The Minkowski question mark function, quasi-modular forms and the Dedekind η-function

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Definition

The function $?(x)$, named after H. Minkowski, is defined as follows:

$$?([0, a_1, a_2, a_3, \ldots]) = 2\sum_{i=1}^{\infty}(-1)^{i+1}2^{-\sum_{j=1}^{i} a_j}, \quad a_i \in \mathbb{N}.$$

$?(x)$ is the distribution function of the Farey tree:

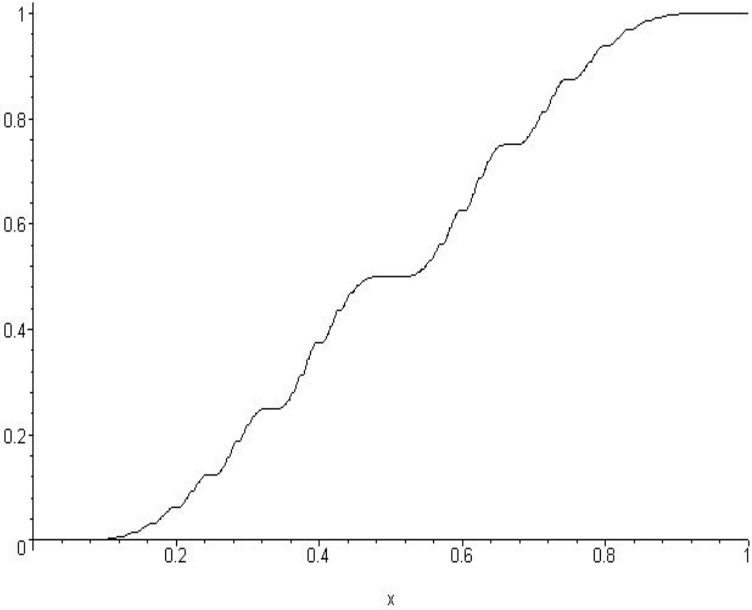
$$x \mapsto \frac{x}{x+1}, \quad \frac{1}{x+1}, \text{ starting from } \frac{1}{2}.$$

The function $?(x)$ is continuous, monotone, singular w.r.t. the Lebesgue measure, and satisfies functional equations

$$?(x) = \begin{cases} 1-?(1-x), \\ 2?(\frac{x}{x+1}). \end{cases}$$

The last hints to "modular world", but how exactly?

The Minkowski ?(x) function



Unsolved/solved arithmetic problems

1) Arithmetic of the moments

Find the arithmetic structure of the moments, defined by

$$m_L = \int_0^1 x^L \,\mathrm{d}?(x), \quad L \geq 1.$$

These are relatives of *periods for Maass wave forms*. In particular, does there exist a series for them, involving known functions, expressions and integrals?

2) Modular connection

Establish an intrinsic and natural connection between the world of $?(x)$ and the world of modular forms.

Stieltjes transform of $?(x)$

The Stieltjes transfrom of $?(x)$ is defined by

$$G(z) = \int_{0}^{1} \frac{x}{1-xz}\,\mathrm{d}?(x) = \sum_{L=1}^{\infty} m_L z^{L-1}.$$

Theorem (GA, 2010)

The function $G(z)$ is an analytic function in $\mathbb{C} \setminus [1,\infty)$. It satisfies the functional equation

$$-\frac{1}{1-z} - \frac{1}{(1-z)^2} G\Big(\frac{1}{1-z}\Big) + 2G(z+1) = G(z),$$

also the symmetry property

$$G(z+1) = -\frac{1}{z^2} G\Big(\frac{1}{z}+1\Big) - \frac{1}{z},$$

and a certain regularity property.

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Homogenuity and the Eisenstein series

Let

$$G_2(z) = \frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z}, G_2(z+1) = G_2(z), G_2(-1/z) = z^2 G_2(z) - 2\pi i z,$$

be the the weight 2 Eisenstein series.
Now, if $f(z) = G(z+1) - \frac{i}{2\pi} G_2(z)$, then this function is uniformely bounded for $\Im(z) > \epsilon > 0$, and

Homogeneous form

$$f(z) = \frac{1}{2} f(z-1) + \frac{1}{2} \frac{1}{(1-z)^2} f\Big(\frac{z}{1-z}\Big).$$

We can recover the quasi-modular part by

$$\lim_{n\to\infty} f(z-n) = -\frac{i}{2\pi} G_2(z).$$

Is this sufficient for the modular connection? $2 \mapsto k$?

Can we extract something periodic from a three-term functional equation above?

Suppose, ψ satisfies (Lewis, Zagier, "Period functions for Maass wave forms", *Ann. of Math.* (2001)):

$$\psi(z) = \psi(z+1) + (z+1)^{-2s}\psi\Big(\frac{z}{z+1}\Big).$$

Then an algebraic identity

$$\begin{aligned} 0 &= \Big[\psi(z+1) - \psi(z) + \frac{1}{(z+1)^{2s}}\psi(\frac{z}{z+1})\Big] \\ &\quad -\frac{1}{(z+1)^{2s}}\Big[\psi(\frac{z}{z+1}) - \psi(-\frac{1}{z+1}) + \Big(\frac{z+1}{z}\Big)^{2s}\psi\Big(-\frac{1}{z}\Big)\Big] \\ &= \Big[\psi(z+1) + (z+1)^{-2s}\psi(-\frac{1}{z+1})\Big] - \Big[\psi(z) + z^{-2s}\psi(-\frac{1}{z})\Big]. \end{aligned}$$

Questions

- Can we similarly define functions, which satisfy the three-term functional equation, and which have a certain (or any of them, prescribed in advance) quasi-modular form, that is, an element of $\mathbb{C}[G_2, G_4, G_4]$ as its periodic part? $\surd$
- Can the last trick of getting a certain periodic function, as a transform of the solution of the three-term functional equation, be applied in this case? $\surd$
- Can both these requirements be accommodated under the same uniform construction? $\surd$

Definition

Definition

Let $k \in 2\mathbb{N}$. The function $f(\varkappa, z)$ is called *a weight k mean-modular form*, if

- it is bivariate holomorphic function and satisfies the functional equation

$$f(\varkappa, z) = \varkappa f(\varkappa, z-1) + \frac{1-\varkappa}{(1-z)^k} f\Big(\varkappa, \frac{z}{1-z}\Big)$$

  for $z \in \mathfrak{h}$, $\varkappa \in \overline{\mathcal{D}}$, where $\mathcal{D} = \{\varkappa \in \mathbb{C} : |\varkappa| < 1, |1-\varkappa| < 1\}$;
- for every $\epsilon > 0$ there exist a constant $C(\epsilon)$ such that $|f(\varkappa, z)| < C(\epsilon)$, $z^{-k}|f(\varkappa, -1/z)| < C(\epsilon)$ for $\Im(z) > \epsilon$, $\varkappa \in \overline{\mathcal{D}}$.

Let $\mathcal{H}$ stand for the ring of functions in variable $\varkappa$, which are holomorphic in $\mathcal{D}$ and continuous in its closure. The set of weight k mean-modular forms make an $\mathcal{H}$-module MMF_k.

Theorem 2 GA, (2017)

Let f be a MMF of weight k. Then the following limit

$$\mathfrak{E}f(\varkappa, z) = \lim_{\substack{n\to\infty \\ n\in\mathbb{Z}}} f(\varkappa, z-n)$$

exists, and ($\widetilde{M}_k = \mathbb{C}[E_2, E_4, E_6]_k$)

$$\mathfrak{E} : \mathsf{MMF}_k \mapsto \widetilde{M}_k \otimes_{\mathbb{C}} \mathcal{H}.$$

Moreover, $\mathfrak{E}$ is an isomorphism. There exists a canonical $\mathbb{C}$-subspace of MMF_k, call it Mmf_k, consisting of mean-modular forms f, characterized by a fact that $\mathfrak{E}(f)$ is a constant function in $\varkappa$. Then one has an isomorphism

$$\mathfrak{E} : \mathsf{Mmf}_k \mapsto \widetilde{M}_k.$$

We call a function $T(z)$ *a mean-modular section*, or *MMS*, of weight k, if there exists a mean-modular form $f(\varkappa, z)$ of weight k such that

$$T(z) = f\Big(\frac{1}{2}, z\Big).$$

Denote the $\mathbb{C}$-linear space of MMS of weight k by Mms_k. Thus:

- if $T(z)$ is a modular form for $\mathrm{PSL}_2(\mathbb{Z})$, then $T(z)$ is a MMS of the same weight; indeed, $T(z)$ is a MMF as a constant function in $\varkappa$.
- "Sporadic" solutions of the three term functional equation (for a specific $\varkappa$), which are also in $M_k(\Gamma(N))$, do not qualify MMS.
- Most importantly,

$$\int\limits_0^1 \frac{x}{1 - x(z+1)}\,\mathrm{d}?(x) - \frac{i}{2\pi} G_2(z).$$

  is a MMS of weight 2.

"Specialization" map

Let, correspondingly, define the map $S : \mathrm{MMF}_k \mapsto \mathrm{Mms}_k$, by

$$S\Big(f(\varkappa, z)\Big) = f\Big(\frac{1}{2}, z\Big).$$

This restricts also to the map $S : \mathrm{Mmf}_k \mapsto \mathrm{Mms}_k$, which we denote by the same letter S. The map $\mathfrak{E} : \mathrm{Mms}_k \mapsto \widetilde{M}_k$, which we denote by the same letter $\mathfrak{E}$ again (this should not cause a confusion) is defined by

$$\mathfrak{E}(T) = \lim_{\substack{n \to \infty \\ n \in \mathbb{Z}}} T(z - n).$$

Corollary

The map $\mathfrak{E} : \mathrm{Mms}_k \mapsto \widetilde{M}_k$ is an isomorphism.

The functional equation at $\varrho = e^{\frac{\pi i}{3}}$ - modular connection

Let us define another map Γ, which maps from MMF_k or Mmf_k, by the following formula:

$$\Gamma\Big(f(\varkappa, z)\Big) = f(\varrho, z) - \frac{1}{z^k} f\Big(\varrho, -\frac{1}{z}\Big).$$

The second main result reads as follows.

Theorem 2, GA (2017)

For any $f \in \mathrm{MMF}_k$ or $f \in \mathrm{Mmf}_k$, there exists a modular form $g \in M_{k-2}$, such that

$$\Gamma(f) = \eta^4 g;$$

Here η is the Dedekind η-function $\eta(z) = e^{\frac{\pi i}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})$.

Diagram

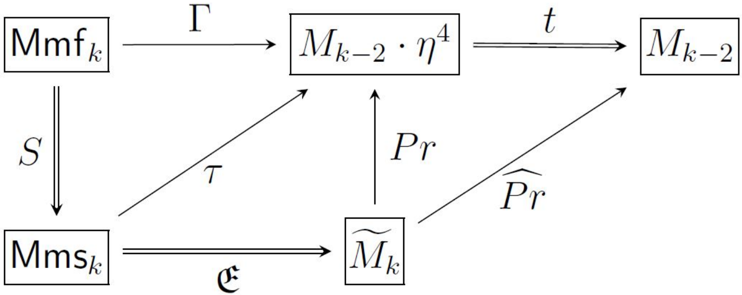


Figure: Commutative diagram of homomorphisms of the following $\mathbb{C}$-vector spaces: mean-modular forms Mmf$_k$; mean-modular sections Mms$_k$; quasi-modular forms $\widetilde{M}_k$; and modular forms M_{k-2}. Double lines denote an isomorphism. $\tau\Big(\int\limits_0^1 \frac{x}{1-x(z+1)}\,\mathrm{d}?(x) - \frac{i}{2\pi}G_2(z)\Big) = C\eta^4$.

Construction of MMF

- The following function is in Mmf_2:

$$G(\varkappa, z+1) = (1-\varkappa) \sum_{\substack{a,b,c,d \geq 0, \\ ad-bc=1}} \frac{\varkappa^{\imath(\frac{a+b}{c+d})}(1-\varkappa)^{\jmath(\frac{a+b}{c+d})}}{\big[(a+c)z-(b+d)\big](cz-d)} - \frac{i}{2\pi} G_2(z).$$

- ("Serre's derivative") If f is a MMF of weight k, then ($E_2 = \frac{3}{\pi^2} G_2$)

$$\vartheta_k f = \frac{1}{2\pi i} \frac{\partial}{\partial z} f(\varkappa, z) - \frac{k}{12} \cdot E_2(z) \cdot f(\varkappa, z)$$

  is a MMF of weight $k+2$.

- If $g(z)$ is a modular form for $\mathrm{PSL}_2(\mathbb{Z})$ of weight r, and $f(\varkappa, z)$ is a MMF of weight k, then $g(z) \cdot f(\varkappa, z)$ is a MMF of weight $k+r$.
- The map ϑ commutes with all the maps $\mathfrak{E}$, S, Γ and t. For this we need the transformation property $z^{-2}E_2(-1/z)$, and the crucial identity

$$\frac{\eta'}{\eta} = \frac{\pi i}{12} E_2(z).$$

The last algebraic identity

$$\begin{aligned} 0 &= \psi(z) - \varrho\psi(z-1) - \frac{\varrho^{-1}}{(1-z)^k}\psi\Big(\frac{z}{1-z}\Big) \\ &- \frac{\varrho^{-2}}{(1-z)^k}\Big[\psi\Big(\frac{1}{1-z}\Big) - \varrho\psi\Big(\frac{z}{1-z}\Big) - \frac{\varrho^{-1}(1-z)^k}{z^k}\psi\Big(-\frac{1}{z}\Big)\Big] \\ &= \Big[\psi(z) + \frac{\varrho^{-3}}{z^k}\psi\Big(-\frac{1}{z}\Big)\Big] - \varrho\Big[\psi(z-1) + \frac{\varrho^{-3}}{(1-z)^k}\psi\Big(\frac{1}{1-z}\Big)\Big]. \end{aligned}$$

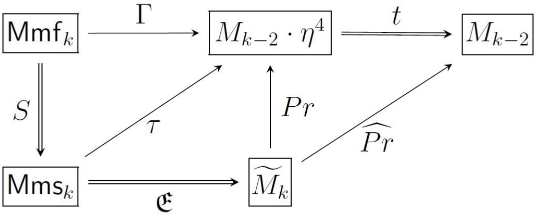
Let

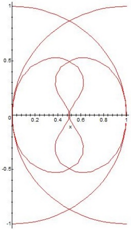
$$\mathscr{W}(z) = \psi(z) - \frac{1}{z^k}\psi\Big(-\frac{1}{z}\Big).$$

$$\mathscr{W}(z+1) = \varrho\mathscr{W}(z), \quad \frac{1}{z^k}\mathscr{W}\Big(-\frac{1}{z}\Big) = -\mathscr{W}(z).$$

$$\begin{cases} \eta(z+1) = e^{\frac{\pi i}{12}}\eta(z), \\ \eta\Big(-\frac{1}{z}\Big) = \sqrt{-iz}\,\eta(z). \end{cases} \implies \mathscr{W} = \eta^4 g, \quad g \in M_{k-2}.$$

Conclusions. Acknowledgements





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Thank You for Your attention!

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