pi$ as $n \to \infty$.

Suppose that $P$ is a polynomial in one variable with positive coefficients and let $q(P)$ be the quotient of the largest and the smallest coefficients of $P$. Let $\mathcal{P}_d$ denote the set of polynomials of degree $d$ with nonnegative coefficients, i.e.,

$$\mathcal{P}_d = \{ \sum_{n=0}^{d} b_n z^n : b_0, \ldots, b_d \geq 0, b_0, b_d > 0 \}.$$  

Consider the quantity

$$m_k(d) = \inf_{P \in \mathcal{P}_d} q(P^k) = \min_{P \in \mathcal{P}_d} q(P^k).$$  

Note that without loss of generality it is sufficient to consider only those polynomials of $\mathcal{P}_d$ which satisfy $b_0 = 1$ and $b_d \geq 1$. Indeed, we can multiply $P$ by a constant. In addition, instead of $P$ we can consider the reciprocal polynomial of $P$, defined by

$$P^*(z) = z^d P(1/z) = b_d + b_{d-1} z + \cdots + b_0 z^d,$$

for which $q(P^*) = q(P^k)$. Also, by compactness, the infimum of $q(P^k)$ is attained.
For example, the quantity $m_2(d)$ evaluates how ‘flat’ can the square of a polynomial with nonnegative coefficients be in terms of its coefficients. It is easy to see that $m_2(1) = 2$ (the minimum is attained at $P(z) = 1 + z$) and $m_2(2) = 2.25$ (the minimum is attained at $P(z) = 1 + z/2 + z^2$). More generally, let us take

$$P(z) = 1 + \frac{1}{2}(z + z^2 + \cdots + z^{d-1}) + z^d \in \mathcal{P}_d.$$

Then each coefficient of $P^2$ is greater than or equal to 1 and two extreme coefficients of $P^2$ are equal to 1. The largest coefficient of $P^2$ is that of $z^d$. It is equal to $(d + 7)/4$. Consequently,

$$m_2(d) \leq q(P^2) = (d + 7)/4 \text{ for each } d \geq 1.$$

This inequality is not optimal for $d \geq 3$. Consider, for instance, the sequence whose first two terms are $y_0 = 1, y_1 = 1/2$ and whose $m$th element $y_m, m \geq 2$, is defined by the following recurrent formulas depending on the parity of $m$:

$$2y_{2k+1}y_0 + 2y_{2k-1}y_1 + \cdots + 2y_{k+1}y_{k-1} + y_k^2 = 1,$$

$$2y_{2k}y_0 + 2y_{2k-1}y_1 + \cdots + 2y_{k+2}y_{k-1} + 2y_{k+1}y_k = 1.$$

Then

$$y_0 = 1, y_1 = \frac{1}{2}, y_2 = \frac{3}{8}, y_3 = \frac{5}{16}, y_4 = \frac{35}{128}, y_5 = \frac{63}{256}, y_6 = \frac{231}{1024}, \ldots.$$  

Consider the following reciprocal polynomial

$$P_d(z) = 1 + y_1 z + y_2 z^2 + y_3 z^3 + \cdots + y_k z^{d-k} + y_{k+1} z^{d-k+1} + z^d.$$

Then

$$P_3(z)^2 = (1 + \frac{1}{2}(x + x^2 + x^3))^2 = 1 + x + \frac{5}{4} x^2 + \frac{5}{2} x^3 + \frac{5}{4} x^4 + x^5 + x^6,$$

so $q(P_3^2) = 5/2$ and $m_3(3) \leq 5/2$. Similarly,

$$P_4(z)^2 = 1 + x + x^2 + \frac{11}{8} x^3 + \frac{169}{64} x^4 + \frac{11}{8} x^5 + x^6 + x^7 + x^8,$$

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which implies that \( q(P_2^4) = \frac{169}{64} \) and \( m_2(4) \leq \frac{169}{64} = 2.640625 \). In the same manner, we obtain

\[
\begin{align*}
m_2(5) &\leq q(P_2^5) = \frac{89}{32} = 2.78125, \\
m_2(6) &\leq q(P_2^6) = \frac{737}{256} = 2.87890625, \\
m_2(7) &\leq q(P_2^7) = \frac{381}{128} = 2.9765625, \\
m_2(8) &\leq q(P_2^8) = \frac{49993}{16384} = 3.05133056\ldots, \\
m_2(9) &\leq q(P_2^9) = \frac{25609}{8192} = 3.12609863\ldots, \\
m_2(10) &\leq q(P_2^{10}) = \frac{208841}{65536} = 3.18666076\ldots, \\
m_2(11) &\leq q(P_2^{11}) = \frac{106405}{32768} = 3.24722290\ldots, \\
m_2(12) &\leq q(P_2^{12}) = \frac{3458321}{1048576} = 3.29811191\ldots.
\end{align*}
\]

Is it true that \( m_2(d) = q(P_d^2) \) for each \( d \)? Is \( m_2(d) \) bounded or unbounded in terms of \( d \)?

A polynomial analogue of the problem raised in Section 1 can be stated as follows. Is it true that, for every \( k \geq 2 \), there is an absolute constant \( b = b(k) \geq 1 \) such that, for each \( d \in \mathbb{N} \), the \( k \)th power of some Newman polynomial of degree \( d \) has all of its coefficients in \([1, b]\)?

Of course, if the quantity \( m_2(d) \) and, more generally, \( m_k(d) \) is unbounded, then the answer to this question is negative. Indeed, the set of all Newman polynomials of degree \( d \) is just a small subset of \( P_d \). For \( k = 2 \), this conjecture about squares of Newman polynomials may be viewed as a polynomial analogue of the Erdős-Turán conjecture.

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References


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