FRIENDLY PATHS

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AMM 11484. The coordinate plane is divided into unit squares in the usual way. We consider broken lines which consist only from the sides of unit squares. The broken line is called *ascending* if it goes only Right or Upwards at every turn. It is called *descending* if it goes only Right or Downwards at every turn.
Given N points with integer coordinates, N being a positive integer. A broken line is called *friendly* if it is either ascending or descending, and such that the amount of given points on the one side of it equals to the amount of given point on the other side (the given points which the broken line passes through do not count).

(1) Prove that if $N = b^2 + a^2 + b + a$ for some positive integers a, b such that $a \leq b \leq a + \sqrt{2a}$, there exists a configuration of N such points that there does not exist a friendly line.

(2*)(*No known solution yet*) Prove that if N is odd then there always exists a friendly line.

Solution. (1) Let $a, b \in \mathbb{N}$ are such that $a \leq b \leq a + \sqrt{2a}$. Consider the configuration of $N = b^2 + a^2 + b + a$ points as shown in Figure 1. Let us call ascending or descending broken line to be *suitable*. We will show that for this configuration there does not exists a friendly line.
(i) Suppose the first point $\mathbf{p}$ and the last point $\mathbf{q}$ which our suitable line encounters while crossing this picture, belong to two neighboring parts: A and B, B and C, C and D, D and A. Obviously, such suitable line cannot be friendly.
(ii) Suppose that point $\mathbf{p}$ belongs to part A, while the point $\mathbf{q}$ belongs to C. Since N is even, there must be even number of points which the friendly line passes through. On the other hand, we see that independently of points $\mathbf{p}$ and $\mathbf{q}$, there are exactly $2a+1$ points on a suitable line. Consequently, it cannot be friendly.
(iii) Suppose $\mathbf{p}$ belongs to part D, while the point $\mathbf{q}$ belongs to B. Similarly, if starting from $\mathbf{p}$ and ending at $\mathbf{q}$ an ascending line does not pass through an unmarked point, there are exactly $2b+1$ marked points on this line and it cannot be friendly. Suppose it passes through unmarked point. As can be easily seen, the minimal difference between number of points (say, ℓ) on the "left shore" and the number of points on the "right shore" (say, r) is achieved for point $\mathbf{f}$ and the broken line indicated in Figure 1. Thus, in this case $r = \frac{a(a-1)}{2} + \frac{b(b-1)}{2} + \frac{(b-a)(b-a-1)}{2}$, while $\ell = \frac{a(a+1)}{2} + \frac{b(b+1)}{2} - \frac{(b-a)(b-a+1)}{2}$. Consequently, there does not exist a friendly line if

$$\ell > r \Leftrightarrow$$

$$\frac{a(a+1)}{2} + \frac{b(b+1)}{2} - \frac{(b-a)(b-a+1)}{2} > \frac{a(a-1)}{2} + \frac{b(b-1)}{2} + \frac{(b-a)(b-a-1)}{2}$$

$$\Leftrightarrow (b-a)^2 < a + b \Leftrightarrow b < a + \sqrt{a+b}.$$

Since $b \leq a + \sqrt{2a} \leq a + \sqrt{a+b}$ (and both cannot be equalities), this proves the desired result. □

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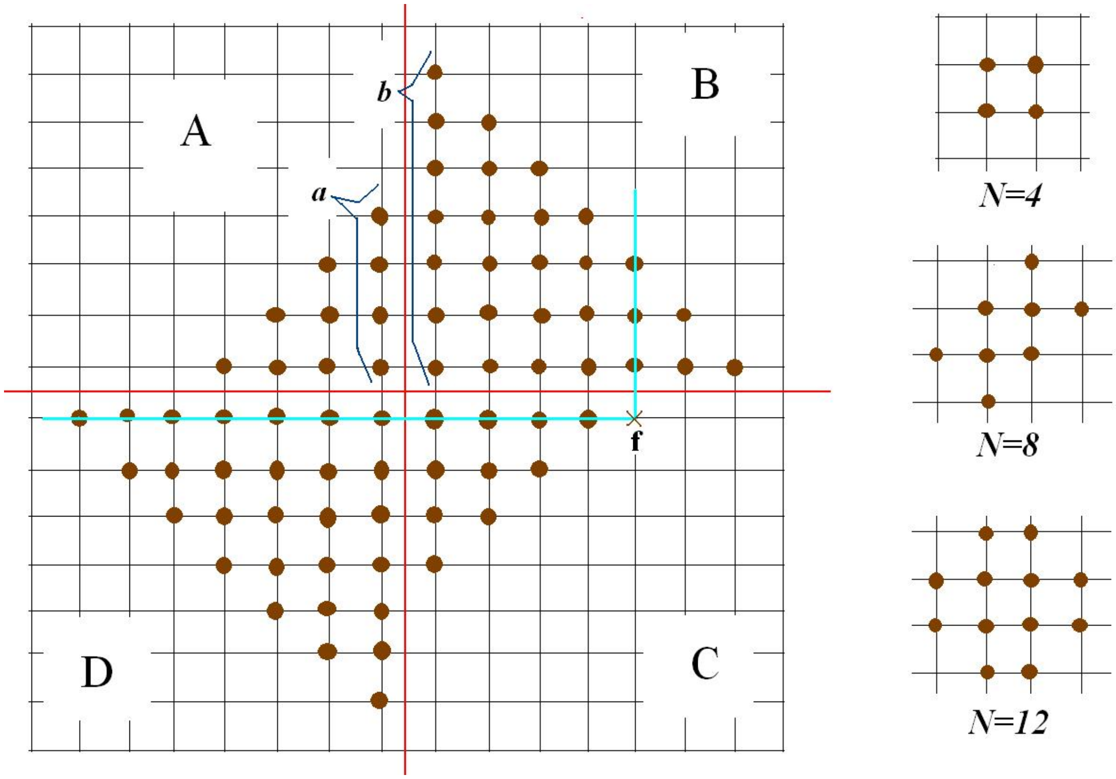

FIGURE 1