EXACT LOWER BOUND FOR THE HOUSE OF ALGEBRAIC INTEGER WITH ABELIAN GALOIS GROUP

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Define the house $\overline{|\alpha|}$ of the algebraic integer α of degree n to be the maximum of the absolute values of its conjugates: $\overline{|\alpha|} = \max\limits_{1\leq j\leq n} |\alpha_j|$. Let θ be the smallest Pisot number, that is, the real root of the equation $x^3 - x - 1 = 0$. In the article [1] the author proves the following theorem.

Theorem ([1]). *Suppose that α is a non-reciprocal algebraic integer of degree n. Then there exists a constant ω such that*

$$\overline{|\alpha|} > 1 + \frac{\omega}{n},$$

where ω is the smallest root of the equation

$$(1 - \frac{\log\theta}{\omega})\log(\frac{\omega}{\ln\theta} - 1) = 2\log\theta + \log(1 - e^{-2\omega}).$$

This bound is valid for all non-reciprocal algebraic integers independently from their arithmetic. Nevertheless, if we refine the arithmetic of the splitting field of the minimal polynomial of α, we can get much better bound, independent from n. For example, in the article [2] the author proves the following theorem.

Theorem ([2]). *Suppose that α is an algebraic integer (not a root of unity). Suppose, $P(x)$ is its minimal polynomial with abelian Galois group. Then the Mahler height of α satisfies the inequality*

$$h(\alpha) \geq \frac{1}{2}\log\frac{\sqrt{5}+1}{2}.$$

Trivially, this gives the bound for the house $\overline{|\alpha|} \geq \sqrt{\frac{\sqrt{5}+1}{2}} \approx 1.2720....$ Despite the fact that the inequality for Mahler height is sharp, the last inequality for the house can be improved. More precisely, the following theorem is valid.

Theorem. *Suppose that α is an algebraic integer (not a root of unity). Suppose, $P(x)$ is its minimal polynomial with abelian Galois group. Then the house of the polynomial satisfies the inequality*

$$\overline{|\alpha|} \geq \sqrt{2} \approx 1.4142....$$

Proof. Let the degree of $P(z)$ be n. Let $K = \mathbf{Q}(\alpha_1) = \mathbf{Q}(\alpha_1, \alpha_2, ..., \alpha_n)$ (in the abelian case, the single root generates the whole splitting field, since $\mathbf{Q}(\alpha_1)$ has the degree n over $\mathbf{Q}$, and the transitive permutation group acting on the set of n elements is abelian only if (but not necessarily if) its order is equal to n). If polynomial has a real root, then α_1 can be chosen to be this root, hence all roots are real, and in this case the needed inequality holds. Suppose, polynomial has no real roots. Then K has an automorphism of complex conjugation $\sigma : z \to \overline{z}$ of order 2. Let K' be its invariant field, that is, $\mathbf{Q} \subset K' \subset K$, the degree $[K : K'] = 2$. In other words, K' is maximal real subfield of K. Since subgroup $\{id, \sigma\}$ is normal (Galois group G is abelian), therefore $K'/\mathbf{Q}$ is a normal extension. hence, $P(z)$ splits over K' into $N = n/2$ polynomials of order 2 and real coefficients:

$$P(z) = R_1(z)R_2(z)...R_N(z).$$

Let α_i and $\overline{\alpha_i}$ be the roots of $R_i(z)$. Each $R_i(z)$ is attained from $R_1(z)$ acting each coefficient under automorphisms from the group $\mathrm{Gal}(K'/\mathbf{Q})$ of order N. Consider an arbitrary polynomial $W(x)$ with integer coefficients. Let $|\alpha_i|^2 = \rho_i \in K'$ for $i = 1, 2, ..., N$. This is an algebraic integer. An expression $\zeta_w^{(i)} = W(\rho_i)$ is also an algebraic integer. More, for $i = 1, 2, ..., N$ they form a full set of conjugates of $\zeta_w^{(1)}$ over $\mathbf{Q}$. Hence, the product $\prod_{i=1}^{N} W(\rho_i) = N_{K'/\mathbf{Q}}(\zeta_w)$ modulo is an integer, hence it is 0 or modulo not less that 1. Therefore, there exists i, for which $|W(\rho_i)|$ is zero or not less than 1. Choose $W(x) = x - 1$. Hence, for a positive ρ we have $|\rho - 1| \geq 1$ or is 0. This gives $\rho = 0$ (impossible case), $\rho = 1$ for all $i = 1, 2, ..., N$ (all roots are on the unit circle, which according to Kronecker's theorem gives only cyclotomic polynomials), or $\rho \geq 2$. Hence the theorem is proved.

For the conclusion we note that the bound is sharp: it is achieved for the polynomial $P(x) = x^2 - 2$.

References

[1] A. Dubickas, *The maximal conjugate of a non-reciprocal algebraic integer*, Lith. Math. J. **37**, No. 2, (1997) 168-174.

[2] A. Schinzel, *On the product of the conjugates outside the unit circle of an algebraic number*, Acta Arith. **24** (1973) 385-399