Minkowski-Alkauskas Constant

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May 6, 2008

In addition to examining [1]

$$?\left(0+\frac{1|}{|a_1}+\frac{1|}{|a_2}+\frac{1|}{|a_3}+\cdots\right)=\sum_{k=1}^{\infty}(-1)^{k-1}2^{-(a_1+a_2+\cdots+a_k-1)},$$

we study [2]

$$F\left(a_0+\frac{1|}{|a_1}+\frac{1|}{|a_2}+\frac{1|}{|a_3}+\cdots\right)=\sum_{k=1}^{\infty}(-1)^{k-1}2^{-(a_0+a_1+a_2+\cdots+a_k)}.$$

The former is the original Minkowski question mark function, a self-map of $[0,1]$; the latter is defined on the nonnegative real line with $2F(x)=?(x)$ for all $x\in[0,1]$. In particular,

$$F(0)=0,\qquad F(\tfrac{1}{2})=\tfrac{1}{4},\qquad F(1)=\tfrac{1}{2},\qquad F(\sqrt{2})=\tfrac{3}{5},$$

$$F(\tfrac{1+\sqrt{5}}{2})=\tfrac{2}{3},\qquad F(2)=\tfrac{3}{4},\qquad F(3)=\tfrac{7}{8},\qquad \lim_{x\to\infty}F(x)=1^-.$$

The distribution F is continuous, strictly increasing, singular, and uniquely determined by the functional equation

$$2F(x)=\begin{cases}F(x-1)+1 & \text{if } x\geq 1,\\ F\left(\dfrac{x}{1-x}\right) & \text{if } 0\leq x<1.\end{cases}$$

Define moments

$$M_\ell=\int_0^\infty x^\ell dF(x),\qquad m_\ell=\int_0^1 x^\ell d?(x)$$

then $m_1=M_1-1=1/2$ follows easily. Similar closed-form expressions for

$$m_2=M_2-4=0.2909264764...,$$

$$m_4 = M_4 - 24m_2 - 100 = 0.1269922584...$$

presently do not exist, although progress has recently been made [3]. It is known that

$$2m_3 = 3m_2 - 1/2 = 2(0.1863897146...),$$

$$2M_3 = 9m_2 + 69/2, \qquad 2m_5 = 5m_4 - 5m_2 + 1$$

and analogous relations hold for higher-order moments. Hence calculating m_2, m_4, $\ldots$ to high precision is important for understanding m_3, m_5, $\ldots$.

Alkauskas [4] proved the following asymptotic formula:

$$\begin{aligned} m_\ell &\sim \sqrt[4]{4\pi^2 \ln(2)} \cdot c \cdot \left(e^{-2\sqrt{\ln(2)}}\right)^{\sqrt{\ell}} \ell^{1/4} \\ &\sim (2.3562298899...)(0.1891699952...)^{\sqrt{\ell}} \ell^{1/4} \end{aligned}$$

as $\ell \to \infty$, where

$$c = \int_0^1 2^x (1 - F(x)) dx = 1.0301995633....$$

This is a fascinating result, especially because m_2, m_4, $\ldots$ remain so mysterious! One would not have expected an asymptotic formula for m_ℓ as such to be possible.

0.1. Addendum. An infinite series for m_ℓ that does not explicitly involve continued fractions was unveiled in [5]:

$$\frac{1}{(\ell-1)!} \sum_{n=0}^{\infty} \int \cdots \int_{[0,\infty)^{n+1}} x_0^\ell \cdot \frac{(x_0 x_n)^{-1/2} \cdot \prod_{j=0}^{n-1} I_1\left(2\sqrt{x_j x_{j+1}}\right)}{\prod_{j=0}^{n} e^{x_j}\left(2e^{x_j} - 1\right)} dx_0 \ldots dx_n$$

where $I_1(z)$ is the modified Bessel function of the first kind. Unfortunately this does not improve upon numerical accuracy found in [3]. Does a simpler formula exist (even if only for $\ell = 2$ or $\ell = 4$)?

Integrals of the form

$$\int_0^1 \cos(2\pi k x)\, d?(x)$$

are evaluated to high precision in [6]; another sample calculation is

$$\pi \int_0^1 (?(x) - x) \cot(\pi x)\, dx = -0.4559592037...$$

which corresponds to the value of an associated zeta function at unity.

References

[1] S. R. Finch, Minkowski-Bower constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 441–443.

[2] G. Alkauskas, The moments of Minkowski question mark function: the dyadic period function, *Glasgow Math. J.* 52 (2010) 41–64; arXiv:0801.0051; MR2587817 (2011d:11007).

[3] G. Alkauskas, The Minkowski question mark function: explicit series for the dyadic period function and moments, *Math. Comp.* 79 (2010) 383–418; addenda/corrigenda 80 (2011) 2445–2454; arXiv:0805.1717; MR2552232 (2010k:11006) and MR2813370 (2012d:11009).

[4] G. Alkauskas, An asymptotic formula for the moments of the Minkowski question mark function in the interval $[0, 1]$, *Lithuanian Math. J.* 48 (2008) 357–367; arXiv:0802.2721; MR2470798 (2009i:11115).

[5] G. Alkauskas, Semi-regular continued fractions and an exact formula for the moments of the Minkowski question mark function, *Ramanujan J.* 25 (2011) 359–367; arXiv:0912.1039; MR2819722.

[6] G. Alkauskas, Fourier-Stieltjes coefficients of the Minkowski question mark function, *Analytic and Probabilistic Methods in Number Theory*, Proc. 2011 Palanga conf., ed. A. Laurinčikas, E. Manstavičius and G. Stepanauskas, TEV, 2012, pp. 19–33; arXiv:1008.4014; MR3025455.