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Dirichlet Series Associated with Strongly q-Multiplicative Functions*

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Abstract. In this work the Dirichlet series $\kappa_f(s) = \sum_{n=1}^{\infty} \frac{f(n-1)}{n^s}$ associated with real strongly q-multiplicative functions $f(n)$ are studied. We will confine ourselves to the case $\sum_{i=0}^{q-1} f(i) = 0$. It is known that in this case the function $\kappa_f(s)$ has an analytic continuation to the whole complex plane as an entire function with trivial zeros on the negative real line. The real function $\Lambda_f(t)$ satisfying the integral equation with delayed argument $\delta_f \Lambda_f(\frac{t}{q}) = \int_0^t \Lambda_f(u)\,du$ for some nonzero real δ_f naturally appears in the representation of the function $\kappa_f(s)$. In this article we find some asymptotic properties of the function $\kappa_f(s)$, prove that $\kappa_f(s)$ is an entire function of order 2, and also prove that in the region $\Re s \leq -k_0$, $|\Im s| \leq \frac{\pi}{2\ln q}$ the function $\kappa_f(s)$ has only trivial zeros which are simple.

Key words: Dirichlet series, strongly q-multiplicative functions

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1. Introduction

For a given integer $q \geq 2$ a nonzero function $f\colon \mathbf{N} \to \mathbf{R}$ is called strongly q-multiplicative if

$$f\left(\sum_{l=0}^{L} \varepsilon_l q^l\right) = \prod_{l=0}^{L} f(\varepsilon_l).$$

Here $\sum_{l=0}^{L} \varepsilon_l q^l$ is the q-ary expansion of the natural number. (Some authors use the expression completely q-multiplicative function). We will deal with real functions for simplicity but one can consider complex functions. Note that $f(0) = 1$. All functions and constants which will appear in the future depend on f, nevertheless, we will omit the index f making an exception only for $\kappa_f(s)$. We will consider only the case

$$\sum_{i=0}^{q-1} f(i) = 0. \tag{1}$$

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The generating power series is

$$T(z)=\sum_{n=0}^{\infty}f(n)z^{n}=\prod_{i=0}^{\infty}\left(\sum_{j=0}^{q-1}f(j)z^{q^{i}j}\right)=\prod_{i=0}^{\infty}P\big(z^{q^{i}}\big).$$

It is obvious that this series converges for $|z|<1$ and has the unit circle as its natural boundary. In fact, suppose ζ is a regular point of $T(z)$, where ζ is a q^{n}-th root of unity. Then from the equality $T(z)=P(z)P(z^{q})\dots P(z^{q^{n-1}})T(z^{q^{n}})$ we deduce that $z=1$ is a regular point also. In this case $\lim_{r\to1}T(r)=A$ is a finite number. From the equalities $P(1)=0$ and $T(z)=P(z)T(z^{q})$ we derive that $0\cdot A=A$, hence $A=0$. Then for any q^{m}-th root of unity ζ', we have $\lim_{r\to1}T(r\zeta')=0$, and since this set is everywhere dense on the unit circle, ζ is not a regular point of $T(z)$. Our aim is to investigate the series

$$\kappa_{f}(s)=\sum_{n=1}^{\infty}\frac{f(n-1)}{n^{s}}\tag{2}$$

which is defined for the complex number $s=\sigma+it$. Let $A=\sup_{i\leq q-1}|f(i)|$. Since trivially $|f(n)|\leq A^{\log_{q}n}$, Dirichlet series (2) is absolutely convergent for $\sigma>1+\gamma$, where $\gamma=\frac{\ln A}{\ln q}$ and thus define an analytic function in this region. This type of Dirichlet series was studied in [5]. The special case $f(n)=\zeta^{s_{q}(n)}$, where ζ is q-th root of unity $\neq 1$, and $s_{q}(n)$ is the sum of the digits of n in the q-ary expansion was studied in [1]. It was shown that, if Condition (1) is satisfied, the function $\kappa_{f}(s)$ admits an analytic continuation to the whole complex plane as an entire function with trivial zeros for all non-positive integers. We will demonstrate an existence of the analytic continuation in a more elementary way. The original proof in [5] uses the Mellin transform approach and this is a more fundamental point of view as proposed below but our method allows to calculate the order of the function $\kappa_{f}(s)$.

2. Analytic continuation

To continue $\kappa_{f}(s)$ analytically, we will use partial summation. For any nonnegative real t define $\rho_{1}(t)=\sum_{1\leq i\leq t}f(i-1)$, where i runs through the integers in the interval $[1,t]$, (for $t<1$ the sum is considered to be 0). Thus summing by parts we get

$$\kappa_{f}(s)=s\int_{0}^{\infty}\frac{\rho_{1}(t)}{t^{s+1}}dt.\tag{3}$$

The integral on the right is absolutely convergent for $\sigma>\gamma$ and uniformly for $\sigma\geq\gamma+\delta$ with positive δ. This gives analytic continuation to this region. In the sequel when we will encounter with a representation of a function as an integral, we will not mention that convergence is uniform in regions which cover the specified region, thus having that this function is analytic. Equation (3) will be the basis of analytic continuation by integrating by part. Define by induction $\rho_{k+1}(t)=\int_{0}^{t}\rho_{k}(u)\,du$. By induction we can prove that there exists a periodic function $\overline{\rho_{k}}(t)$ with a period equal to q^{k} satisfying $\rho_{k}(t)=f(\frac{t}{q^{k}})\overline{\rho_{k}}(t)$. Here we as usually write $f(\frac{t}{q^k})$ for $f([\frac{t}{q^k}])$. From the integral expression connecting $\rho_{k+1}(t)$ and $\rho_k(t)$ and from the definition of the function $f(n)$ it is easy to check the inductive proposition. Note that $\rho_k(t) = 0$ for $0 \le t \le 1$. Hence integrating Eq. (3) by parts k times, we obtain

$$\kappa_f(s) = s(s+1)\dots(s+k)\int_0^\infty \frac{\rho_k(t)}{t^{s+k+1}}dt \tag{4}$$

This is valid for $\sigma > \gamma$ but the integral is absolutely convergent for $\sigma > -k+\gamma$, hence it provides with the analytic continuation of the function $\kappa_f(s)$ in this region. Here we deduce that $\kappa_f(n) = 0$ for all integers $n \le 0$. Using an analogous method to obtain the analytic continuation of the Riemann ζ function to the whole complex plane demands a more detailed study of the analogous sequence of functions $\widehat{\rho_k}(t)$, approximating them by the Bernoulli polynomials $B_k(t)$.

Now we can investigate functions $\rho_k(t)$ and prove that this sequence, after renormalization, uniformly converges to a certain function $\Lambda(t)$ in any finite interval. To simplify calculations we can assume that

$$\Delta = -\sum_{i=0}^{q-1} i \cdot f(i) \neq 0.$$

In fact, there exists $m \le q-1$ for which $\sum_{i=0}^{q-1} i^m \cdot f(i) \neq 0$, because the determinant of the matrix $(i^m)_{i,m=0}^{q-1}$ is not zero. (We define $0^0 = 1$). Choosing the least natural m with this property we could make analogous calculations. So in case $\Delta \neq 0$ we can prove that for constants $c_k = (\Delta)^{k-1} \cdot q^{(k-2)(k-1)/2}$ the functions

$$\Lambda_k(t) = c_k^{-1} \cdot \rho_k(q^k t)$$

uniformly converge to a certain function $\Lambda(t)$ in any finite interval. For the economy of space we will omit the proof, which uses only elementary means. So we have $\int_0^t \Lambda_k(u)\,du = \frac{\Delta}{q}\Lambda_{k+1}(t)$ and hence

$$\int_0^t \Lambda(u)\,du = \frac{\Delta}{q}\Lambda\left(\frac{t}{q}\right). \tag{5}$$

The last equation gives

$$\Lambda^{(n)}(t) = \Lambda(q^n t)\frac{q^{n^2/2+3n/2}}{\Delta^n}.$$

Since $\Lambda^{(n)}(0) = 0$ for all positive integers n, then for a positive r we have the Taylor formula $\Lambda(r) = \frac{\Lambda^{(n)}(\eta)}{n!}r^n$ for a certain $0 \le \eta \le r$. Let $0 \le \varepsilon \le 1$. Then $\Lambda(\varepsilon q^{-k}) = \Lambda(q^k\eta')q^{-k^2/2+3k/2}\varepsilon^k\Delta^{-k}(k!)^{-1}$ for some $\eta' \le \varepsilon q^{-k}$. Since $0 \le q^k\eta' \le 1$, this gives

$$\left|\Lambda\left(\frac{\varepsilon}{q^k}\right)\right| \le C\frac{q^{-k^2/2+3k/2}}{|\Delta|^k k!}. \tag{6}$$

There exist a constant B for which $|\Lambda_k(t)| \le B|f(t)|$. Also we can prove that $\Lambda_k(\frac{j}{q}) = \sum_{i<j} f(i)$, $j \in \mathbf{N}$. It is convenient to replace $\rho_k(t)$ in the expression of $\kappa_f(s)$ by $c_k \cdot \Lambda_k(\frac{t}{q^k})$. Hence we have

$$\frac{\kappa_f(s)\Gamma(s)}{\Omega(s)} = \frac{q}{\Delta} \cdot \frac{\Gamma(s+k+1)}{\Omega(s+k)} \cdot \int_0^\infty \frac{\Lambda_k(t)}{t^{s+k+1}}\,dt. \tag{7}$$

This is valid for $\sigma > -k + \gamma$. Here $\Omega(s) = q^{s^2/2+3s/2}(\Delta)^{-s}$. (For negative value of Δ we write $(\Delta)^s$ for$|\Delta|^s \cdot e^{\pi i s}$).

3. The order of the function $\kappa_f(s)$

Define a new function

$$\Psi(s) = \int_0^\infty \Lambda(t)t^{-s-1}\,dt.$$

Using (5) we can integrate this integral by parts and get a functional equation for $\Psi(s)$, namely $\Psi(s) = (s+1)\cdot\Delta\cdot q^{-s-2}\cdot\Psi(s+1)$ and hence $\Psi(s) = p(s)\cdot\Omega(s)\cdot\Gamma^{-1}(s+1)$, where $p(s)$ is an analytic periodic function: $p(s) = p(s+1)$. Now we choose any compact set E of the region $\sigma > \gamma$ and take in (7) $s = s' - k$ and let $k \to \infty$. From the evaluation of $\Lambda_k(t)$ we get that $\int_0^\infty \Lambda_k(t)t^{-s-1}dt$ uniformly converges to $\int_0^\infty \Lambda(t)t^{-s-1}dt$ in E. Hence we have the following statement.

Theorem 1. *There exists an analytic periodic function $p(s)$, with period 1, such that*

$$\frac{\kappa_f(s-k)\Gamma(s-k)}{\Omega(s-k)} \to p(s) \quad \text{as } k \to \infty \tag{8}$$

The convergence is uniform in any compact set of the complex plane.

As we will see later, the Fourier expansion of the function $p(s) = \sum_{n\in\mathbf{Z}} a_n e^{2\pi i n s}$ has infinitely many nonzero coefficients for negative and positive n. The statement of Theorem 1 is of some interest, since it shows that the function on the left of (7) has some features of a periodic function. Note that an analogous theorem for the Riemann ζ function is a direct consequence of the functional equation. In fact, let $\Gamma(s/2)\cdot\Gamma^{-1}((1-s)/2)\cdot\pi^{-s}\cdot\sqrt{\pi} = \Phi(s)$. Then $\zeta(s-k)\Phi(s-k) \to 1$ uniformly in any compact set E. Hence in the case of $\kappa_f(s)$ we have a more interesting situation.

Now we are able to calculate the order of the function $\kappa_f(s)$.

Definition. The order of the entire function $F(z)$ is the number $\rho = \limsup_{r\to\infty} \frac{\ln\ln M(r)}{\ln r}$, where $M(r) = \sup_{|z|=r} |F(z)|$.

Theorem 2. *The order of the entire function $\kappa_f(s)$ is 2.*

Proof: Let us evaluate $|\kappa_f(s)|$ for $\sigma < -2-\gamma$. Choose a natural k such that $\gamma+1 \le \sigma+k < \gamma+2$. Then $|s+i| \le |s|$ for $i = 1, 2, \ldots k$ and $k \le 2|\sigma|$, hence (4) implies $|\kappa_f(s)| \le A^k|s|^{k+1}q^{k^2/2} \le G\exp(C|s|\ln|s| + D|s|^2)$ (we have in mind the values of the constants c_k). So for $\sigma < -2-\gamma$ we have $|\kappa_f(s)| < \exp(H|s|^2)$. We use analogous considerations to evaluate $\kappa_f(s)$ in the strip $-\gamma-2 \le \sigma \le \gamma+2$. In the half plane $\sigma > \gamma+2$ $\kappa_f(s)$ is bounded, hence the order of $\kappa_f(s)$ is not greater than 2. Now choose r from the interval $[\frac{1}{4}, \frac{3}{4}]$ such that $p(r) \neq 0$ (as will be clear from the argument what follows, $p(s)$ is not a constant). From the functional equation of the Γ function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

and from the Stirling formula we get $|\Gamma(-k+r)| < A\exp(-\frac{1}{2}k\ln k)$ for all $k > k_0$ with an appropriate positive A. Now considering (8) for $s = r$, we deduce that the order of the $\kappa_f(s)$ is not less than 2. Theorem is proved. □

4. The function $p(s)$

Let us consider the function $\Psi(s)$ more closely. We have

$$h(s) = \Psi(s-1)\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}\,dx \int_0^\infty \Lambda(t)t^{-s}dt, \quad \sigma > 1+\gamma.$$

Since $e^{-x}x^{s-1}\Lambda(t)t^{-s}$ is an integrable function in the first quadrant for $\sigma > 1+\gamma$, we can write the above as a double integral by Fubini theorem. Changing variables $t = t$, $x = \alpha t$ we get

$$h(s) = \int_0^\infty g(\alpha)\alpha^{s-1}d\alpha = \ln q \int_{-\infty}^\infty S(\alpha)q^{\alpha s}d\alpha$$

where $g(\alpha) = \int_0^\infty e^{-\alpha t}\Lambda(t)\,dt$ and $S(\alpha) = g(q^\alpha)$. Hence $g(\alpha)$ is the Laplace transform of $\Lambda(t)$ and $h(s)$ is the Mellin transform of $g(\alpha)$. We have

$$\begin{aligned} S(\alpha) &= \sum_{i=0}^\infty e^{-q^\alpha i}\int_0^1 e^{-q^\alpha t} f(i)\Lambda(t)\,dt = \sum_{i=0}^\infty f(i)e^{-q^\alpha i}\int_0^1 e^{-q^\alpha t}\Lambda(t)\,dt \\ &= T\left(e^{-q^\alpha}\right)\int_0^1 e^{-q^\alpha t}\Lambda(t)\,dt. \end{aligned}$$

Define $\int_0^1 e^{-q^\alpha t}\Lambda(t)\,dt$ by $F(q^\alpha)$. The function $S(\alpha)$ is defined for α for which $|e^{-q^\alpha}| < 1$. That is, $S(\alpha)$ is defined for α, for which $\Re q^\alpha > 0$. Let us investigate this function. $S(\alpha) = \int_0^\infty e^{-q^\alpha t}\Lambda(t)\,dt = \int_0^\infty e^{-q^\alpha t}d\frac{\Delta}{q}\Lambda(\frac{t}{q}) = q^\alpha\Delta\int_0^\infty e^{-q^{\alpha+1}t}\Lambda(t)\,dt = q^\alpha\Delta S(\alpha+1)$. Hence $S(\alpha) = q^{-\alpha^2/2+\alpha/2}Q(\alpha)\Delta^{-\alpha}$, where Q is an analytic function, satisfying the relation $Q(\alpha) = Q(\alpha+1)$ for those α for which $\Re q^\alpha > 0$, that is, for $-\frac{\pi}{2}+2\pi l < \Im\alpha\ln q < \frac{\pi}{2}+2\pi l$, $l \in \mathbf{Z}$. Hence we have

$$h(s) = \ln q \int_{-\infty}^\infty q^{-\alpha^2/2+\alpha/2}Q(\alpha)\Delta^{-\alpha}q^{\alpha s}d\alpha \tag{9}$$

The equation $S(\alpha-1)=q^{\alpha-1}\Delta S(\alpha)$, written in the terms of F becomes $F(q^{\alpha})=\frac{P(e^{-q^{\alpha-1}})}{\Delta q^{\alpha-1}}F(q^{\alpha-1})$. Since $F(q^{\alpha-k})\to\int_0^1\Lambda(t)\,dt=\frac{\Delta}{q}$ as $k\to\infty$, we have

$$F(q^{\alpha})=\frac{\Delta}{q}\prod_{i=1}^{\infty}\frac{P(e^{-q^{\alpha-i}})}{\Delta\cdot q^{\alpha-i}}$$

with the convergent infinite product.

For completeness we give the explicit expression of $Q(\alpha)$. From the considerations above it can be deduced that

$$Q(\alpha)=q^{\alpha^2/2-\alpha/2}\cdot T(e^{-q^{\alpha}})\cdot\frac{\Delta}{q}\cdot\prod_{i=1}^{\infty}\frac{P(e^{-q^{\alpha-i}})}{\Delta\cdot q^{\alpha-i}}\cdot\Delta^{\alpha}.$$

We can check directly that this function is periodic. For $|\Im\alpha|<\frac{\pi}{2\ln q}$, we have the Fourier expansion $Q(\alpha)=\sum_{n\in\mathbf{Z}}b_n e^{2\pi in\alpha}$. As is clear from above, the lines $t=-\frac{\pi}{2\ln q}$ and $t=\frac{\pi}{2\ln q}$ are natural boundary of this Fourier series, hence it has infinitely many nonzero coefficients for negative and positive n. Then we can integrate (9) term by term obtaining $h(s)=\ln q\sum_{n\in\mathbf{Z}}b_n h_n(s)$, where

$$h_n(s)=\int_{-\infty}^{\infty}q^{-\alpha^2/2+\alpha/2}e^{2\pi in\alpha}\Delta^{-\alpha}q^{\alpha s}\,d\alpha$$

The integrating term by term can be based on the Lebesgue theorem: the integral $h_n(s)$ absolutely converges and the series $\sum_{n\in\mathbf{Z}}|b_n|$ is also absolutely convergent.

Changing variables $\alpha=\alpha'+s+\frac{2\pi in}{\ln q}$ we obtain

$$h_n(s)=\Delta^{-s}q^{s^2/2+s/2}e^{2\pi ins}q^{-\frac{2\pi^2n^2}{\ln^2 q}}\xi_n\int_{-\infty}^{\infty}q^{-\alpha^2/2+\alpha/2}\Delta^{-\alpha}\,d\alpha \tag{10}$$

In fact, limits should be from $-\infty-it-\frac{2\pi in}{\ln q}$ to $\infty-it-\frac{2\pi in}{\ln q}$, but since the integrable function has no poles in the considered region and since vertical integrals tend to zero, we can change the limits. Here $|\xi_n|=1$. Note that the integral on the right of (10) is a nonzero constant. Hence up to a constant multiplier the Fourier coefficients of $p(s)$ and $Q(s)$ are related by $a_n=q^{-\frac{2\pi^2n^2}{\ln^2 q}}\xi_n b_n$ and the statement about the coefficients of the function $p(s)$ is proved.

5. The zeros of the functions $p(s)$ and $\kappa_f(s)$

First we will prove that the function $p(s)$ has no zeros in the region $|t|\le\frac{\pi}{2\ln q}$ and this will allow us to make some conclusions about the zeros of $\kappa_f(s)$.

Theorem 3. *Function $p(s)$ has no zeros in the region $|t|\le\frac{\pi}{2\ln q}$.*

Proof: From the equation $\Psi(s-1) = p(s) \cdot \Omega(s-1) \cdot \Gamma^{-1}(s)$, where

$$\Psi(s) = \int_0^\infty \frac{\Lambda(x)}{x^s}\, dx$$

we find that the zeros of $p(s)$ and $\Psi(s)$ in the region $\sigma > 1+\gamma$ coincide. Hence we will prove that $\Psi(s)$ has no zeros in the region $|t| \le \frac{2\pi}{\ln q}$ and $\sigma > 1+\gamma$. Suppose it has. Let s be a zero of $\Psi(s)$. Then also $s+n$, $n \in \mathbf{N}$, is a zero. We suppose that $t > 0$. The case $t < 0$ is analogous and the case $t = 0$ is simpler. Making the substitutions $x \to \frac{1}{x}$ and $s \to s+2$, we get that the imaginary part is also zero:

$$\int_0^\infty \Lambda\left(\frac{1}{x}\right) x^\sigma \sin(t \ln x)\, dx = 0.$$

Since the integral is additive, then, for all polynomials $P(x)$, we have

$$\int_0^\infty \Lambda\left(\frac{1}{x}\right) x^\sigma P(x) \sin(t \ln x)\, dx = 0. \tag{11}$$

Our plan is the following: we will construct polynomials for which (11) fails, thus proving the theorem. The function $\Lambda(x)$ vanishes in the interval [0, 1] in a finite set of real numbers $x_1, x_2, \ldots, x_m$ (even more, $m \le q$) (we do not consider these points for which $\Lambda(x_i - \varepsilon)$ and $\Lambda(x_i + \varepsilon)$ for small ε have the same sign). Let $W(x) = (x - x_1^{-1})(x - x_2^{-1}) \ldots (x - x_m^{-1})$. The function $\sin(t \ln x)$ for $x > 1$ has simple zeros at the points $\exp(\frac{\pi k}{t})$, $k \in \mathbf{N}$. Now define a function $R(x)$ and polynomials $P_T(x)$, $T \in \mathbf{N}$:

$$R(x) = \prod_{k=1}^{\infty} \left(1 - x \cdot \exp\left(-\frac{\pi k}{t}\right)\right), \quad P_T(x) = \prod_{k=1}^{T} \left(1 - x \cdot \exp\left(-\frac{\pi k}{t}\right)\right).$$

Note that the infinite product converges, thus the function $R(x)$ is defined correctly. Since $R(x)$ is strongly positive in the interval [0, 1] and finite, $0 < R(x) \le c$ in this interval. Trivially $|P_T(x)| \le 1$ for $0 \le x \le 1$. Since $R(x \exp(-\frac{\pi T}{t})) \cdot P_T(x) = R(x)$, hence

$$|P_T(x)| \ge c^{-1} |R(x)|, \quad 0 \le x \le \exp\left(\frac{\pi T}{t}\right). \tag{12}$$

Let $\delta(T)$ be any natural number. We will choose later $\delta(T)$ so that $\delta(T) \to \infty$. Note that $|P_T(x)| < d_T x^T$ for $x \ge \exp(\frac{\pi T}{t})$, where $d_T = \exp(-\frac{\pi(T^2+T)}{t})$, and trivially $|W(x)| < C x^m$

for all positive x. Now evaluate the following integral (we use (6)):

$$\left|\int_{\exp(\frac{\pi T}{t})}^{\infty}\Lambda\left(\frac{1}{x}\right)x^{\sigma+\delta(T)}W(x)\sin(t\ln x)P_T(x)\,dx\right|$$
$$< Cd_T\int_{\exp(\frac{\pi T}{t})}^{\infty}\Lambda\left(\frac{1}{x}\right)x^{\sigma+\delta(T)+T+m}\,dx$$
$$< \hat{C}d_T\sum_{k=[\frac{\pi T}{t\ln q}]}^{\infty}\frac{q^{(k+1)(\sigma+\delta(T)+T+m)-k^2/2+3k/2}}{|\Delta|^k k!}.$$

Since $0 < t \le \frac{\pi}{2\ln q}$, then $k > 2T-1$, and also $kT - k^2/2 < k/2$. We can continue:

$$\left|\int_{\exp(\frac{\pi T}{t})}^{\infty}\cdots dx\right| < \hat{C}d_T q^{T+\sigma+\delta(T)+m}\sum_{k>2T-1}\frac{q^{k(\sigma+\delta(T)+m+2)}}{\Delta^k k!}$$
$$< \hat{C}\exp\left(-\frac{\pi(T^2+T)}{t}\right)q^{T+\sigma+\delta(T)+m}\exp\left(q^{\sigma+\delta(T)+m+2-\log_q(|\Delta|)}\right).$$

If we now choose $\delta(T) = [\log_q T]$ in the last inequality we will have

$$\left|\int_{\exp(\frac{\pi T}{t})}^{\infty}\Lambda\left(\frac{1}{x}\right)x^{\sigma+\delta(T)}W(x)\sin(t\ln x)P_T(x)\,dx\right| \to 0 \text{ as } T\to\infty. \tag{13}$$

Next, evaluation of integral (11) for $P(x) = x^{\delta(T)}W(x)P_T(x)$ in the interval $[0, 1]$ is easy (note that $\Lambda(x) < Bx^{\varepsilon}$ for an appropriate ε for $x \ge 1$):

$$\left|\int_0^1\Lambda\left(\frac{1}{x}\right)x^{\sigma+\delta(T)}P_T(x)W(x)\sin(t\ln x)\,dx\right|$$
$$< \frac{C}{\sigma+\delta(T)+1-\varepsilon} \to 0 \text{ as } T\to\infty. \tag{14}$$

At last, $\sin(t\ln x)P_T(x)$ and $\Lambda(\frac{1}{x})W(x)$ have constant sign in the interval $[1, \exp(\frac{\pi T}{t})]$, hence using(12) we have:

$$\left|\int_1^{\exp(\frac{\pi T}{t})}\Lambda\left(\frac{1}{x}\right)x^{\sigma+\delta(T)}W(x)P_T(x)\sin(t\ln x)\,dx\right|$$
$$\ge c^{-1}\left|\int_1^{\exp(\frac{\pi T}{t})}\Lambda\left(\frac{1}{x}\right)x^{\sigma+\delta(T)}W(x)R(x)\sin(t\ln x)\,dx\right| \to \infty \text{ as } T\to\infty \tag{15}$$

Now (13), (14), (15) are inconsistent with (11). Theorem 3 is proved. □

The following corollary partly gives the answer to the question posed in [1] about the simplicity of the trivial zeros of the function $\kappa_f(s)$.

Corollary. *There exists the positive integer k_0 such that in the region $\Re s \leq -k_0$ and $|\Im s| \leq \frac{\pi}{2\ln q}$ the function $\kappa_f(s)$ has only trivial zeros and they are simple.*

Proof: Let $\inf_{s\in E} |p(s)| = u$, where E is the set $0 \leq \Re s \leq 1$, $|\Im s| \leq \frac{\pi}{2\ln q}$. Then $u > 0$ and Theorem 1 shows that $|\frac{\kappa_f(s-k)\Gamma(s-k)}{\Omega(s-k)}| > u/2$ in this region for $k \geq k_0$. That is $|\frac{\kappa_f(s)\Gamma(s)}{\Omega(s)}| > u/2$ in the region described in the formulation of the corollary. Hence $\kappa_f(s)$ has a zero only for those s in this region, where Γ has a pole, that is only at $s = -k$. The pole is simple, hence the zero is also simple. □

For the conclusion we give the following proposition without proof and we believe that this is the general case. For the 2-multiplicative function $f(0) = 1$, $f(1) = -1$ (for the Thue-Morse sequence) the statement is correct.

Proposition (*unproved*).

$$\frac{\kappa_f(s)\Gamma(s)}{\Omega(s)} \approx p(s)X(q^s).$$

Here $X(z)$ is an analytic in some circle $|z| < \delta$. The sign $\approx$ has the same meaning as in Theorem 1, that is

$$q^{kn}\left(\frac{\kappa_f(s-k)\Gamma(s-k)}{\Omega(s-k)} - p(s-k)X_n(q^{s-k})\right) \to d_n q^{sn} \text{ as } k \to \infty$$

Here $X_n(z)$ is the polynomial defined via the first n terms of the Taylor series of $X(z)$, d_n is the coefficient of z^n. The convergence is uniform in any compact set E of the complex plane.

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