Transfer operator for the Gauss' continued fraction map. Structure of the eigenvalues and trace formulas.

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Definition 1

Warning

In order to simplify a bit, this talk, though directly and intricately related to the theory of continued fractions, dynamical systems and modular forms, will not mention these topics. Also, there will be almost no functional analysis!

Many mathematicians contributed to this topic: C.F. Gauss, R.O. Kuzmin, P. Lévy, A.Y. Khinchin, E. Wirsing, K.I. Babenko, S.P. Jur'ev, D. Knuth, D. Mayer, Ph. Flajolet, B. Vallée, D. Zagier, D. Hensley, L. Vepštas, A.J. MacLeod, P. Sebah, K. Briggs.

In the talk, we fix the notation

$$\phi = \frac{1+\sqrt{5}}{2}.$$

Definition 2

Let $\mathbb{D}$ be the disc $\{z \in \mathbb{C} : |z-1| < \frac{3}{2}\}$. Let $\mathbf{V}$ be the Banach space of functions which are analytic in $\mathbb{D}$ and are continuous in its closure, with the supremum norm.
The Perron-Frobenius, or *the transfer operator* for the Gauss' continued fraction map, (also called *the Gauss-Kuzmin-Wirsing operator*), is defined for functions $f \in \mathbf{V}$ by

$$\mathcal{L}[f](z) = \sum_{m=1}^{\infty} \frac{1}{(z+m)^2} f\Big(\frac{1}{z+m}\Big).$$

Our chief interest is the point spectrum of this operator. As was shown by Babenko, this operator is of trace class and is nuclear of order 0: $\sum\limits_{n=1}^{\infty} |\lambda_n|^{\varepsilon} < +\infty$.
Throughout the presentation, "n" is reserved for the index of eigenvalue (canonical or not)?

Numerical values

n	$(-1)^{n+1}\lambda_n$	n	$(-1)^{n+1}\lambda_n$
1	1.0000000000000000	9	0.00024413146552451558
2	0.3036630028987326	10	0.000091689083768593376
3	0.1008845092931040	11	0.000034516546163854253
4	0.03549615902165984	12	0.000013017697877023030
5	0.01284379036244026	13	0.0000049167823024644912
6	0.004717777511571031	14	0.0000018593073515090423
7	0.001748675124305511	15	0.00000070381134308703980
8	0.0006520208583205029	16	0.00000026664134344795640

Table : The eigevalues λ_n for $1 \leq n \leq 16$.

The nature of the eigenvalues λ_n for $n \geq 2$ is unknown. It is widely believed that these constants are unrelated to other most important constants in mathematics - but note the appearance of extended ring of periods (Zagier-Kontesevich) further in our work! Now, more that 480 digits of λ_2 have been calculated by K. Briggs, but one can get rigorous certificates only for the several first few digits of λ_2 and λ_3.
On the other hand, the trace of the operator $\mathcal{L}$ can be given explicitly. As was shown by D. Mayer and K.I. Babenko (also, Ph. Flajolet and B. Vallée), we have

Trace formulas

$$\begin{aligned}
\mathrm{Tr}(\mathcal{L}) &= \sum_{n=1}^{\infty} \lambda_n = \int\limits_0^{\infty} \frac{J_1(2x)}{e^x - 1}\,\mathrm{d}x = \sum_{\ell=1}^{\infty} \frac{1}{\xi_\ell^{-2} + 1} \\
&= \frac{1}{2} - \frac{1}{2\sqrt{5}} + \frac{1}{2}\sum_{k=1}^{\infty} (-1)^{k-1} \binom{2k}{k} (\zeta(2k) - 1), \\
\mathrm{Tr}(\mathcal{L}^2) &= \sum_{n=1}^{\infty} \lambda_n^2 = \int\limits_0^{\infty}\int\limits_0^{\infty} \frac{J_1(2\sqrt{xy})^2}{(e^x - 1)(e^y - 1)}\,\mathrm{d}x\,\mathrm{d}y = \sum_{i,j=1}^{\infty} \frac{1}{(\xi_{i,j}\xi_{j,i})^{-2} - 1}; \\
\xi_\ell &= \frac{1}{\ell +}\,\frac{1}{\ell +}\,\frac{1}{\ell +}\cdots, \quad \xi_{i,j} = \frac{1}{i +}\,\frac{1}{j +}\,\frac{1}{i +}\,\frac{1}{j +}\cdots, \quad \ell, i, j \in \mathbb{N}.
\end{aligned}$$

Proposition. Trace of the square of the operator

We have an identity:

$$\begin{aligned}\mathrm{Tr}(\mathcal{L}^2) &= \frac{2}{5+3\sqrt{5}}+\frac{1}{3+2\sqrt{3}}+\frac{4}{21+5\sqrt{21}}+\frac{3}{16+12\sqrt{2}}\\ &+\sum_{k=2}^{\infty}(-1)^k\binom{2k-2}{k-2}\Big(\zeta^2(k)-1-\frac{2}{2^k}-\frac{2}{3^k}-\frac{3}{4^k}\Big)\\ &= 1.103839653617_{+}.\end{aligned}$$

The kth term of this series is asymptotically $\frac{1}{2}(\frac{4}{5})^k(\pi k)^{-1/2}$.

The formulas for trace of the square of the operator(s) are crucial in our proof: we construct eigenvalues *explicitly* by a special method (without relying on functional analysis), and show that the given collection amounts to all eigenvalues, not a single is missed, since that would invalidate the trace formulas for the square.

Conjectures (1988-1995)

The following three statements are true:

i) Simplicity. The eigenvalues are simple, $|\lambda_n|$ strictly decreases.

ii) Sign. The eigenvalues have alternating sign: $(-1)^{n+1}\lambda_n > 0$.

iii) Ratio. There exists $\lim\limits_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = -\frac{3+\sqrt{5}}{2} = -\phi^2$.

The first two were raised by Mayer and Roepstorff (1988), reiterated by MacLeod. The last was raised by MacLeod (1992) (with a constant ≈ -2.6), seconded by Flajolet and Vallée (1995) (with the constant $-\phi^2$).

The ratio conjecture is very cautious, since nobody posed a question on true asymptotics of λ_n. Motivation:

$$\mathcal{L}_0[f](z) = \frac{1}{(z+1)^2} f\Big(\frac{1}{z+1}\Big).$$

These are true, and much more!

Theorem 1 [GA] (Asymptotics)

We have the formula

$$(-1)^{n+1}\lambda_n = \phi^{-2n} + C \cdot \frac{\phi^{-2n}}{\sqrt{n}} + d(n) \cdot \frac{\phi^{-2n}}{n},$$

where the constant $C = \dfrac{\sqrt[4]{5} \cdot \zeta(3/2)}{2\sqrt{\pi}} = 1.1019785625880999_{+}$;

here $\zeta(\star)$ is the Riemann zeta function, and the function $d(n)$ is bounded.

Based on high precision numerical computations of P. Sebah, who computed λ_n for $1 \leq n \leq 150$, we have:

$$d(1) = 0.5160, d(2) = 0.6042, d(3) = 0.5222, d(4) = 0.4662, d(5) = 0.4343, d(10) = 0.3850,$$
$$d(20) = 0.3688, d(30) = 0.3646, d(40) = 0.3626, d(50) = 0.3615, d(70) = 0.3604,$$
$$d(100) = 0.3595, d(130) = 0.3590, d(148) = 0.35887, d(149) = 0.35886, d(150) = 0.35885.$$

Theorem 2 [GA] (Arithmetic and decomposition of trace formulas)

(i) There exist positive functions $W_v(n)$, $v \geq 0$, $n \geq 1$, defined by $W_0(n) = 1$, $W_1(n) = \frac{5}{4} \cdot \phi^{-2n} P_{n-1}^{(0,1)}(3/2)$ (Jacobi polynomials), and then by a certain explicit recurrence, such that for $n \geq 1$,

$$(-1)^{n+1}\lambda_n = \phi^{-2n}\sum_{\ell=0}^{\infty} W_\ell(n), \quad W_\ell(n) = \frac{\sqrt[4]{5}}{2\sqrt{\pi}\cdot \ell^{3/2}\sqrt{n}} + \frac{B}{\ell^{3/2} n} \text{ for } \ell, n \geq 1,$$

for a bounded function $B = B(\ell, n)$.

(ii) This decomposition is compatible and gives the decomposition of trace formulas for the powers of $\mathcal{L}$: for the first, the second, and the third powers, we have, respectively,

$$\sum_{n=1}^{\infty}(-1)^{n+1}\phi^{-2n}W_{\ell-1}(n) = \frac{1}{\xi_\ell^{-2}+1},$$

$$\sum_{i+j=\ell}\sum_{n=1}^{\infty}\phi^{-4n}W_{i-1}(n)W_{j-1}(n) = \sum_{i+j=\ell}\frac{1}{(\xi_{i,j}\xi_{j,i})^{-2}-1},$$

$$\sum_{i+j+k=\ell}\sum_{n=1}^{\infty}(-1)^{n+1}\phi^{-6n}W_{i-1}(n)W_{j-1}(n)W_{k-1}(n) = \sum_{i+j+k=\ell}\frac{1}{(\xi_{i,j,k}\cdots)^{-2}+1}.$$

Decomposition of trace formulas

Recall once more that

$$\mathrm{Tr}(\mathcal{L}) = \sum_{n=1}^{\infty} \lambda_n = \sum_{\ell=1}^{\infty} \frac{1}{\xi_\ell^{-2} + 1}.$$

So, in the trace formulas now we are able to crystallize the contribution of each individual eigenvalue. Thus, this defines an infinite matrix

$$\Big((-1)^{n+1}\phi^{-2n}W_{\ell-1}(n)\Big)_{n,\ell=1}^{\infty},$$

whose elements in rows add up to eigenvalues, elements in columns add up to $(\xi_\ell^{-2}+1)^{-1}$, and the sum of all real numbers in the matrix is equal to $\mathrm{Tr}(\mathcal{L})$.
This is canonical, and similar decomposition holds for $\mathrm{Tr}(\mathcal{L}^k)$, $k \in \mathbb{N}$ also (as given above).

Continuation of this work

Transfer operator for the Gauss' continued fraction map. II. Fine arithmetic of the decomposition formulas (*in preparation*).

Theorem 2 (continued)

(iii)$\star$ Let $\mathbf{a} = \{\ell_1, \ell_2, \ldots, \ell_k\}$, $\ell_i \in \mathbb{N}$. Let us define

$$\Omega_{\mathbf{a}}(w) = \sum_{n=1}^{\infty} \Big(\prod_{i=1}^{k} W_{\ell_i}(n) \Big) w^n, \quad |w| < 1.$$

Then all $\Omega_{\mathbf{a}}(w) = \Omega_{\ell_1, \ell_2, \ldots, \ell_k}(w)$ are "arithmetic" functions.

For example,

$$\Omega_{\varnothing}(w) = \frac{w}{1-w}, \quad \Omega_1(w) = \frac{\phi^2 + w}{2\big((\phi^4 - w)(1-w)\big)^{1/2}} - \frac{1}{2}.$$

Examples 1

Identities involving $W_0(n) = 1$, and $W_1(n)$:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\phi^{-2n} = \frac{1}{\phi^2+1} = \frac{1}{\xi_1^{-2}+1},$$

$$\frac{5}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\phi^{-4n}P_{n-1}^{(0,1)}(3/2) = \frac{1}{2\sqrt{2}+4} = \frac{1}{\xi_2^{-2}+1},$$

$$\frac{5}{4}\sum_{n=1}^{\infty}\phi^{-6n}P_{n-1}^{(0,1)}(3/2) = \frac{1}{4\sqrt{3}+6} = \frac{1}{(\xi_{1,2}\xi_{2,1})^{-2}-1},$$

$$\frac{5}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\phi^{-8n}P_{n-1}^{(0,1)}(3/2) = \frac{1}{6\sqrt{10}+20} = \frac{1}{(\xi_{1,1,2}\xi_{1,2,1}\xi_{1,1,2})^{-2}+1}.$$

Identities involving $W_2(n)$. Though $W_2(n)$ is given by a series, not in a closed form, we have:

$$\sum_{n=1}^{\infty}(-1)^{n+1}W_2(n)\phi^{-2n} = 0.0839748528310781_{+} = \frac{2}{13+3\sqrt{13}} = \frac{1}{\xi_3^{-2}+1}.$$

This can be double-checked by computer, so the above Theorem 2 must hold true! We are left only with filling all the needed technical details ☺.

Examples 2

A still more interesting example occurs for higher powers. Numerically, we have:

$$\begin{aligned} A &= \sum_{n=1}^{\infty} W_2(n)\phi^{-4n} = 0.0428848639793538_{+} \in \mathcal{P}\left[\frac{1}{\pi}\right], \\ B &= \sum_{n=1}^{\infty} W_1^2(n)\phi^{-4n} = 0.0356498091111648_{+} \in \mathcal{P}\left[\frac{1}{\pi}\right], \end{aligned}$$

$$\begin{aligned} C &= \frac{2}{21+5\sqrt{21}} = \frac{1}{(\xi_{1,3}\xi_{3,1})^{-2}-1} = 0.0455447255899809_{+}, \\ D &= \frac{1}{16+12\sqrt{2}} = \frac{1}{(\xi_{2,2}\xi_{2,2})^{-2}-1} = 0.0303300858899106_{+}. \end{aligned}$$

And thus, we get the correct identity

$$2A + B = 2C + D = 0.1214195370698725_{+}.$$

Direct relations

This is not an exotic topic!

- Trace formulas of D. Mayer were crucial in proving his fundamental result that for a certain generalization of $\mathcal{L}$, call it $\mathcal{L}_s$, to a complex parameter s, one has:

$$Z(s) = \det(1 - \mathcal{L}_s^2) = \prod_{n=1}^{\infty}(1 - \lambda_n^2(s)),$$

  where $Z(s)$ is the Selberg zeta function for $\mathrm{PSL}_2(\mathbb{Z})$.
- Maass forms, period functions of modular forms, the work by J. Lewis & D. Zagier, *Ann. Math. (2)*, 2001, 191-258.
- Every factor $1 - \lambda_n(s)$ vanishes at (halfs) of some Riemann zeros and some spectral parameters associated with even Maass forms. Every factor $1 + \lambda_n(s)$ vanishes at some set of zeros of $Z(s)$, spectral parameters of odd Maass forms.

The main functional equation

Let us now forget everything said, and focus on the following

Question. How can one solve the functional equation?

$$\lambda f(z) = \frac{1}{(z+1)^2} f\Big(\frac{1}{z+1}\Big) + \lambda f(z+1),$$

$\lambda \in \mathbb{R}$, $f(z)$ is defined in the cut plane $\mathbb{C} \setminus (-\infty, -1)$, is not identically 0, with the regularity condition

$$\sup_{\Re(z) \geq -\frac{1}{2}} |(z+1)f(z)| < +\infty.$$

The main idea, which occurred to me in 2008, was first implemented in [1] G. Alkauskas, The Minkowski question mark function: explicit series for the dyadic period function and moments, *Math. Comp.* **79** (269) (2010), 383-418; *Math. Comp.* **80** (276) (2011), 2445-2454.

Solution (2012 Clausthal, Bonn)

Introduce the operator, depending on ω, $|\omega| \leq 1$; it has the same nice properties as $\mathcal{L}$.

$$\mathcal{L}_{\omega}[f(\omega,t)](z) = \sum_{m=1}^{\infty} \frac{\omega^{m-1}}{(z+m)^2} f\Big(\omega, \frac{1}{z+m}\Big).$$

This gives the functional equation for the eigenfunction:

$$\lambda(\omega) f(\omega,z) = \frac{1}{(z+1)^2} f\Big(\omega, \frac{1}{z+1}\Big) + \omega\lambda(\omega) f(\omega, z+1).$$

I arrived at this after month of calculations in Clausthal. It is very easy, and does **not** include complex dynamics (good!), and this operator was investigated before! The point is, that if You write it now, the question arises. "It does not give anything new, just a trivial introduction of ω. **Where does it reduce to linear algebra an finite calculations**"?

Crucial trick 1

Trick, which solves the problem. Let us rewrite the above functional equation for the eigenfunction f in terms of a function g, where

$$f(z) = \frac{1}{(z+\phi)^2}\, g\Big(\frac{z-\phi^{-1}}{z+\phi}\Big).$$

Here

$$x = \frac{z-\phi^{-1}}{z+\phi}, \quad z = \frac{x\phi+\phi^{-1}}{1-x}, \quad \Re(z) > -\frac{1}{2} \text{ corresponds to } |x| < 1.$$

Crucial trick 2

We get an avatar of the functional equation:

Equation for the eigenfunction and eigenvalue

$$\lambda(\omega)g(\omega,x)=\phi^{-2}g(\omega,-\phi^{-2}x)+\frac{5\,\omega\lambda(\omega)}{(-x+2\phi)^2}\,g\Big(\omega,\frac{2x\phi^{-1}+1}{-x+2\phi}\Big),$$

$g(\omega,x)$ is defined and is analytic in $|\omega|\leq 1$, $|x|<1$, satisfies a certain regularity condition.

If we look attentively, we see that the last term is delayed (contains ω), and the second term is just a scaling of the first. So, if we expand in terms of Taylor coefficients of $g(\omega,x)$ and $\lambda(\omega)$, we obtain not a system of infinite linear equations ("functional analysis"), but a finite recurrence ("linear algebra"!).

This one tricks "cracks" the whole problem.

Main recurrence 1

Let us denote the Taylor coefficients

$$\Lambda(\omega, n) = \phi^{-2n} \cdot \sum_{\ell=0}^{\infty} \omega^{\ell} \cdot W_{\ell}(n), \quad W_0(n) = 1,$$

$$G(n, \omega, x) = \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} \quad q_n^{(j)}(n) \cdot x^{j-1} \cdot \omega^{\ell}, \quad q_0^{(j)}(n) = \delta_j^n.$$

Plug into the functional equation $\lambda_n(\omega) = (-1)^{n+1}\Lambda(\omega, n)$, $g_n(\omega, x) = G(n, \omega, x)$,
$\lambda_n(0) = (-1)^{n+1}\Lambda(0, n) = (-1)^{n+1}\phi^{-2n}$, $g_n(0, x) = G(n, 0, x) = x^{n-1}, \quad n \geq 1.$

We do not prove that these functions are analytic in ω, we construct solutions to the functional equation explicitly, and trace formulas show that there are no other!

Note that $\mathrm{Tr}(\mathcal{L}_{\omega}) = \sum_{m=1}^{\infty} \frac{\omega^{m-1}}{\xi_m^{-2} + 1} = \sum_{\ell=0}^{\infty} \omega^{\ell} \sum_{n=1}^{\infty} \phi^{-2n}(-1)^{n+1} W_{\ell}(n).$

Main recurrence 2

If we compare the Taylor coefficients at $x^{j-1}\omega^{v}$ of the functional equation for g, we get

The main recurrence. For $v \geq 0$, $j \geq 1$, one has

$$\sum_{r=0}^{v} q_{v-r}^{(j)} W_r = q_v^{(j)}(-1)^{n+j}\phi^{2n-2j} + \sum_{r=0}^{v-1}\sum_{i=1}^{\infty} q_{v-r-1}^{(i)} K(j;i) W_r.$$

Here $K(j;\ell)$ is the Taylor coefficient at x^{j-1} of $5 \cdot \frac{(2x\phi^{-1}+1)^{\ell-1}}{(-x+2\phi)^{\ell+1}}$. These can be expresses in terms of standard Jacobi polynomials. The recurrence allows us to calculate W_v and $q_v^{(j)}$, except for $q_v^{(n)}$, $v \geq 1$, which can be chosen almost arbitrarily (we always choose 0). This nicely fits to the fact that $g_n(\omega, x)$ is uniquely defined up to "the constant factor", which is an arbitrary analytic function of the variable ω, $|\omega| \leq 1$. So, our whole problem, asymptotics and decomposition formulas are governed by the collection of explicit positive real numbers $K(j;\ell)$, $j,\ell \geq 1$.

Eigenvalue simplicity property is a direct consequence of this explicit construction and trace formulas, it does not depend on asymptotic results! Unfortunatelly, for eigenvalue sign conjecture we must use asymptotics and explicit bounds.

Thm 2 (Arithmetic and decomposition of trace formulas) - repeat

(i) There exist positive functions $W_v(n)$, $v \geq 0$, defined by $W_0(n) = 1$, $W_1(n) = \frac{5}{4} \cdot \phi^{-2n} P_{n-1}^{(0,1)}(3/2)$, and then by *the above explicit recurrence*, such that for $n \geq 1$,

$$(-1)^{n+1}\lambda_n = \phi^{-2n} \sum_{\ell=0}^{\infty} W_\ell(n), \quad W_\ell(n) = \frac{\sqrt[4]{5}}{2\sqrt{\pi} \cdot \ell^{3/2}\sqrt{n}} + \frac{B}{\ell^{3/2} n} \text{ for } \ell, n \geq 1,$$

for a bounded function $B = B(\ell, n)$.

(ii) This decomposition is compatible and gives the decomposition of trace formulas for the powers of $\mathcal{L}$: for the first, the second, and the third powers, we have, respectively,

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^{-2n} W_{\ell-1}(n) &= \frac{1}{\xi_\ell^{-2} + 1}, \\ \sum_{i+j=\ell} \sum_{n=1}^{\infty} \phi^{-4n} W_{i-1}(n) W_{j-1}(n) &= \sum_{i+j=\ell} \frac{1}{(\xi_{i,j}\xi_{j,i})^{-2} - 1}, \\ \sum_{i+j+k=\ell} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^{-6n} W_{i-1}(n) W_{j-1}(n) W_{k-1}(n) &= \sum_{i+j+k=\ell} \frac{1}{(\xi_{i,j,k} \cdots)^{-2} + 1}. \end{aligned}$$

Bibliography, acknowledgements

G. Alkauskas, Transfer operator for the Gauss' continued fraction map. I. Structure of the eigenvalues and trace formulas, http://arxiv.org/abs/1210.4083.

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Thank You for Your attention!