Abstract. We construct one-term Edgeworth expansions to distributions of $U$ statistics and Studentized $U$ statistics, based on stratified samples drawn without replacement. Replacing the cumulants defining the expansions by consistent jackknife estimators we obtain Empirical Edgeworth expansions. The expansions provide second order approximations that improve upon the normal approximation. Theoretical results are illustrated by a simulation study where we compare various approximations to the distribution of the commonly used Gini’s mean difference estimator.

Key Words: Edgeworth expansion, jackknife, sampling without replacement, $U$-statistic, normal approximation.

Mathematics Subject Classification 2000: 62E20, 62D05, 62F12, 62F40, 62G20, 62G09

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1 INTRODUCTION

Stratified simple random sampling without replacement (STSRS for short) is widely used in surveys. It generalizes sampling with replacement and without replacement. In order to assess the precision of survey estimates it is important to develop approximations to distributions of various classes of estimators based on STSRS samples.

Here we study estimators which are $U$ statistics. It is well known that, generally, non-degenerate $U$-statistics are asymptotically normal, see [18]. An improvement upon the normal approximation is provided by an Edgeworth expansion, see [14]. In this paper we construct the one-term Edgeworth expansions for asymptotically linear $U$ statistics and Studentized $U$ statistics.

In the case where the population parameters (cumulants) defining the expansion are unknown it is convenient to use Empirical Edgeworth expansions where unknown parameters are replaced by their estimators ([3], [16], [17], [22]). We define the jackknife estimators and show their consistency.

Since general symmetric statistics can be approximated by $U$ statistics up to second order (e.g., via Hoeffding’s decomposition, see [2], [9]) our results extend to general asymptotically linear symmetric statistic.

In a simulation study we compare the accuracy of the normal approximation, Edgeworth expansion and empirical Edgeworth expansion for the commonly used STSRS estimator of Gini’s mean difference. The simulation demonstrates that the Empirical Edgeworth expansion outperforms the normal approximation, but tends to be less accurate than the true Edgeworth expansion.

The outline of the paper is as follows. Results on Edgeworth expansions and Empirical Edgeworth expansions are given in Sections 2 and 3. Results of a simulation study are referred to in Section 4. Proofs and technical details are given in the Appendix.

2 Edgeworth expansion

Consider the population $\mathcal{X} = \{x_1, \ldots, x_N\}$ and assume that we want to estimate the population parameter $u = \sum_{1 \leq i < j \leq N} t(x_i, x_j)$, where $t$ is a symmetric function (i.e., $t(x, y) = t(y, x)$). Suppose that the population is divided in $h$ non-intersecting strata $\mathcal{X} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_h$. Here $\mathcal{X}_k = \{x_{k,1}, \ldots, x_{k,N_k}\}$ and $N_1 + \ldots + N_k = N$. From every $\mathcal{X}_k$ we draw (without replacement) the simple random sample $X_k = \{X_{k,1}, \ldots, X_{k,n_k}\}$, $k = 1, \ldots, h$. 
so that the samples $X_1, \ldots, X_h$ are independent. The statistic, based on the stratified sample without replacement $X = (X_1, \ldots, X_h),$

\[
\hat{u} = \hat{u}(X) = \sum_{1 \leq j \leq h} \sum_{(x,y) \in X_j} w_j t(x,y) + \sum_{1 \leq j \leq h} \sum_{x \in X_j} \sum_{y \in X_j} w_{jr} t(x,y),
\]

is an unbiased estimator of the parameter $u$. Here

\[
w_j = w_j(X) = \left( \frac{N_j}{2} \right) \left( \frac{n_j}{2} \right)^{-1}, \quad w_{jr} = w_{jr}(X) = N_j N_r (n_j n_r)^{-1}.
\]

Note that in the case where $t(x,y) = |x-y|$ and $x_i, i = 1, \ldots, N$, denote real valued measurements, we obtain the estimator of Gini’s mean difference.

Our analysis of the distribution of $\hat{u}$ uses linearization by means of Hoeffding’s ([18]) decomposition. We approximate $\hat{u}$ by a linear statistic $L$ of the form

\[
L = \sum_{k=1}^{h} L_k, \quad L_k = \sum_{x \in X_k} g_k(x).
\]

Choosing (first order influence) functions $g_k$ that minimize the expectation $E|\hat{u} - u - L|^2$ and denoting the remaining quadratic part $Q$ we obtain Hoeffding’s decomposition

\[
\hat{u} = u + L + Q.
\]

Note that the linear part $L$ and quadratic part $Q$ in (3) are uncorrelated. Explicit formulas for $L$ and $Q$ are given in (20) below.

We shall assume that the variance $\sigma_L$ of the linear part $L$ is positive, $\sigma_L^2 > 0$. By the central limit theorem, for large $n = n_1 + \ldots + n_h$ and $N$, the distribution of $L/\sigma_L$ can be approximated by the standard normal distribution, see [4], [15]. In the case where the linear part dominates the statistic and we have $\sigma^2/\sigma_L^2 \approx 1$, the normal approximation applies to $(\hat{u} - u)/\sigma$ as well. Here $\sigma^2$ denotes the variance of $\hat{u}$.

Assuming that sample sizes $n_k$ are large for every $k$, we construct the one-term Edgeworth expansion for the distribution function $F_u(x) = P\{\hat{u} - u \leq \sigma x\}$ and for the distribution function $F_S(x) = P\{\hat{u} - u \leq xS\}$ of Studentized $U-$ statistic. $S^2$ denotes the classical (delete-one) jackknife estimator of the variance $\sigma^2$,

\[
S^2 = \sum_{k=1}^{h} q_k \frac{n_k - 1}{n_k} v_k^2, \quad v_k^2 = \sum_{i=1}^{n_k} (\hat{u}(X_{k|i}) - \bar{u}_k)^2, \quad \bar{u}_k = \frac{1}{n_k} \sum_{k=1}^{n_k} \hat{u}(X_{k|i}).
\]
Here $X_k|_i$ denotes the STSRS sample obtained from $X$ by removing the observation $X_{k,i}$.

**Theorem 2.1.** The one-term Edgeworth expansions of $F_u(x)$ and $F_S(x)$ are functions $G_u(x)$ and $G_S(x)$,

$$
G_u(x) = \Phi(x) - \frac{\alpha + 3\kappa}{6\sigma^3} \Phi'(x)(x^2 - 1),
$$

(5)

$$
G_S(x) = \Phi(x) + \frac{\alpha + \alpha'x^2 + 3\kappa(x^2 + 1)}{6\sigma^3} \Phi'(x).
$$

(6)

Here $\Phi$ denotes the standard normal distribution function and $\Phi'$ denotes its derivative. The moments $\alpha, \alpha'$ and $\kappa$ are given in (8) below.

Denote

$$
\tau^2_k = N_k p_k q_k, \quad p_k = n_k/N_k, \quad q_k = (N_k - n_k)/N_k
$$

(7)

and introduce the moments

$$
\alpha = \sum_{r=1}^{h} (q_r - p_r) \tau^2_r \alpha_r, \quad \alpha' = \sum_{r=1}^{h} (1 + q_r) \tau^2_r \alpha_r,
$$

(8)

$$
\kappa = \sum_{r=1}^{h} \tau^4_r \kappa_{rr} + 2 \sum_{1 \leq k < r \leq h} \tau^2_k \tau^2_r \kappa_{kr}.
$$

Here

$$
\alpha_k = \mathbf{E}g^2_k(X_{k,1}) = \frac{1}{N_k} \sum_{x \in X_k} g^2_k(x),
$$

$$
\kappa_{kk} = \mathbf{E}\psi_k(X_{k,1}, X_{k,2})g_k(X_{k,1})g_k(X_{k,2}) = \frac{1}{(N_k/2)} \sum_{(x,y) \in X_k} \psi_k(x,y)g_k(x)g_k(y),
$$

$$
\kappa_{kr} = \mathbf{E}\psi_{kr}(X_{k,1}, X_{r,1})g_k(X_{k,1})g_r(X_{r,1}) = \frac{1}{N_k N_r} \sum_{x \in X_k, y \in X_r} \psi_{kr}(x,y)g_k(x)g_r(y).
$$

Functions $\psi_k(X_{k,i}, X_{k,j})$ and $\psi_{kr}(X_{k,i}, X_{r,j})$ (sometimes called the second order influence functions) reflect the contribution of pairs of observations $\{X_{k,i}, X_{k,j}\}$ and $\{X_{k,i}, X_{r,j}\}$. Explicit expressions of $g_k$ and $\psi_k, \psi_{kr}$ are given in (21) and (22) below.

**Remark 2.1.** Theorem 2.1 provides *formal* Edgeworth expansions in the asymptotic framework where the number of strata remains bounded and, for every $k$, sample sizes $n_k$ diverge to infinity and $\sigma^2_L/\sigma^2 \to 1$. To be more precise assume that we have a sequence of stratified populations
\[ X^{(\nu)} = X_1^{(\nu)} \cup \ldots \cup X_h^{(\nu)} \] of sizes \( N^{(\nu)} = N_1^{(\nu)} + \ldots + N_h^{(\nu)} \). Assume that for every \( k = 1, \ldots, h \) the numbers \( n^{(\nu)} k = \min\{n_k^{(\nu)}, N_k^{(\nu)} - n_k^{(\nu)}\} \) diverge to infinity and \( \sigma_2^2 / \sigma^2 \to 1 \) as \( \nu \to \infty \). In this case one can prove that the expansions (5), (6) provide approximations to the distribution functions \( F_u \) and \( F_S \) that improve upon the normal approximation if, in addition, the linear part \( L \) satisfies a Cramér type condition and the kernel \( t \) satisfies appropriate moment conditions. Pushing harder on the methods used in this paper it is possible to establish \( O(n_*^{-1}) \) bound for the remainder \( \sup_x |F_u(x) - G_u(x)| \) (under \( 4+\delta \) moment conditions) and \( o(n_*^{-1/2}) \) bound for the remainder \( \sup_x |F_S(x) - G_S(x)| \) (under \( 6+\delta \) moment conditions) in the particular case where \( n_* k \approx n_* \), \( 1 \leq k < r \leq h \). Here \( n_* = n_* 1 + \ldots + n_* h \) denotes the "actual sample size". Rigorous proofs of such bounds in the case of simple random sampling (\( h = 1 \)) are given in [7], [10].

Remark 2.2. Normal approximation of \( U \)-statistics based on samples without replacement was studied in [8], [21], [25]. One term Edgeworth expansion for finite population \( U \)-statistics based on simple random samples without replacement was established in [20], see also [10]. A similar result for Studentized \( U \)-statistic was shown in [7]. Theorem 2.1 extends these results to stratified samples. The extension is non-trivial: a more complex structure of Hoeffding’s decomposition for STSRS samples yields new formulae for cumulants. In particular, the cumulant \( \alpha + 3\kappa \) defining the Edgeworth correction term of (5) includes the sum \( \sum_{r<k} \tau_r^2 \kappa_r \kappa_r \), see (8), reflecting the contribution of cross strata interactions.

One term Edgeworth expansions for a linear STSRS statistic were obtained in [11], [12]. Theorem 2.1 extends these results to a more general class of \( U \) statistics.

Remark 2.3. Edgeworth expansions also provide a theoretical tool for analysis and design of resampling approximations like bootstrap and empirical Edgeworth expansions, see [1], [5], [12], [16], [17], [22], [24]. In the next section we construct empirical Edgeworth expansions. The bootstrap approximation of STSRS \( U \)-statistics will be studied elsewhere.

3 Empirical Edgeworth expansion

Empirical Edgeworth expansions of statistics of independent observations were studied in [3], [5], [16], [17], [22]. Here we construct empirical Edgeworth expansions that are suited to \( U \)-statistics based on STSRS samples.
Let us define the jackknife estimators \( \hat{\alpha}_k \) and \( \hat{\kappa}_{kr} \) of the moments \( \alpha_k \) and \( \kappa_{kr} \). Substitution of these estimators in (8) give jackknife estimators \( \hat{\alpha}, \hat{\alpha}', \hat{\kappa} \) of \( \alpha, \alpha', \kappa \). Put

\[
\hat{\alpha}_k = \frac{1}{n_k} \sum_{i=1}^{n_k} V_{k|i}^2, \quad \hat{\kappa}_{kk} = \frac{1}{\binom{n_k}{2}} \sum_{1 \leq i < j \leq n_k} W_{k|ij} V_{k|i} V_{k|j}, \quad 1 \leq k \leq h, \tag{9}
\]

\[
\hat{\kappa}_{kr} = \frac{1}{n_k n_r} \sum_{i=1}^{n_k} \sum_{j=1}^{n_r} W_{kr|ij} V_{k|i} V_{r|j}, \quad 1 \leq k < r \leq h. \tag{10}
\]

Here we denote

\[
V_{k|i} = \tau_k - \hat{u}(X_{k|i}), \tag{11}
\]

\[
W_{k|ij} = \tilde{u}_k - \frac{n_k - 1}{n_k} \tau_{k(k)|i} - \frac{n_k - 1}{n_k} \tau_{k(k)|j} + \frac{n_k - 2}{n_k} \hat{u}(X_{k|ij}), \tag{12}
\]

\[
W_{kr|ij} = \tilde{u}_{kr} - \tau_{k(r)|i} - \tau_{r(k)|j} + \hat{u}(X_{kr|ij}) \tag{13}
\]

and, for \( 1 \leq k, r \leq h, k \neq r \) we write

\[
\hat{u}_k = \frac{1}{n_k} \sum_{1 \leq i < j \leq n_k} \hat{u}(X_{k|i}), \quad \hat{u}_{kr} = \frac{1}{n_k n_r} \sum_{i=1}^{n_k} \sum_{j=1}^{n_r} \hat{u}(X_{kr|ij}), \tag{14}
\]

\[
\tau_{k(k)|i} = \frac{1}{n_k} \sum_{1 \leq j \leq n_k, j \neq i} \hat{u}(X_{k|i}), \quad \tau_{k(r)|i} = \frac{1}{n_r} \sum_{j=1}^{n_r} \hat{u}(X_{kr|ij}).
\]

Here \( X_{k|i} \) denotes the STSRS sample obtained from \( X \) by removing the observations \( X_{k,i} \) and \( X_{k,j} \) from \( X_k \); \( X_{kr|ij} \) denotes the STSRS sample obtained from \( X \) by removing the observation \( X_{k,i} \) from \( X_k \) and \( X_{r,j} \) from \( X_r \).

Define \( \hat{\alpha}, \hat{\alpha}' \) and \( \hat{\kappa} \) by formulas (8), where \( \alpha_k \) and \( \kappa_{kr} \) are replaced by their estimators \( \hat{\alpha}_k \) and \( \hat{\kappa}_{kr} \). One can show under appropriate moment conditions, see Appendix below, that as \( n_k \to \infty \)

\[
\frac{\hat{\alpha}}{S^3} - \frac{\alpha}{\sigma^3} = o_P(1), \quad \frac{\hat{\alpha}'}{S^3} - \frac{\alpha'}{\sigma^3} = o_P(1), \quad \frac{\hat{\kappa}}{S^3} - \frac{\kappa}{\sigma^3} = o_P(1). \tag{15}
\]

Therefore the empirical Edgeworth expansions

\[
H_u(x) = \Phi(x) - \frac{\hat{\alpha} + 3\hat{\kappa}}{6S^3} \Phi'(x)(x^2 - 1), \tag{16}
\]

\[
H_S(x) = \Phi(x) + \frac{\hat{\alpha} + \hat{\alpha}'x^2 + 3\hat{\kappa}(x^2 + 1)}{6S^3} \Phi'(x). \tag{17}
\]

match \( G_u \) and \( G_S \) (in probability). In this case the empirical Edgeworth expansions provide approximations which are second-order correct in probability.
4 Simulation study

In this section, we examine the performance of the Edgeworth expansion (6) and the empirical Edgeworth expansion (17) by simulation. Two stratified populations \(X^{(r)} \subset \mathbb{R}, r = 1, 2,\) are generated, each consisting of three strata of sizes \(N_1 = N_2 = 500\) and \(N_3 = 200.\) For Gini’s mean difference estimator \(\hat{u}\) (defined in (1) and below) based on a stratified sample without replacement with sizes \(n_1 = n_2 = n_3 = 50,\) we evaluate approximation errors \(\Delta^{(r)}(x) = F_S(x) - \Phi(x),\) \(\Delta^{(r)}_E(x) = F_S(x) - \tilde{G}_S(x),\) and \(\Delta^{(r)}_{EE}(x) = F_S(x) - \tilde{H}_S(x).\) Here \(F_S\) denotes the distribution function of the Studentized statistic \((\hat{u} - u)/S\) obtained by a Monte-Carlo simulation and \(\tilde{G}_S,\) respectively, \(\tilde{H}_S\) denote the distribution functions which are obtained from \(G_S\) and \(H_S\) via the transformation \(\tilde{f}(x) := \sup_{y \leq x} ((0 \lor f(y)) \land 1).\)

Table 1. Approximation accuracy

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-2.33)</th>
<th>(-1.65)</th>
<th>(-1.29)</th>
<th>(-0.68)</th>
<th>(0.0)</th>
<th>(0.68)</th>
<th>(1.29)</th>
<th>(1.65)</th>
<th>(2.33)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Phi(x))</td>
<td>(0.01)</td>
<td>(0.05)</td>
<td>(0.1)</td>
<td>(0.25)</td>
<td>(0.5)</td>
<td>(0.75)</td>
<td>(0.9)</td>
<td>(0.95)</td>
<td>(0.99)</td>
</tr>
<tr>
<td>(10^3 \Delta^{(1)}(x))</td>
<td>(7.76)</td>
<td>(12.9)</td>
<td>(13.8)</td>
<td>(9.63)</td>
<td>(3.68)</td>
<td>(7.87)</td>
<td>(10.7)</td>
<td>(8.97)</td>
<td>(3.60)</td>
</tr>
<tr>
<td>(10^3 \Delta^{(1)}_E(x))</td>
<td>(2.29)</td>
<td>(2.05)</td>
<td>(2.0)</td>
<td>(1.69)</td>
<td>(1.08)</td>
<td>(0.41)</td>
<td>(-0.91)</td>
<td>(-1.96)</td>
<td>(-1.86)</td>
</tr>
<tr>
<td>(10^3 \Delta^{(2)}_{EE}(x))</td>
<td>(19.5)</td>
<td>(32.9)</td>
<td>(36.2)</td>
<td>(24.9)</td>
<td>(10.2)</td>
<td>(17.9)</td>
<td>(25.3)</td>
<td>(20.5)</td>
<td>(7.22)</td>
</tr>
<tr>
<td>(10^3 \Delta^{(2)}_E(x))</td>
<td>(6.38)</td>
<td>(6.19)</td>
<td>(6.93)</td>
<td>(4.78)</td>
<td>(0.92)</td>
<td>(-2.26)</td>
<td>(-3.94)</td>
<td>(-6.23)</td>
<td>(-2.7)</td>
</tr>
<tr>
<td>(10^3 \Delta^{(2)}_{EE}(x))</td>
<td>(13.3)</td>
<td>(20.6)</td>
<td>(23.1)</td>
<td>(17.2)</td>
<td>(8.24)</td>
<td>(10.1)</td>
<td>(12.25)</td>
<td>(8.18)</td>
<td>(1.01)</td>
</tr>
</tbody>
</table>

Table 1 demonstrates that the Edgeworth expansion improves upon the normal approximation (as expected). The empirical Edgeworth expansion outperforms the normal approximation, but tends to be less accurate than the true Edgeworth expansion.

Table 2 gives values of the ratios \(\alpha/\sigma^3,\) \(\alpha'/\sigma^3\) and \(\kappa/\sigma^3\) which define the Edgeworth expansion \(G_S.\) It also provides estimated values of the mean square errors of their estimators \(\alpha_* = \hat{\alpha}/S^3,\) \(\alpha'_* = \hat{\alpha}'/S^3\) and \(\kappa_* = \hat{\kappa}/S^3,\) cf. (15). The latter define the empirical Edgeworth expansion \(H_S.\)

Table 2. Moments and estimated MSE of jackknife estimators

<table>
<thead>
<tr>
<th>(r)</th>
<th>(\alpha/\sigma^3)</th>
<th>(\alpha'/\sigma^3)</th>
<th>(\kappa/\sigma^3)</th>
<th>(\text{MSE}(\alpha_*))</th>
<th>(\text{MSE}(\alpha'_*))</th>
<th>(\text{MSE}(\kappa_*))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.11</td>
<td>0.29</td>
<td>-0.023</td>
<td>0.0004</td>
<td>0.0028</td>
<td>0.00001</td>
</tr>
<tr>
<td>2</td>
<td>0.21</td>
<td>0.59</td>
<td>-0.023</td>
<td>0.0028</td>
<td>0.0223</td>
<td>0.00004</td>
</tr>
</tbody>
</table>

We conclude this section with a short description of the study populations \(X^{(r)} = X_1^{(r)} \cup X_2^{(r)} \cup X_3^{(r)},\) \(r = 1, 2.\) The elements of the first popu-
lation are realized values of independent normal variables: \( X_1^{(1)} \sim \mathcal{N}(2, 1); \)
\( X_2^{(1)} \sim \mathcal{N}(3, 4); \)
\( X_3^{(1)} \sim \mathcal{N}(6, 9). \) The second population combines symmetric
and skewed strata: \( X_1^{(2)} = \{1 + 0.004(i - 1), 1 \leq i \leq N_1\} \) (uniform distribution
on \([1, 3]\)); \( X_2^{(2)} = 2 - \ln(1 - 0.002(i - 1)), 1 \leq i \leq N_2\) (exponential
distribution); \( X_3^{(2)} \sim \chi^2 + 3. \)

5 Proof of (5)

Before the proof we collect some useful facts about Hoeffding’s decomposition
for STSRS samples, see [8], [9] for proofs and details.

5.1. Hoeffding decomposition. Let us consider a general
\( U - \) statistic
\[
U = U(X) = \sum_{1 \leq k \leq h} \sum_{x, y \in X_k} t_k(x_i, x_j) + \sum_{1 \leq k < r \leq h} \sum_{x \in X_k, y \in X_r} t_{kr}(x_i, y_j), \tag{18}
\]
where \( t_k \) are symmetric functions. Hoeffding’s decomposition
\[
U = EU + L + Q, \tag{19}
\]
decomposes \( U - EU \) into the sum of the linear part \( L \) and quadratic part \( Q \)
which are centered and uncorrelated. Here
\[
L = \sum_{1 \leq k \leq h} L_k, \quad Q = \sum_{1 \leq k \leq r \leq h} Q_{kr}, \tag{20}
\]
\[
L_k = \sum_{x \in X_k} g_k(x), \quad Q_{kk} = \sum_{(x, y) \in X_k} \psi_k(x, y), \quad Q_{kr} = \sum_{x \in X_k, y \in X_r} \psi_{kr}(x, y).
\]
The functions \( g_k, \psi_k \) and \( \psi_{kr}, \) for \( k < r, \) are defined as follows. Denote
\( \tilde{t}_k(x, y) = t_k(x, y) - EU(X_k, X) \) and \( \tilde{t}_{kr}(x, y) = t_{kr}(x, y) - EU(X_k, X_r). \)
We have
\[
g_k(x) = (n_k - 1)t^*_k(x) + \sum_{1 \leq j \leq h, j \neq k} n_j t^*_{kj}(x), \tag{21}
\]
where
\[
t^*_k(x) = \frac{N_k - 1}{N_k - 2} \mathbb{E}(\tilde{t}_k(X_{k, 1}, X_{k, 2})|X_{k, 1} = x),
\]
and, for \( k < r, \)
\[
t^*_{kr}(x) = \mathbb{E}(\tilde{t}_{kr}(X_{k, 1}, X_{r, 1})|X_{k, 1} = x), \quad t^*_{rk}(x) = \mathbb{E}(\tilde{t}_{kr}(X_{k, 1}, X_{r, 1})|X_{r, 1} = x).
\]
Furthermore, we have for $k < r$

$$\psi_k(x, y) = \tilde{t}_k(x, y) - t_k^*(x) - t_k^*(y), \quad \psi_{kr}(x, y) = \tilde{t}_{kr}(x, y) - t_{kr}^*(x) - t_{kr}^*(y).$$

(22)

For $r > k$ we denote $\psi_{rk}(y, x) := \psi_{kr}(x, y)$.

Note that for every $k$ and $r$ we have

$$E g_k(X_{k,1}) = 0, \quad E \psi_k(X_{k,1}, X_{k,2}) = 0, \quad E \psi_{kr}(X_{k,1}, X_{r,1}) = 0. \quad (23)$$

Moreover, the following identities hold

$$E(\psi_k(X_{k,i}, X_{k,j})|X_{k,j}) = 0, \quad i \neq j, \quad (24)$$

$$E(\psi_{kr}(X_{k,i}, X_{r,j})|X_{r,j}) = 0, \quad E(\psi_{kr}(X_{k,i}, X_{r,j})|X_{k,i}) = 0. \quad (25)$$

It follows from (24) and (25) that the parts $L$ and $Q$ are uncorrelated and the variances $\sigma^2_U$, $\sigma^2_L$ and $\sigma^2_Q$ of $U$, $L$ and $Q$ satisfy

$$\sigma^2_U = \sigma^2_L + \sigma^2_Q. \quad (26)$$

Let $\sigma^2(L_k)$, $\sigma^2(Q_{kk})$, and $\sigma^2(Q_{kr})$ denote the variance of $L_k$, $Q_{kk}$, and $Q_{kr}$. Using (24) and (25) one can show (see Bloznelis (2003b)) that

$$\sigma^2_L = \sum_{1 \leq k \leq h} \sigma^2(L_k), \quad \sigma^2_Q = \sum_{1 \leq k \leq r \leq h} \sigma^2(Q_{kr}), \quad (27)$$

$$\sigma^2(L_k) = \frac{N_k}{N_k - 1} r_k^2 \sigma_k^2, \quad \sigma^2(Q_{kk}) = \left(\frac{n_k - n_i}{2}\right) \left(\frac{n_k}{2}\right) \sigma_{kk}^2,$$

$$\sigma^2(Q_{kr}) = \frac{N_k}{N_k - 1} \frac{N_r}{N_r - 1} r_k^2 r_r^2 \sigma_{kr}^2,$$

where we denote

$$\sigma_k^2 = E g_k^2(X_{k,1}), \quad \sigma_{kk}^2 = E \psi_{kk}^2(X_{k,1}, X_{k,2}), \quad \sigma_{kr}^2 = E \psi_{kr}^2(X_{k,1}, X_{r,1}). \quad (28)$$

**Remark 5.1.** Note that every $U$-statistic defined by (18) can be written in the form (1). Just denote $t(x, u) := t_j(x, y) w_j^{-1}$, for $\{x, y\} \subset X_j$ and $t(x, y) = t_{jr}(x, y) w_{jr}^{-1}$ for $x \in X_j, y \in X_r, j < r$.

**5.2. Proof of (5).**

We shall show that

$$G_U(x) = \Phi(x) - \frac{\alpha + 3\kappa}{6\sigma_U^2} \Phi'(x)(x^2 - 1), \quad (29)$$

where
is the one-term Edgeworth expansion of the probability distribution function 
\( F_U = \mathbb{P}\{U - \mathbb{E}U \leq \sigma_U x\} \) of a general \( U \)-statistics given by (18). The quantities \( \alpha, \kappa \) are defined by formula (8) applied to the kernels (21) and (22).

We assume without loss of generality that \( \mathbb{E}U = 0 \) and \( \sigma^2_U = 1 \). In order to construct an Edgeworth expansion of \( F_U \) we write an asymptotic expansion to the characteristic function 
\( f_U(t) = \mathbb{E} \exp\{itU\} \)
and apply Fourier inversion formula. We shall show that the function 
\( g(t) = g_L(t) + g_Q(t), \)
where
\[
\begin{align*}
g_L(t) &= \exp\{-\frac{t^2}{2}\} \left(1 + \frac{(it)^3}{6} \alpha \right), \\
g_Q(t) &= \exp\{-\frac{t^2}{2}\} \frac{(it)^3}{2} \kappa
\end{align*}
\]
is an one-term expansion of the function \( f_U(t) \). An application of the inversion formula (see, e.g. Chung (1974), page 159) gives the one-term asymptotic expansion 
\[
G_u(x) = \frac{1}{2} + \frac{i}{2\pi} \lim_{M \to \infty} \mathbf{V.P.} \int_{|t| \leq M} e^{-itx} g(t) t^{-1} dt.
\]
Here \( \mathbf{V.P.} \int_R = \lim_{h \to 0} (\int_{-\infty}^{-h} + \int_{h}^{\infty}) \) denotes Cauchy’s Principal Value.

Let us prove that \( g(t) \) is the one-term asymptotic expansion of \( f_U(t) \). For simplicity we consider the case of two strata only. It is convenient to represent the characteristic function \( f_U(t) \) in the Edrős-Rényi (1959) form, see (35) below. Using this representation we replace \( f_U(t) \) by \( f_L(t) + 2^{-1} \kappa (it)^3 f_L(t) \), see Lemma 8.1 below. Here \( f_L(t) = \mathbb{E} \exp\{itL\} \) denotes the characteristic function of the random variable \( L \). Finally we replace \( f_L(t) \) by its asymptotic expansion \( g_L(t) \). To see that \( g_L(t) \) is indeed an asymptotic expansion of \( f_L(t) \) we split \( f_L(t) \) and, for every \( k = 1, 2 \), we replace \( f_{L_k}(t) = \mathbb{E} \exp\{itL_k\} \) by its one-term asymptotic expansion (see, Robinson (1978), formula (13))
\[
f_{L_k}(t) \approx \exp\{-t^2 \frac{\sigma^2(L_k)}{2}\} (1 + \frac{(it)^3}{6} (q_k - p_k) \tau^2_k \alpha_k).
\]
Collecting the main terms and using the approximation \( \sigma^2_L \approx \sigma^2_L \) we obtain
\[
f_L(t) \approx \exp\{-t^2 \sigma^2_L/2\} (1 + \frac{(it)^3}{6} \alpha) \approx g_L(t),
\]
10
thus completing the proof of the fact that \( g(t) \) is the one-term expansion of \( f_U(t) \).

In the remaining part of the proof we construct the Erdős-Rényi representation for \( f_U(t) \) and prove Lemma 8.1. The proof of the lemma is given in Section 8.

Write \( U = \mathcal{X} \cup \mathcal{Y} \), where \( \mathcal{X} = \{ x_1, \ldots, x_{N_1} \} \) and \( \mathcal{Y} = \{ y_1, \ldots, y_{N_2} \} \). Let \( X^* = (X_1, \ldots, X_{N_1}) \) and \( Y^* = (Y_1, \ldots, Y_{N_2}) \) denote random permutations of the sequences \( \{ x_i \} \) and \( \{ y_j \} \). We assume that \( X^* \) and \( Y^* \) are independent.

The random vector \((X, Y)\), where \( X = \{ x_1, \ldots, x_{N_1} \} \) and \( Y = \{ y_1, \ldots, y_{N_2} \} \), represents the STSI sample drawn from the population \( U \).

We shall write \( g_x, g_y, \psi_x, \psi_y, \psi_{xy} \) and \( \sigma_x, \sigma_y, \sigma_{xx}, \sigma_{xy}, \sigma_{yy}, \kappa_x, \kappa_y, \kappa_{xy} \) instead of \( g_1, g_2, \psi_1, \psi_2, \psi_{12} \) and \( \sigma_1, \sigma_2, \sigma_{11}, \sigma_{22}, \sigma_{12}, \kappa_1, \kappa_2, \kappa_{12} \). Similarly, we write \( L_x, L_y, Q_x, Q_y, Q_{xy} \) instead of \( L_1, L_2, Q_{11}, Q_{22}, Q_{12} \) and \( p_x, q_x, p_y, q_y \) instead of \( p_1, q_1, p_2, q_2 \).

Let \( \nu = \{ \nu_1, \ldots, \nu_{N_1} \} \) and \( \eta = \{ \eta_1, \ldots, \eta_{N_2} \} \) be independent sequences of independent Bernoulli random variables with probabilities
\[
P(\nu_i = 1) = 1 - P(\nu_i = 0) = p_x, \quad P(\eta_j = 1) = 1 - P(\eta_j = 0) = p_y,
\]
for \( 1 \leq i \leq N_1 \) and \( 1 \leq j \leq N_2 \). After we write \( U \) in the form (19), i.e., \( U = L_x + L_y + Q_x + Q_y + Q_{xy} \), it is easy to see that the distribution of \( U \) coincides with the conditional distribution of the sum
\[
U^* = L_x^* + L_y^* + Q_x^* + Q_y^* + Q_{xy}^*, \tag{32}
\]
given the events \( \{ \nu_1 + \ldots + \nu_{N_1} = n_1 \} \) and \( \{ \eta_1 + \ldots + \eta_{N_2} = n_2 \} \). Here
\[
L_x^* = \sum_{i=1}^{N_1} g_x(X_i)\nu_i, \quad L_y^* = \sum_{j=1}^{N_2} g_y(Y_j)\eta_j, \quad Q_{xy}^* = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \psi_{xy}(X_i, Y_j)\nu_i\eta_j,
\]
and
\[
Q_x^* = \sum_{1 \leq i < j \leq N_1} \psi_x(X_i, X_j)\nu_i\nu_j, \quad Q_y^* = \sum_{1 \leq i < j \leq N_2} \psi_y(Y_i, Y_j)\eta_i\eta_j.
\]

Denote \( \omega_i = \nu_i - p_x \) and \( \xi_j = \eta_j - p_y \) and let \( L_x, L_y, Q_x, Q_y, Q_{xy} \) be defined in the same way as \( L_x^*, L_y^*, Q_x^*, Q_y^*, Q_{xy}^* \) but with \( \nu_i \) and \( \eta_j \) replaced by \( \omega_i \) and \( \xi_j \) respectively. It follows from (24), (25) that almost surely
\[
L_x = L_x^*, \quad L_y = L_y^*, \quad Q_x = Q_x^*, \quad Q_y = Q_y^*, \quad Q_{xy} = Q_{xy}^*.
\]
In particular, from (32) we have \( U^* = \mathbb{U} \), where

\[ \mathbb{U} = \mathbb{L} + \mathbb{Q}, \quad \text{where} \quad \mathbb{L} = \mathbb{L}_x + \mathbb{L}_y, \quad \mathbb{Q} = \mathbb{Q}_x + \mathbb{Q}_y + \mathbb{Q}_{xy}. \]  

(33)

Denote \( S_x = \sum_{i=1}^{N_1} w_i \) and \( S_y = \sum_{j=1}^{N_2} \xi_j \) and write

\[ H = t\mathbb{U} + (s_1/\tau_1)S_x + (s_2/\tau_2)S_y, \quad \tilde{\mathbb{L}} = t\mathbb{L} + (s_1/\tau_1)S_x + (s_2/\tau_2)S_y. \]  

(34)

Note that the distribution of \( U \) coincides with the conditional distribution of \( U \) given the events \( \{S_x = 0\} \) and \( \{S_y = 0\} \). Using this fact we can write the characteristic function \( f_U(t) \) in the following way, see Erdős and Rényi (1959),

\[ f_U(t) = \lambda_1 \lambda_2 \int_{-\pi\tau_1}^{\pi\tau_1} ds_1 \int_{-\pi\tau_1}^{\pi\tau_1} ds_2 \exp\{iH\}. \]  

(35)

Here

\[ \lambda_1^{-1} = 2\pi\tau_1 \mathbb{P}\{S_x = 0\}, \quad \lambda_2^{-1} = 2\pi\tau_2 \mathbb{P}\{S_y = 0\}. \]

6 Proof of (6)

The proof goes along the same lines as that of Theorem 1 of Bloznelis (2003a), who considered the case of the simple random sample drawn without replacement.

We assume without loss of generality that \( \sigma^2 = 1 \), where \( \sigma^2 \) denotes the variance of \( \hat{u}(X) \).

In order to fix notation let \( g_k, \psi_k \) and \( \psi_{kr} \) denote the kernels of the decomposition \( \hat{u} = u + L + Q \), see (3), defined by (21) and (22).

The proof consists of two steps. Firstly we replace \( S \) by the short stochastic expansion \( 1 + 2^{-1} L^* \), where

\[ L^* = \sum_{k=1}^{h} L_k^*, \quad L_k^* = \sum_{i=1}^{n_k} f_k(X_{k,i}). \]  

(36)

Here \( f_k = f_{k,1} + f_{k,2} + f_{k,3} \) with

\[ f_{k,1}(x) = g_k(g_k^2(x) - \sigma_k^2), \quad \sigma_k^2 = \mathbb{E}g_k^2(X_{k,1}), \]

\[ f_{k,2}(x) = 2g_k n_k \frac{n_k - 1}{n_k - 2} \frac{N_k - 1}{N_k - 2} \mathbb{E}(\psi_k(X_{k,1}, X_{k,2})g_k(X_{k,2})|X_{k,1} = x), \]

\[ f_{k,3}(x) = 2 \sum_{r=1}^{h} \mathbb{I}_{\{r \neq k\}} g_r \frac{\sigma_r^2}{n_r - 1} \mathbb{E}(\psi_{kr}(X_{k,1}, X_{r,1})g_r(X_{r,1})|X_{k,1} = x). \]
Secondly, we apply (29) to the probability
\[ P\{ \hat{u} - u \leq x(1 + 2^{-1}L^*) \} = P\{ \hat{U} \leq x \}, \]
where
\[ \hat{U} = \hat{u} - u - 2^{-1}xL^* \]
is a U-statistic of the form (18).

6.1. Step 1. Here we replace \( S \) by \( 1 + 2^{-1}L^* \). It is convenient to represent the difference \( \bar{u}_k - \hat{u}(X_{k|i}) \) in the following way
\[
\bar{u}_k - \hat{u}(X_{k|i}) = Z_k(i) + W_k(i) + M_k(i),
\]
\[
Z_k(i) = \frac{n_k}{n_k - 1} \sum_{1 \leq j \leq n_k} g_k(X_{k,j})(\mathbb{1}_{\{i=j\}} - \frac{1}{n_k}),
\]
\[
W_k(i) = \frac{n_k}{n_k - 2} \sum_{1 \leq j < r \leq n_k} \psi_k(X_{k,j}, X_{k,r})(\mathbb{1}_{\{i \in \{j,r\}} - \frac{2}{n_k}),
\]
\[
M_k(i) = \frac{n_k}{n_k - 1} \sum_{r=1}^{h} \sum_{s=1}^{n_r} \sum_{j=1}^{n_k} \psi_{kr}(X_{k,j}, X_{r,s})(\mathbb{1}_{\{i=j\}} - \frac{1}{n_k}).
\]

Write \( v^2_k = \tilde{v}_k + r_k \), where
\[
\tilde{v}_k = \sum_{i=1}^{n_k} Z_k^2(i) + 2 \sum_{i=1}^{n_k} Z_k(i)(W_k(i) + M_k(i)), \quad r_k = \sum_{k=1}^{n_k} (W_k(i) + M_k(i))^2.
\]

A calculation shows that
\[
\sum_{i=1}^{n_k} Z_k^2(i) = \frac{n_k}{n_k - 1} \sum_{i=1}^{n_k} g_k^2(X_{k,i}) - \frac{2n_k}{(n_k - 1)^2} H_{k,1},
\]
\[
\sum_{i=1}^{n_k} Z_k(i)W_k(i) = \frac{n_k}{(n_k - 1)(n_k - 2)} (H_{k,4} - \frac{2}{n_k} H_{k,2} H_{k,3}),
\]
\[
\sum_{i=1}^{n_k} Z_k(i)M_k(i) = \frac{n_k^2}{(n_k - 1)^2} (H_{k,5} - \frac{1}{n_k} H_{k,2} H_{k,6}).
\]

Here we denote
\[
H_{k,1} = \sum_{1 \leq i < j \leq n_k} g_k(X_{k,i})g_k(X_{k,j}),
\]
\[
H_{k,2} = \sum_{i=1}^{n_k} g_k(X_{k,i}), \quad H_{k,3} = \sum_{1 \leq i < j \leq n_k} \psi_k(X_{k,i}, X_{k,j}),
\]

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\[ H_{k,4} = \sum_{1 \leq i < j \leq n_k} \psi_k(X_{k,i}, X_{k,j})(g_k(X_{k,i}) + g_k(X_{k,j})), \]
\[ H_{k,5} = \sum_{r=1}^{h} \sum_{i=1}^{n_r} \psi_{kr}(X_{k,i}, X_{r,s}) g_k(X_{k,i}), \]
\[ H_{k,6} = \sum_{r=1}^{h} \sum_{i=1}^{n_r} \psi_{kr}(X_{k,i}, X_{r,s}). \]

Collecting these formulas and (37) in (4) we obtain
\[ S^2 = 1 + \sum_{k=1}^{h} \sum_{i=1}^{n_k} f_{k,1}(X_{k,i}) + Q^* + Q^* + R, \] (38)
\[ Q^* = \sum_{k=1}^{h} Q_k^*, \quad Q_k^* = 2 q_k \frac{n_k}{n_k} - H_{k,4}, \]
\[ Q^* = \sum_{k=1}^{h} Q_k^*, \quad Q_k^* = 2 q_k \frac{n_k}{n_k} - 1 H_{k,5}. \]

Here \( R = R_1 - R_2 - R_3 - R_4 - R_5 \), where
\[ R_1 = \sum_{k=1}^{h} q_k \frac{n_k - 1}{n_k} r_k, \quad R_2 = 1 - \sum_{k=1}^{h} q_k n_k \sigma_k^2, \]
\[ R_3 = 2 \sum_{k=1}^{h} q_k \frac{1}{n_k - 1} H_{k,1}, \quad R_4 = 4 \sum_{k=1}^{h} q_k \frac{1}{n_k - 2} H_{k,2} H_{k,3}, \]
\[ R_5 = 2 \sum_{k=1}^{h} q_k \frac{1}{n_k - 1} H_{k,2} H_{k,6}. \]

Furthermore, using Hoeffding’s decomposition (see (19)) we approximate \( Q^* \) and \( Q^* \) by the linear statistics
\[ Q^* = \sum_{k=1}^{h} \sum_{i=1}^{n_k} f_{k,2}(X_{k,i}) + R_7, \] (39)
\[ Q^* = \sum_{k=1}^{h} \sum_{i=1}^{n_k} f_{k,3}(X_{k,i}) + R_8. \] (40)

Under appropriate moment conditions one can show that the remainders \( R_i, \)
\( 1 \leq i \leq 7 \) are negligible, see Bloznelis (2003a). In order to show that the
remainder $R_8$ is negligible it is convenient to write $Q^*$ in the form

$$Q^* = \sum_{k<R} Q^*_{kr},$$

$$Q^*_{kr} = \sum_{i=1}^{n_k} \sum_{j=1}^{n_r} 2\psi_{kr}(X_{k,i}, X_{r,j}) \left( q_k \frac{n_k}{n_k - 1} g_k(X_{k,i}) + q_r \frac{n_r}{n_r - 1} g_r(X_{r,j}) \right).$$

Then we approximate $Q^*$ by the linear part of decomposition (see (19)) and obtain the approximation (40).

Finally, combining (38) and (39), (40) we obtain $S^2 \approx 1 + L^*$. Furthermore, for small $L^*$ we can write $S \approx \sqrt{1 + L^*} \approx 1 + 2^{-1}L^*$.

6.2. Step 2. Here we construct the one-term Edgeworth expansion to the distribution function $P\{\tilde{U} \leq x\}$. We are going to apply (29). Firstly, we write Hoeffding’s decomposition of $\tilde{U}$, $\tilde{U} = \tilde{L} + Q$. Here $\tilde{L} = L - 2^{-1}xL^*$ denotes the linear part and $Q$ denotes the quadratic part, where $L$ and $Q$ are the same as in (3). Let $\hat{\sigma}^2_U$ and $\hat{\sigma}^2_L$ denote the variances of $\tilde{U}$ and $\tilde{L}$ and let $\hat{\alpha}, \hat{\alpha}_r, \hat{\kappa}, \hat{\kappa}_r$ denote moments of various parts of $\tilde{U}$ defined by (8).

Substituting $\hat{\alpha}, \hat{\kappa}, \hat{\sigma}^2_U$ in (29) we obtain the one-term Edgeworth expansion $\tilde{G}(x)$ of the probability $P\{\tilde{U} \leq \hat{\sigma} U x\}$. A calculation shows that $\hat{\sigma} = \sigma + R(\sigma), \hat{\alpha} = \alpha + R(\alpha), \hat{\kappa}_k = \alpha_k + R(\alpha_k), \hat{\kappa}_r = \kappa_r + R(\kappa_r)$, where $R(\sigma), \ldots, R(\kappa)$ are negligible remainders. Therefore, $\tilde{G}(x)$ can be replaced by $G_u(x)$ (= the one-term Edgeworth expansion of $F_u(x) = P\{\tilde{u} - E\tilde{u} \leq x\}$, see (5)). It follows that $G_u(x/\hat{\sigma} U)$ is the one-term Edgeworth expansion of $P\{\tilde{U} \leq x\}$. In the remaining part of the proof we shall show that $G_u(x/\hat{\sigma} U) = G_S(x) + R$, where $G_S$ is given by (6) and where $R$ is a negligible remainder.

Denote $\delta = \hat{\sigma}^2_U - 1$ and expand using Taylor’s formula

$$G_u\left(\frac{x}{\hat{\sigma} U}\right) = G_u\left(x + x\left(\frac{1}{\hat{\sigma} U} - 1\right)\right) = G_u(x) + G'_u(x)x\left(\frac{1}{\hat{\sigma} U} - 1\right) + R_1$$

$$= G_u(x) + \Phi'(x)x\left(\frac{1}{\hat{\sigma} U} - 1\right) + R_1 + R_2$$

$$= G_u(x) - 2^{-1}\delta \Phi'(x)x + R_1 + R_2 + R_3. $$

In the last step we used the approximation (via Taylor’s expansion) $\hat{\sigma}^2_U - 1 \approx -\delta/2$. Here $R_i, i = 1, 2, 3$ are negligible remainders. In order to evaluate the factor $\delta$ we apply variance decomposition formula (see, (26)),

$$\hat{\sigma}^2_U = \hat{\sigma}^2_L + \sigma^2_Q + \hat{\sigma}^2_L - \sigma^2_L = 1 + \hat{\sigma}^2_L - \sigma^2_L. $$

(41)
Therefore, \( \delta = \bar{\sigma}^2_L - \sigma^2_L \). From (27) we have

\[
\delta = \sum_{k=1}^{h} \tau_k N_k \left( \bar{\sigma}^2 - \sigma^2 \right),
\]

where \( \bar{\sigma}^2 = \mathbb{E}(g_k(X_{k,1}) - 2^{-1} x f_k(X_{k,1}))^2 \).

A calculation shows that

\[
\bar{\sigma}^2 = \sigma^2 - x \mathbb{E}g_k(X_{k,1})f_k(X_{k,1}) + r_k,
\]

where

\[
\mathbb{E}g_k(X_{k,1})f_k(X_{k,1}) = \alpha_k g_k + 2 \sum_{r=1}^{h} \tau_r \kappa_{kr}.
\]

We obtain

\[
\delta = -x \sum_{k=1}^{h} \tau_k^2 \alpha_k - 2x \sum_{k=1}^{h} \tau_r^4 \kappa_{kk} - 4x \sum_{1 \leq k < r \leq h} \tau_k^2 \tau_r \kappa_{kr} + R_4.
\]

Here \( R_4 = \sum r_k \) denotes the remainder. One can shown that \( R_4 \) is negligible. Substitution of this formula in (41) gives the desired result \( G_u(x/\bar{\sigma}_U) = G_S(x) + R \) with negligible remainder \( R \). The proof of (6) is complete.

### 7 Consistency of jackknife estimators

We sketch the proof of (15). Assume without loss of generality that \( \sigma^2 = 1 \). Substitution of (3), (20) in (11), (12), (13) gives

\[
\begin{align*}
V_{k|i} &= g_k(X_{k,i}) + R_{k|i}, \\
W_{k|ij} &= \psi_k(X_{k,i}, X_{k,j}) + R_{k|ij}, \\
W_{kr|ij} &= \psi_{kr}(X_{k,i}, X_{r,j}) + R_{kr|ij},
\end{align*}
\]

where

\[
R_{k|i} = -\frac{1}{n_k - 1} \sum_{x \in \mathcal{X}_{k|i}} g_k(x)
\]

\[
+ \sum_{x \in \mathcal{X}_{k|i}} \psi_k(X_{k,i}, x) - \frac{2}{n_k - 2} \sum_{\{x, y\} \subset \mathcal{X}_{k|i}} \psi_k(x, y)
\]

\[
+ \sum_{1 \leq s \leq h, s \neq k} \frac{1}{n_k - 1} \sum_{y \in \mathcal{X}_s} \psi_{ks}(x, y),
\]

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\[ R_{k|ij} = \frac{1}{(n_k-2)} \sum_{\{x,y\} \subset X_{k|ij}} \psi_k(x, y) \]
\[ \quad - \frac{1}{n_k - 2} \sum_{x \in X_{k|ij}} \psi_k(X_k, x) - \frac{1}{n_k - 2} \sum_{x \in X_{k|ij}} \psi_k(X_k, x), \]
\[ R_{kr|ij} = \frac{1}{n_k - 1} \sum_{x \in X_{k|ij}} \sum_{y \in X_{r|ij}} \psi_{kr}(x, y) \]
\[ \quad - \frac{1}{n_k - 1} \sum_{x \in X_{k|ij}} \sum_{y \in X_{r|ij}} \psi_{kr}(X_k, x) - \frac{1}{n_r - 1} \sum_{x \in X_{r|ij}} \psi_{kr}(X_k, x). \]

Here we denote \( X_{k|ij}' = X_k \setminus \{X_{k,i}\} \) and \( X_{r|ij}' = X_k \setminus \{X_{k,j}\} \). One can show (under appropriate moment conditions on \( \psi_k, \psi_{kr} \) and \( g_k \)) that the remainder terms \( R_{k|ij}, R_{kr|ij}, R_{kr|ij} \) can be neglected. Therefore, we obtain from (9) and (10) that

\[ \hat{\kappa}_k \approx \frac{1}{n_k} \sum_{x \in X_k} g_k^3(x), \quad \hat{\kappa}_{kk} \approx \frac{1}{(n_k^2)} \sum_{\{x, y\} \subset X_k} \psi_k(x, y)g_k(x)g_k(y), \]
\[ \hat{\kappa}_{kr} \approx \frac{1}{n_k} \frac{1}{n_r} \sum_{x \in X_k} \sum_{y \in X_r} \psi_{kr}(x, y)g_k(x)g_r(y). \]

Finally, (15) follows by the law of large numbers. A rigorous proof in the case of a simple random sample \( (h = 1) \) is given in [6].

## 8 Expansions

Denote, for \( r > 0 \),

\[ B_{xr} = \sigma_x^{-r} E|g_x(X_1)|^r, \quad B_{yr} = \sigma_y^{-r} E|g_y(Y_1)|^r, \]
\[ \gamma_x = \tau_1^6 \sigma_x^2 E\psi_x^2(X_1, X_2)\psi_y^2(X_1, Y_1), \quad \gamma_y = \tau_2^6 \sigma_y^2 E\psi_y^2(Y_1, Y_2)\psi^2_{xy}(X_1, Y_1). \]

**Lemma 8.1.** There exist (large) absolute constants \( C, c, c_T > 0 \) such that for \( \tau_1, \tau_2, N_1^{1/2}B_{x3}^{-1}, N_2^{1/2}B_{y3}^{-1} > c \) and we have, for \( |t| \leq T \),

\[ |f_U(t) - f_L(t) - 2^{-1}\kappa(it)^3 f_L(t)| \leq C|t|(1 + |t|^3)(R_1 + R_2 + R_3). \quad (42) \]

Here

\[ T^{-1} = c_T \max\{B_{x3}^2\sigma_x^2; B_{y3}^2\sigma_y^2\} \quad (43) \]
and
\[
\begin{align*}
R_1 &= \sigma_Q^2 \left(1 + \frac{\sigma^2(Q_x) + \sigma^2(Q_{xy})}{\tau_1 \sigma_x^2} + \frac{\sigma^2(Q_y) + \sigma^2(Q_{xy})}{\tau_2 \sigma_y^2}\right), \\
R_2 &= \left(\frac{\gamma_x}{\tau_1^2} \left(1 + \frac{1}{\sigma_x^2 \tau_2^2}\right) + \frac{\gamma_y}{\tau_2^2} \left(1 + \frac{1}{\sigma_y^2 \tau_1^2}\right)\right), \\
R_3 &= \left(B_{x4}^{1/2} \frac{\tau_1}{\tau_2} \sigma(Q_{xy}) + \sigma(Q_x) + B_{y4}^{1/2} \frac{\tau_1}{\tau_2} \sigma(Q_{xy}) + \sigma(Q_y)\right).
\end{align*}
\]

For typical examples of statistics we have \(R_1 + R_2 + R_3 = o(|\alpha| + |\kappa|)\) as \(\alpha, \kappa \to 0\). Note that the bound (42) is not sufficient to establish the validity of the expansion (29). For this purpose one needs an improved version of (42) where the factor \(|t|(1 + |t|^3)\) in the right were replaced by a function which is integrable over the interval \(|t| > 0\) with respect to the measure \(\frac{dt}{|t|}\).

Before the proof of Lemma 8.1 we introduce some notation

8.1. Notation. In what follows \(c\) and \(C\) with indices or without denote generic absolute constants. For complex numbers \(a, b\) we write \(a \ll b\) if \(|a| \leq c|b|\). Given a complex valued function \(z(t, s_1, s_2)\) write
\[
J(z) = \lambda_1 \lambda_2 \int_{-\pi \tau_1}^{\pi \tau_1} ds_1 \int_{-\pi \tau_2}^{\pi \tau_2} ds_2 z(t, s_1, s_2).
\]

For a number \(A > 0\) and complex valued functions \(z_1(t, s_1, s_2)\) and \(z_2(t, s_1, s_2)\) write \(z_1 \prec A\) if \(|J(z_1)| \ll A\) and write \(z_1 \sim z_2\) if \(z_1 - z_2 \prec \mathcal{R}\), where \(\mathcal{R} = |t|(1 + |t|^3)(R_1 + R_2 + R_3)\).

For \(k = 1, 2, \ldots\) denote \(\Omega_k = \{1, \ldots, k\}\) and \(\Omega_{k|2} = \{(i, j) : 1 \leq i < j \leq k\}\). For sums
\[
S = \sum_{1 \leq i \leq k} s_i, \quad T = \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq r} t_{ij}, \quad V = \sum_{1 \leq i < j \leq k} v_{ij}
\]
and sets \(A \subset \Omega_k, B \subset \Omega_k \times \Omega_r, D \subset \Omega_{k|2}\) write for short
\[
S(A) = \sum_{i \in A} s_i, \quad T(B) = \sum_{(i, j) \in B} t_{ij}, \quad V(D) = \sum_{(i, j) \in D} v_{ij}.
\]

For integers \(1 < m_1 < N_1\) and \(1 < m_2 < N_2\) put
\[
\begin{align*}
A_1^n &= \Omega_{m_1}, & A_2^n &= \Omega_{N_1} \setminus \Omega_{m_1}, & A_1^y &= \Omega_{m_2}, & A_2^y &= \Omega_{N_2} \setminus \Omega_{m_2}, \\
B_1 &= \Omega_{m_1} \times \Omega_{m_2}, & B_2 &= \Omega_{m_1} \times (\Omega_{N_2} \setminus \Omega_{m_2}), \\
B_3 &= (\Omega_{N_1} \setminus \Omega_{m_1}) \times \Omega_{m_2}, & B_4 &= (\Omega_{N_1} \setminus \Omega_{m_1}) \times (\Omega_{N_2} \setminus \Omega_{m_2}), \\
D_1^x &= \Omega_{m_1|2}, & D_2^x &= \Omega_{m_1} \times (\Omega_{N_1} \setminus \Omega_{m_1}), & D_3^x &= \Omega_{N_1|2} \setminus (D_1^x \cup D_2^x), \\
D_1^y &= \Omega_{m_2|2}, & D_2^y &= \Omega_{m_2} \times (\Omega_{N_2} \setminus \Omega_{m_2}), & D_3^y &= \Omega_{N_2|2} \setminus (D_1^y \cup D_2^y).
\end{align*}
\]
Split

\[ \tilde{L} = \tilde{L}_1 + \tilde{L}_2, \quad \tilde{L}_1 = \tilde{L}_1^x + \tilde{L}_1^y, \quad \tilde{L}_2 = \tilde{L}_2^x + \tilde{L}_2^y, \]
\[ \tilde{L}_k^x = t \sum_{\{i \leq 4\}} Q_{xy}(B_i), \quad \tilde{L}_k^y = t \sum_{\{i \leq 3\}} Q_x(D_i^x), \quad \tilde{L}_k^x = t \sum_{\{i \leq 3\}} Q_y(D_i^y), \quad k = 1, 2, \]
\[ L_{xy} = \sum_{1 \leq i \leq 4} Q_{xy}(B_i), \quad L_x = \sum_{1 \leq i \leq 3} Q_x(D_i^x), \quad L_y = \sum_{1 \leq i \leq 3} Q_y(D_i^y), \]

and write \( H = L_1 + F_1 + F \), where \( L_1 = L_{1x} + L_{1y} \),
\[ L_{1x} = \tilde{L}_1^x + t Q_x(D_1^x) + t Q_{xy}(B_2), \quad L_{1y} = \tilde{L}_1^y + t Q_y(D_1^y) + t Q_{xy}(B_3), \]
\[ F_1 = t Q_x(D_1^x) + t Q_y(D_1^y) + t Q_{xy}(B_1), \quad F = \tilde{L}_2 + F_2, \quad F_2 = t Q_x(D_2^x) + t Q_y(D_2^y) + t Q_{xy}(B_4). \]

Note that the random variable \( F \) does not depend on \( \omega_i, i \leq m_1 \) and \( \xi_j, j \leq m_2 \). Furthermore we have
\[ L_{1x} = \sum_{1 \leq i \leq m_1} a_i \omega_i, \quad L_{1y} = \sum_{1 \leq j \leq m_2} b_j \xi_j, \quad (44) \]

where
\[ a_i = t g_x(X_i) + s_1 \tau_1^{-1} + t v_i^x + t u_i^x, \quad b_j = t g_y(Y_j) + s_2 \tau_2^{-1} + t v_j^y + t u_j^y, \]
\[ v_i^x = \sum_{m_1 < j \leq N_1} \psi_x(X_i, X_j) \omega_j, \quad u_i^x = \sum_{m_2 < j \leq N_2} \psi_{xy}(X_i, Y_j) \xi_j, \]
\[ v_j^y = \sum_{m_2 < j \leq N_2} \psi_y(Y_j, Y_i) \xi_i, \quad u_j^y = \sum_{m_1 < i \leq N_1} \psi_{xy}(X_i, Y_j) \omega_i. \]

Note that conditionally given all the random variables but \( \{\omega_i, i \leq m_1\} \) and \( \{\xi_j, j \leq m_2\} \) the random variable \( L_{1x} \) (respectively \( L_{1y} \)) is a linear statistic of random variables \( \omega_1, \ldots, \omega_{m_1} \) (respectively \( \xi_1, \ldots, \xi_{m_2} \)).

Given \( B \subset \Omega_{m_1} \) denote
\[ \tilde{J}_{xy}(B) = \exp\{it L_{xy}(B) + it Q_{xy}(B \times (\Omega_{N_1} \setminus \Omega_{m_1})) \}
+ \exp\{it Q_{xy}(B \times (\Omega_{N_2} \setminus \Omega_{m_2})) + i(s_1/\tau_1)S_x(B) \}, \]
\[ \tilde{J}_y(B) = \exp\{it L_{xy}(B) + it Q_{xy}(B \times (\Omega_{N_2} \setminus \Omega_{m_2})) + i(s_1/\tau_1)S_y(B) \}, \]
\[ \tilde{J}_x(B) = \exp\{it L_{xy}(B) + it Q_{xy}(B \times (\Omega_{N_1} \setminus \Omega_{m_1})) + i(s_1/\tau_1)S_x(B) \} \]
\[ \tilde{J}(B) = \exp\{it L_{xy}(B) + i(s_1/\tau_1)S_x(B) \}. \]
Let $J^x_{xy}(B)$ (respectively $J^y_y(B)$, $J^x_y(B)$ and $J^x(B)$) denote the absolute value of the conditional expectation of $J^x_{xy}(B)$ (respectively $J^y_y(B)$, $J^x_y(B)$ and $J^x(B)$) given all the random variables but $\{\omega_j, j \in B\}$. Write

$$
\zeta^x_{xy}(B) = \frac{1}{|B|} \sum_{j \in B} (v_j^x + u_j^x)^2, \quad \zeta^x_x(B) = \frac{1}{|B|} \sum_{j \in B} (v_j^x)^2, \quad \zeta^x_y(B) = \frac{1}{|B|} \sum_{j \in B} (u_j^x)^2.
$$

Given $D \subset \Omega_{m_2}$ denote

$$
\tilde{J}^y_{xy}(D) = \exp\{itL_y(D) + it\mathbb{Q}_y(D \times (\Omega_{N_2} \setminus \Omega_{m_2}))
+ it\mathbb{Q}_{xy}((\Omega_{N_1} \setminus \Omega_{m_1}) \times D) + i(s_2/\tau_2)S_x(D)\},
$$

$$
\tilde{J}^y_x(D) = \exp\{itL_x(D) + it\mathbb{Q}_{xy}((\Omega_{N_1} \setminus \Omega_{m_1}) \times D) + i(s_2/\tau_2)S_x(D)\},
$$

$$
\tilde{J}^y_y(D) = \exp\{itL_y(D) + it\mathbb{Q}_y(D \times (\Omega_{N_2} \setminus \Omega_{m_2})) + i(s_2/\tau_2)S_y(D)\}
$$

$$
\tilde{J}^y(D) = \exp\{itL_y(D) + i(s_2/\tau_2)S_x(D)\}.
$$

Let $J^y_{xy}(D)$ (respectively $J^y_y(D)$, $J^y_x(D)$ and $J^y(D)$) denote the absolute value of the conditional expectation of $\tilde{J}^y_{xy}(D)$ (respectively $\tilde{J}^y_y(D)$, $\tilde{J}^y_x(D)$ and $\tilde{J}^y(D)$) given all the random variables but $\{\xi_j, j \in D\}$. Furthermore, write

$$
\zeta^y_{xy}(D) = \frac{1}{|D|} \sum_{j \in D} (v_j^y + u_j^y)^2, \quad \zeta^y_x(D) = \frac{1}{|D|} \sum_{j \in D} (v_j^y)^2, \quad \zeta^y_y(D) = \frac{1}{|D|} \sum_{j \in D} (u_j^y)^2.
$$

8.2. Proof of Lemma 8.1. Note that

$$
f_U(t) = J(\mathbb{E}\exp\{iH\}), \quad \text{and} \quad f_L(t) = J(\mathbb{E}\exp\{i\bar{L}\}).
$$

Therefore in order to prove the lemma we need to show that

$$
\mathbb{E}\exp\{iH\} \sim (1 + 2^{-1} \kappa(it)^3)\mathbb{E}\exp\{i\bar{L}\}.
$$

We shall prove this relation in two steps

$$
\mathbb{E}\exp\{iH\} \sim \mathbb{E}\exp\{i\bar{L}\} + it\mathbb{E}\exp\{i\bar{L}\}Q, \quad (45)
$$

$$
it\mathbb{E}\exp\{i\bar{L}\}Q \sim 2^{-1} \kappa(it)^3\mathbb{E}\exp\{i\bar{L}\}. \quad (46)
$$

8.2.1. Proof of (45). Introduce the integer valued functions $m_1 = m_1(s_1)$ and $m_2 = m_2(s_2)$,

$$
m_1 \approx C'N_1 \frac{\ln(3 + s_1^2)}{3 + s_1^2}, \quad m_2 \approx C'N_2 \frac{\ln(3 + s_2^2)}{3 + s_2^2},
$$

$$
(47)
$$
where $C' > 0$ is a sufficiently large constant. Denote
\[ m_3 = \lfloor m_1/2 \rfloor, \quad m_4 = \lfloor m_2/2 \rfloor. \]

Write
\[ f_1 = E \exp \{ iH \}, \quad f_2 = EJ, \quad f_3 = iEJF_1, \quad J = \exp \{ i(L_1 + F) \}. \]

In order to prove (45) we shall show that
\[ f_1 \sim f_2 + f_3 + f_4, \quad (48) \]
\[ f_2 \sim E \exp \{ i\tilde{L} \} + iE \exp \{ i\tilde{L} \} (tQ - F_1), \quad (49) \]
\[ f_3 \sim iE \exp \{ i\tilde{L} \} F_1. \quad (50) \]

Let us prove (48). Expanding the exponent in powers of $itQ_{xy}(B_1)$ we obtain
\[ f_1 = f_6 + f_7 + f_8 + f_9, \quad (51) \]

where we denote
\[ R_2 = itEJQ_{xy}(B_1)(\exp \{ itQ_y(D_1^y) \} - 1), \quad R_3 = itEJQ_{xy}(B_1)(\exp \{ itQ_x(D_1^x) \} - 1), \]
\[ R_4 = itEJQ_{xy}(B_1)(\exp \{ itQ_y(D_1^y) \} - 1)(\exp \{ itQ_x(D_1^x) \} - 1). \]

Expanding the exponent in $f_4$ in powers of $itQ_x(D_1^x)$ we obtain
\[ f_4 = f_7 + f_8 + f_9, \quad (53) \]

where $f_7 := EJ \exp \{ itQ_x(D_1^x) \}$ and $f_8 := itEJ \exp \{ itQ_x(D_1^x) \} Q_y(D_1^y)$ and
\[ f_9 := (it)^2 EJ \exp \{ itQ_x(D_1^x) \} Q_y^2(D_1^y) \theta_y, \quad |\theta_y| \leq 1. \]

The random variable $\theta_y$ is a function of $Q_y(D_1^y)$. Furthermore, expanding in powers of $itQ_x(D_1^x)$ we obtain
\[ f_7 = f_2 + itEJQ_x(D_1^x) + R_5, \quad (54) \]

where
\[ R_5 := (it)^2 EJQ_x^2(D_1^x) \theta_x. \]
where \(|\theta_x| \leq 1\) is a function of \(Q_x(D_1^x)\). Similarly, we get

\[
f_8 = it\mathbf{E} J_{Q_y}(D_1^y) + R_6, \quad |R_6| \leq t^2|Q_x(D_1^x)Q_y(D_1^y)|. \tag{55}
\]

Finally,

\[
f_9 = R_7 + R_8, \quad R_7 := (it)^2\mathbf{E} J^2_{Q_y}(D_1^y)\theta_y, \quad |R_8| \leq |t|^3|Q_x(D_1^x)|Q_y^2(D_1^y). \tag{56}
\]

Combining (51)-(56) we obtain

\[
f_1 = f_2 + f_3 + R, \quad R := R_1 + \ldots + R_8. \tag{57}
\]

In order to prove (48) we show that

\[
R_k \prec \mathcal{R}, \quad k = 1, \ldots, 8. \tag{58}
\]

For \(k = 1, 4, 6, 8\) the proof is simple. We have

\[
|R_4| \leq t^2\mathbf{E}|Q_x(D_1^x)|^{1/2}|Q_y(D_1^y)|^{1/2}|Q_{xy}(B_1)|
\leq t^2\mathbf{E}|Q_x(D_1^x)Q_y(D_1^y)| + t^2\mathbf{E}Q_{xy}^2(B_1)
\leq t^2(\mathbf{E}Q_x^2(D_1^x))^{1/2}(\mathbf{E}Q_y^2(D_1^y))^{1/2} + t^2\mathbf{E}Q_{xy}^2(B_1),
\]

\[
|R_6| \leq t^2(\mathbf{E}Q_x^2(D_1^x))^{1/2}(\mathbf{E}Q_y^2(D_1^y))^{1/2}.
\]

Combining these inequalities, (47) and (120) we obtain (58) for \(k = 1, 4, 6, 8\). Let us prove (58) for \(k = 2\). Write \(B'_1 = (\Omega_{m_1} \setminus \Omega_{m_2}) \times \Omega_{m_2}\). By symmetry, it suffices to show (58) for

\[
R'_2 := it\mathbf{E} J_{Q_{xy}}(B'_1) (\exp\{itQ_y(D_1^y)\} - 1) = (it)^2\mathbf{E} J_{Q_{xy}}(B'_1) Q_y(D_1^y)\theta_y^*,
\]

where \(|\theta_y^*| \leq 1\) is a function of \(Q_y(D_1^y)\). We have

\[
R'_2 \leq t^2\mathbf{E} J_{Q_{xy}}(\Omega_{m_3})|Q_{xy}(B'_1)Q_y(D_1^y)|.
\]

Invoking inequalities \(|J_{xy}(\Omega_3)| \leq 1\) and \(|ab| \leq a^2 + b^2\) we write

\[
R'_2 \leq t^2R_{21} + t^2R_{22}, \quad R_{21} := \mathbf{E}Q_{xy}^2(B'_1), \quad R_{22} := \mathbf{E}J_{xy}(\Omega_{m_3})Q_y^2(D_1^y). \tag{59}
\]

Lemma 9.1 implies the bound \(t^2R_{21} \prec \mathcal{R}\). Let us prove that \(t^2R_{22} \prec \mathcal{R}\). Write

\[
R_{22} = \mathbf{E}f_*, \quad f_* := J_{xy}(\Omega_{m_3})Q_y^2(D_1^y) \tag{60}
\]
Combining bounds for Invoking the simple bound Similarly, By symmetry, (119) and (120) we have Here we denote The number \( \alpha > 0 \) is defined in (123) below. By Chebyshev’s inequality and symmetry,

\[
\mathbb{E} f_s \mathbb{I}_x \leq \left( \alpha / \sigma_x^2 \right) \mathbb{E} \chi_{xy}(\Omega_{m3}) Q_y^2(D_1^y) = \left( \alpha / \sigma_x^2 \right) \mathbb{E} (v_1^2 + u_1^2) Q_y^2(D_1^y) = \left( \alpha / \sigma_x^2 \right) (V_1 + V_2).
\]

By symmetry, (119) and (120) we have

\[
V_1 = \mathbb{E} (v_1^2) Q_y^2(D_1^y), \quad V_2 = \mathbb{E} (u_1^2) Q_y^2(D_1^y).
\]

Similarly,

\[
V_2 = (N_2 - m_2)p_q q_y \mathbb{E} \psi_{xy}^2(X_1, Y_{N_2}) Q_y^2(D_1^y) \leq \tau_2^2 m_2^2 q_y^2 \mathbb{E} \psi_{xy}^2(X_1, Y_{N_2}) \psi_y^2(Y_1, Y_2).
\]

Invoking the simple bound \( \mathbb{E} \psi_{xy}^2(X_1, Y_{N_2}) \psi_y^2(Y_1, Y_2) \leq \sigma_{xy}^2 \sigma_{yy}^2 \) and (119) we obtain

\[
V_2 \leq \tau_1^{-2} (m_2/N_2)^2 \sigma_x^2 (Q_{xy}) \sigma_y^2 (Q_y).
\]

Combining bounds for \( V_1 \) and \( V_2 \) we obtain,

\[
\mathbb{E} f_s \mathbb{I}_x \ll (m_2/N_2)^2 \tau_1^{-2} \sigma_x^2 (Q_y) (\sigma_x^2 (Q_{xy}) + \sigma_x^2 (Q_x)) \quad (62)
\]

\[
< \tau_1^{-1} \sigma_x^2 (Q_y) (\sigma_x^2 (Q_{xy}) + \sigma_x^2 (Q_x)).
\]

In the last step we used the simple bound \( \int \mathbb{I}_{|s| \leq \tau_1} ds \leq \tau_1 \).

Furthermore, combining (132) and (120) we obtain

\[
\mathbb{E} f_s \mathbb{I}_x \leq \mathbb{E} Q_y^2(D_1^y) \mathbb{E} X_1 (\Omega_3) \mathbb{I}_x \ll \sigma^2 (Q_y). \quad (63)
\]

Finally, collecting (62) and (63) in (60) we obtain the bound \( t^2 R_{22} \ll \mathcal{R} \), thus completing the proof of (58) for \( R_2 \). The proof of (58) for \( R_3, R_5, R_7 \) is almost the same. We arrive to (48).

Let us prove (50). We shall show that

\[
it \mathbb{E} J_{Q_{xy}}(B_1) \sim \text{it} \mathbb{E} \exp \{ i \bar{L} \} Q_{xy}(B_1), \quad (64)
\]

\[
it \mathbb{E} J_{Q_x}(D_1^x) \sim \text{it} \mathbb{E} \exp \{ i \bar{L} \} Q_x(D_1^x), \quad (65)
\]

\[
it \mathbb{E} J_{Q_y}(D_1^y) \sim \text{it} \mathbb{E} \exp \{ i \bar{L} \} Q_y(D_1^y). \quad (66)
\]

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Write
\[ J = J_0 \exp\{iF_3\}, \quad J_0 = \exp\{i(\bar{L} + F_2)\}, \quad (67) \]
\[ F_3 = tQ_x(D_2^x) + tQ_y(D_2^y) + tQ_{xy}(B_2) + tQ_{xy}(B_3). \]

In the proof of (64), (65), (66) we replace \( J \) by \( J_0 \) and then replace \( J_0 \) by \( \exp\{i\bar{L}\} \).

Let us prove (64). Denote \( B_{11} = (\Omega_{m_1} \setminus \Omega_{m_3}) \times (\Omega_{m_2} \setminus \Omega_{m_4}) \). By symmetry it suffices to show that
\[ \exp \{ iT \} Q_{xy}(B_{11}) \sim \exp \{ i\bar{L} \} Q_{xy}(B_{11}). \]

We shall prove that
\[ \exp \{ iT \} Q_{xy}(B_{11}) \sim \exp \{ i\bar{L} \} Q_{xy}(B_{11}), \quad \exp \{ iT \} Q_{xy}(B_{11}) \sim \exp \{ i\bar{L} \} Q_{xy}(B_{11}). \]

Write
\[ F_3 = H_{31} + tQ_{xy}(B_2), \quad H_{31} = H_{32} + tQ_{xy}(B_3), \]
\[ H_{32} = H_{33} + tQ_y(D_2^y), \quad H_{33} = tQ_x(D_2^x). \]

Expanding in powers of \( \exp \{ iT \} Q_{xy}(B_2) \) we obtain
\[ 1 \quad \exp \{ iH_{31} \} Q_{xy}(B_{11}) + iR_1, \]
\[ R_1 : = \exp \{ iH_{31} \} Q_{xy}(B_{11})Q_{xy}(B_2)\theta_1, \]
where the random variable \(|\theta_1| \leq 1 \) is a function of \( Q_{xy}(B_2) \). Similarly, expanding in powers of \( \exp \{ iT \} Q_{xy}(B_3) \), \( \exp \{ iT \} Q_x(D_1^x) \) and \( \exp \{ iT \} Q_y(D_1^y) \) we obtain
\[ \exp \{ iH_{31} \} Q_{xy}(B_{11}) = \exp \{ iH_{31} \} Q_{xy}(B_{11}) + iR_2, \]
\[ R_2 : = \exp \{ iH_{31} \} Q_{xy}(B_{11})Q_{xy}(B_2)\theta_2, \]
\[ \exp \{ iH_{32} \} Q_{xy}(B_{11}) = \exp \{ iH_{32} \} Q_{xy}(B_{11}) + iR_3, \]
\[ R_3 : = \exp \{ iH_{32} \} Q_{xy}(B_{11})Q_y(D_2^y)\theta_3, \]
\[ \exp \{ iH_{33} \} Q_{xy}(B_{11}) = \exp \{ iH_{33} \} Q_{xy}(B_{11}) + iR_4, \]
\[ R_4 : = \exp \{ iH_{33} \} Q_{xy}(B_{11})Q_x(D_1^x)\theta_4. \]

In order to prove the first part of (69) we shall show that
\[ i^2R_k \ll \mathcal{R}, \quad k = 1, 2, 3, 4. \]
Let us prove (70) for \( k = 1 \). We have

\[
R_1 \leq E J_{xy}^y(\Omega_{m4})|Q_{xy}(B_{11})Q_{xy}(B_2)| = R_{11} + R_{12}, \quad (71)
\]

\[
R_{11} = E J_{xy}^y(\Omega_{m4})|Q_{xy}(B_{11})Q_{xy}(B_2)| I_y,
\]

\[
R_{12} = E J_{xy}^y(\Omega_{m4})|Q_{xy}(B_{11})Q_{xy}(B_2)| \bar{I}_y.
\]

Here

\[
I_y := 1 - \bar{I}_y, \quad \bar{I}_y := \{\alpha_{xy}(\Omega_{m4}) < \sigma_y^2\}.
\]

Random variables \( v_i^y \) and \( u_i^y \) are defined in (44) and the constant \( \alpha \) is defined in (123). We shall show that

\[
t^2 R_{11} \prec R, \quad t^2 R_{12} \prec R. \quad (72)
\]

Let us prove the second bound. Using the simple inequality \( J_{xy}^y(\Omega_{m4}) \leq 1 \) and Cauchy-Schwartz we obtain

\[
R_{12} \leq R_{13}^{1/2} R_{14}^{1/2}, \quad R_{13} := E Q_{xy}^2(B_{11}) I_y, \quad R_{14} := E Q_{xy}^2(B_2). \quad (73)
\]

Lemma 9.1 implies \( R_{14} \ll (m_1/N_1)\sigma^2(Q_{xy}) \). Furthermore, by Chebyshev’s inequality and symmetry

\[
R_{13} \ll \sigma_y^{-2} E Q_{xy}^2(B_{11}) x_{xy}^y(\Omega_{m4}) = \sigma_y^{-2} E Q_{xy}^2(B_{11}) ((v_1^y)^2 + (u_1^y)^2) = \sigma_y^{-2}|B_{11}| p_x q_x p_y q_y (V_1 + V_2). \quad (74)
\]

Here \( |B_{11}| < m_1 m_2 \) denotes the number of elements of the set \( B_{11} \) and

\[
V_1 := E \psi_{xy}^2(X_{m1}, Y_{m2}) (v_1^y)^2 = (N_2 - m_2) p_y q_y E \psi_{xy}^2(X_{m1}, Y_{m2}) \psi_{xy}^2(Y_1, Y_{N_2}) \ll \tau_2 \sigma_x \sigma_y^2 \ll \tau_1^{-2} \tau_2^{-4} \sigma^2(Q_{xy}) \sigma^2(Q_y),
\]

\[
V_2 := E \psi_{xy}^2(X_{m1}, Y_{m2}) (u_1^y)^2 = (N_1 - m_1) p_x q_x E \psi_{xy}^2(X_{m1}, Y_{m2}) \psi_{xy}^2(X_{N_1}, Y_1) \ll \tau_1^2 \sigma_y \ll \tau_1^{-2} \tau_2^{-4} \sigma^2(Q_{xy}).
\]

We obtain

\[
R_{13} \ll (m_1 m_2/N_1 N_2) \tau_2^{-2} \sigma_y^{-2} \sigma^2(Q_{xy}) (\sigma^2(Q_{xy}) + \sigma^2(Q_y)).
\]

Collecting the bounds for \( R_{13} \) and \( R_{14} \) in (73) we obtain \( t^2 R_{12} \ll R \).

Let us prove the first bound of (72). We have

\[
R_{11} \leq R_{15} + R_{16}, \quad R_{15} := E Q_{xy}^2(B_{11}), \quad R_{16} := E J_{xy}^y(\Omega_{m4}) I_y Q_{xy}^2(B_2),
\]

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where $t^2 R_{15} \ll t^2 (m_1 m_2 / N_1 N_2) \sigma^2(Q_{xy}) \sim \mathcal{R}$. Furthermore, by symmetry,

$$R_{16} = m_1 p_y q_y \mathbb{E} J_{xy}^y(\Omega_{m_4}) || y D^2, \quad D := \sum_{j=m_2+1}^{N_2} \psi_{xy}(X_1, Y_j) \xi_j. \quad (75)$$

We have, see (134),

$$J_{xy}^y(\Omega_{m_4}) || y \leq W_1 W_2 \xi_y, \quad (76)$$

where

$$W_1^2 := \prod_{k=1}^{m_4} u_{1,y} (tg_y(Y_k) + s_2 / \tau_2), \quad W_2^2 := \prod_{k=1}^{m_4} v_{1,y} (tu_{1} + tu_{2}).$$

Invoking the bound, which can be shown in the same way as (135),

$$W_2 \xi_y \leq r_1, \quad r_1 := \exp \{16^{-1} \Theta(1)(m_4 / N_2) \sigma^2 \tau_2^2 t^2\}, \quad (77)$$

we obtain, by symmetry,

$$\mathbb{E} J_{xy}^y(\Omega_{m_4}) || y D^2 \leq r_1 \mathbb{E} W_1 D^2 = r_1 (N_2 - m_2) p_y q_y \mathbb{E} W_1 \psi_{xy}^2(X_1, Y_{N_2}). \quad (78)$$

The conditional expectation of $W_1 r_1$ given $Y_{N_2}$ is not greater than $(3 + s_2^2)^{-10}$, see (139). Therefore, we obtain

$$\mathbb{E} J_{xy}^y(\Omega_{m_4}) || y D^2 \leq (3 + s_2^2)^{-10} r_1^2 \sigma^2_{xy}. \quad (79)$$

Substituting this bound in (75) we obtain

$$R_{16} \leq (3 + s_2^2)^{-10} (m_1 / N_1) r_1^2 \sigma^2_{xy}. \quad (79)$$

This implies $t^2 R_{16} \sim \mathcal{R}$ thus completing the proof of (72). We have shown (70) for $k = 1$. For $k = 2, 3, 4$ the proof is much the same.

Let us prove the second part of (69). Expanding the exponent of $J_0$ in powers of $tF_2$ we obtain

$$it \mathbb{E} J_0 Q_{xy}(B_{11}) = it \mathbb{E} \exp \{itL \} Q_{xy}(B_{11}) + (it)^2 R, \quad R := \mathbb{E} \exp \{itL \} Q_{xy}(B_{11}) F_2 \theta_F,$$

where the random variable $|\theta_F| \leq 1$ is a function of $F_2$. We shall show that $t^2 R \sim \mathcal{R}$.

We have $|R| \leq R_1 + R_2 + R_3$, where

$$R_1 := \mathbb{E} \tilde{J}_x(\Omega_{m_3}) \tilde{J}_y(\Omega_{m_4}) \| Q_{xy}(B_{11}) Q_{xy}(B_4)),
$$

$$R_2 := \mathbb{E} \tilde{J}_x(\Omega_{m_3}) \tilde{J}_y(\Omega_{m_4}) \| Q_{xy}(B_{11}) Q_{xy}(D_{2})),
$$

$$R_3 := \mathbb{E} \tilde{J}_x(\Omega_{m_3}) \tilde{J}_y(\Omega_{m_4}) \| Q_{xy}(B_{11}) Q_{xy}(D_{3})).$$
We shall show that
\[ t^2 R_k \ll \mathcal{R}, \quad k = 1, 2, 3. \]  
(80)

For \( k = 1 \) we write
\[ R_1 \leq R_{11} + R_{12}, \quad R_{11} := \mathbb{E}Q_{xy}(B_{11}), \quad R_{12} := \mathbb{E}\tilde{J}^x(\Omega_{m_3})\tilde{J}^y(\Omega_{m_4})Q_{xy}^2(B_4). \]

Here we use simple inequalities \( \tilde{J}^x(\Omega_{m_3}) \leq 1 \) and \( \tilde{J}^y(\Omega_{m_4}) \leq 1 \). Clearly, \( t^2 R_{11} \ll \mathcal{R} \). Furthermore, by symmetry,
\[ R_{12} = (N_1 - m_1)(N_2 - m_2)p_xq_xp_yq_y\mathbb{E}\tilde{J}^x(\Omega_{m_3})\tilde{J}^y(\Omega_{m_4})\psi_{xy}^2(X_{N_1}, Y_{N_2}). \]

Invoking (132), (133) we obtain
\[ R_{12} \ll \tau_1^2 \tau_2^2 \sigma_{xy}^2(3 + \sigma_1^2)^{-10}(3 + \sigma_2^2)^{-10}. \]

This implies \( t^2 R_{12} \ll \mathcal{R} \) thus proving (80) for \( k = 1 \). The proof for \( k = 2, 3 \) is much the same. The proof of (69) is complete. We arrived at (68).

Let us prove (65). We shall show that
\[ it\mathbb{E}JQ_x(D_f^1) \sim it\mathbb{E}_0 \exp\{i\tilde{L}\}Q_x(D_f^1). \]

(81)

Write
\begin{align*}
F_3 &= H_{31}^* + tQ_y(D_2^y), \quad H_{31}^* := H_{32}^* + tQ_{xy}(B_3), \\
H_{32}^* &= H_{33}^* + tQ_{xy}(B_2), \quad H_{33}^* := tQ_x(D_2^x).
\end{align*}

Let us prove that
\[ it\mathbb{E}JQ_x(D_f^1) \sim it\mathbb{E}_0 \exp\{iH_{31}^*\}Q_x(D_f^1). \]

(82)

Expanding the exponent in \( J \) in powers of \( itQ_y(D_2^y) \) we obtain
\begin{align*}
\mathbb{E}JQ_x(D_f^1) &= \mathbb{E}_0 \exp\{iH_{31}^*\}Q_x(D_f^1) + itR_1 + t^2 R_2, \\
R_1 : &= \mathbb{E}_0 \exp\{iH_{31}^*\}Q_x(D_f^1)Q_y(D_2^y), \quad |R_2| \leq \mathbb{E}|Q_x(D_f^1)|Q_y(D_2^y).
\end{align*}

We have, by symmetry and independence,
\[ |R_2| \leq \mathbb{E}|Q_x(D_f^1)||\mathbb{E}Q_y^2(D_2^y) \leq (\mathbb{E}Q_x^2(D_f^1))^{1/2} \mathbb{E}Q_y^2(D_2^y) \]
\[ = (\frac{m_1}{2})^2 p_x^2 q_x^2 \sigma_{xx}^2 \left(1/2\right) m_1 m_2 (N_2 - m_2) p_y^2 q_y^2 \sigma_{yy}^2 \]
\[ \ll (m_1/N_1)(m_2/N_2) \sigma(Q_x) \sigma(Q_y). \]

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Therefore, \( t^3 R_2 < \mathcal{R} \).

Let us prove that \( t^2 R_1 < \mathcal{R} \). By symmetry, it suffices to show that \( t^2 R_1^* < \mathcal{R} \), where we denote

\[
R_1^* = \mathbf{E} J_0 \exp \{ i \mathcal{H}_{31} \} Q_x (D^*_1) Q_y (D^*_y), \quad D^*_y = (\Omega_{m_2} \setminus \Omega_{m_4}) \times (\Omega_{N_2} \setminus \Omega_{m_2}).
\]

We have

\[
| R_1^* | \leq \mathbf{E} J_y^y (\Omega_{m_4}) | Q_x (D_1^f) Q_y (D_y^*) | \leq (1 + s_1^2) R_{11}^* + (1 + s_1^2)^{-1} R_{12}^*;
\]

where \( R_{11}^* := \mathbf{E} J_y^y (\Omega_{m_4}) Q_x^2 (D_1^f) \) and \( R_{12}^* := \mathbf{E} Q_y^2 (D_y^*) \). Here we applied the simple inequality \( ab \leq a^2 c^2 + b^2 c^{-2} \) and the inequality \( J_y^y (\Omega_{m_4}) \leq 1 \). Since \( R_{12}^* \ll (m_2 / N_2) \sigma^2 (Q_y) \) we obtain \( t^2 (1 + s_1^2)^{-1} R_{12}^* < \mathcal{R} \). We complete the proof of (82) by showing that

\[
t^2 (1 + s_1^2) R_{11}^* < \mathcal{R}. \tag{83}
\]

Write \( R_{11}^* = R_{2}^* + R_{3}^* \), where we denote

\[
R_2^* = \mathbf{E} J_y^y (\Omega_{m_4}) Q_x^2 (D_1^f) I_y^y, \quad R_3^* = \mathbf{E} J_y^y (\Omega_{m_4}) Q_x^2 (D_1^f) I_y^t,
\]

\[
I_y^y = \mathbf{1}_{\{ \alpha \mathcal{H}_y < \sigma_y^2 \}}, \quad I_y^t = 1 - I_y^y
\]

and the constant \( \alpha \) is defined in (123) below. Recall that the random variable \( \mathcal{H}_y^y (\Omega_{m_4}) \) is introduced in the subsection 8.1. We shall show that

\[
t^2 (1 + s_1^2) R_k^* < \mathcal{R}, \quad k = 2, 3. \tag{84}
\]

We have, by Chebyshev’s inequality and symmetry,

\[
R_3^* \ll \sigma_y^{-2} \mathbf{E} Q_x^2 (D_1^f) \mathcal{H}_y^y (\Omega_{m_4}) = \sigma_y^{-2} \mathbf{E} Q_x^2 (D_1^f) (u_1^y)^2
\]

\[
= \sigma_y^{-2} \left( \frac{m_1}{2} \right) (N_1 - m_1) \rho_x^3 \mathbf{E} \psi_x^2 (X_1, X_2) \psi_{xy}^2 (X_N, Y_1)
\]

\[
\ll \sigma_y^{-2} (m_1 / N_1)^2 \sigma_x^{-2} \sigma_y^2 (Q_x) \sigma_x^2 (Q_{xy}).
\]

This implies (84) for \( k = 3 \). In order to bound \( R_2^* \) we apply (133) and use symmetry,

\[
R_2^* \ll \frac{1}{(3 + s_2^2)^{10}} \mathbf{E} Q_x^2 (D_1^f) = \frac{(m_1)^2}{(3 + s_2^2)^{10}} \rho_x^3 q_x^2 \sigma_x^2 \ll \frac{m_1^2}{N_1^2} \frac{1}{(3 + s_2^2)^{10}} \sigma_x^2 (Q_x).
\]

We obtain (84), thus completing the proof of (82).
Let us prove that
\[ it \mathbf{E} J_0 \exp \{ i H_{31}^* \} \mathbb{Q}_x(D_1^x) \sim it \mathbf{E} J_0 \exp \{ i H_{32}^* \} \mathbb{Q}_x(D_1^x). \]  
(85)

Expanding the exponent in powers of \( it \mathbb{Q}_{xy}(B_3) \) we obtain
\[
\mathbf{E} J_0 \exp \{ i H_{31}^* \} \mathbb{Q}_x(D_1^x) = \mathbf{E} J_0 \exp \{ i H_{32}^* \} \mathbb{Q}_x(D_1^x) + it R_1 + t^2 R_2,
\]
where
\[
R_1 = \mathbf{E} J_0 \exp \{ i H_{32}^* \} \mathbb{Q}_x(D_1^x) \mathbb{Q}_{xy}(B_3),
\]
\[
| R_2 | \leq \mathbb{E} | \mathbb{Q}_x(D_1^x) \mathbb{Q}_{xy}(B_3) |. 
\]

Invoking the inequality \( a \leq c^2 + a^2 c^{-2} \) we obtain
\[
| R_2 | \leq (1 + s_1^2)^{-1} \mathbb{E} Q_{xy}^2(B_3) + (1 + s_1^2) \mathbb{E} Q_x^2(D_1^x) \mathbb{Q}_{xy}(B_3). \]  
(86)

By symmetry,
\[
\mathbb{E} Q_{xy}^2(B_3) = m_2 (N_1 - m_1) p_x p_y q_x q_y \sigma_{xy}^2 \ll (m_2 / N_2) \sigma^2(\mathbb{Q}_{xy}),
\]
\[
\mathbb{E} Q_x^2(D_1^x) \mathbb{Q}_{xy}(B_3) = \left( \frac{m_1}{2} \right) m_2 (N_1 - m_1) p_x^2 p_y q_y \mathbb{E} \psi_x^2(X_1, X_2) \psi_{xy}(X_{N_1}, Y_1)
\]
\[
\ll \frac{m_1^2 m_2}{N_1^2 N_2} \tau_3 \sigma^2 \sigma_{xy}^2 \ll \frac{m_1^2 m_2}{N_1^2 N_2} \sigma^2(\mathbb{Q}_x) \sigma^2(\mathbb{Q}_{xy}).
\]

Collecting these bounds in (86) we obtain that \( t^3 R_2 \ll \mathcal{R} \).

Let us show that \( t^2 R_1 \ll \mathcal{R} \). By symmetry, it suffices to show that
\[
R_1^* = \mathbf{E} J_0 \exp \{ i H_{32}^* \} \mathbb{Q}_x(D_1^x) \mathbb{Q}_{xy}(B'_3), \quad B'_3 = (\Omega_{N_1} \setminus \Omega_{m_1}) \times (\Omega_{m_2} \setminus \Omega_{m_4}).
\]

We have
\[
| R_1^* | \leq \mathbf{E} J^u(\Omega_{m_4}) | \mathbb{Q}_x(D_1^x) \mathbb{Q}_{xy}(B'_3) |
\leq (1 + s_1^2)^{-1} \mathbf{E} J^u(\Omega_{m_4}) \mathbb{E} Q_x^2(D_1^x) + (1 + s_1^2)^{-1} \mathbb{E} Q_{xy}^2(B'_3).
\]

In the last step we used the simple inequality \( ab \leq a^2 c^2 + b^2 c^{-2} \). Using the simple bound,
\[
\mathbb{E} Q_{xy}(B'_3) \leq (m_2 - m_4)(N_1 - m_1) p_x p_y q_x q_y \sigma_{xy}^2 \leq (m_2 / N_2) \sigma^2(\mathbb{Q}_{xy})
\]
we obtain
\[
t^2 (1 + s_1^2)^{-1} \mathbb{E} Q_{xy}^2(B'_3) \ll \mathcal{R}. \]  
(87)

Furthermore, invoking inequality (133) and using symmetry we obtain
\[
\mathbf{E} J^u(\Omega_{m_4}) \mathbb{E} Q_x^2(D_1^x) \ll \frac{1}{(3 + s_2^2)^{10}} \mathbb{E} Q_x^2(D_1^x) \ll \frac{1}{(3 + s_2^2)^{10}} \frac{m_1^2}{N_1^2} \sigma^2(\mathbb{Q}_x).
\]
This bound implies
\[ t^2(1 + s_1^2)EJ^y(\Omega_{m_4})Q_x^2(D_1^p) \prec R. \] (88)

Combining (87) and (88) we obtain \( t^2 R_1^* \prec R \) thus completing the proof of (85).

The proof of the formulas
\begin{align*}
&\, it EJ_0 \exp\{iH_{33}^*\}Q_x(D_1^r) \sim \, it EJ_0 \exp\{iH_{33}^*\}Q_x(D_1^r), \\
&\, it EJ_0 \exp\{iH_{33}^*\}Q_x(D_1^r) \sim \, it EJ_0 Q_x(D_1^r)
\end{align*}
is similar to that of (85) but simpler. Combining these bounds and (82), (85) we obtain the first formula of (81).

Let us prove the second formula of (81). Expanding the exponent in powers of \( iF_2 \) we obtain
\[ EJ_0 Q_x(D_1^r) = E \exp\{i\tilde{L}\}Q_x(D_1^r) + iR, \quad R := E \exp\{i\tilde{L}\}Q_x(D_1^r)F_2 \theta_F, \]
where the random variable \( |\theta_F| \leq 1 \) is a function of \( F_2 \). We have
\[ |R| \leq EJ^y(\Omega_{m_2})|Q_x(D_1^r)F_2| \leq (1 + s_1^2)R_1 + (1 + s_1^2)^{-1}R_2, \] (89)
where \( R_1 := EJ^y(\Omega_{m_2})Q_x^2(D_1^r) \) and \( R_2 := EJ^y(\Omega_{m_2})F_2^2 \). By symmetry and independence,
\[ R_1 = \left(\frac{m_1}{2}\right)p_2^2 q_x^2 \sigma_x^2 EJ^y(\Omega_{m_2}) \ll \frac{m_1^2}{N_l^2} \sigma_x^2 \left(\frac{1}{1 + s_2^2}\right)^{10}. \] (90)

Furthermore, we have \( R_2 = t^2(R_{21} + R_{22} + R_{23}) \), where
\begin{align*}
R_{21} : &= EJ^y(\Omega_{m_2})Q_{xy}^2(D_3^r) \ll \sigma_x^2(EJ^y(\Omega_{m_2}) \ll \frac{\sigma_x^2}{(1 + s_2)^{10}}, \quad \text{(91)} \\
R_{22} : &= EJ^y(\Omega_{m_2})Q_{xy}^2(D_3^r) \ll \sigma_x^2(EJ^y(\Omega_{m_2}) \ll \frac{\sigma_x^2}{(1 + s_2)^{10}}, \quad \text{(91)} \\
R_{23} : &= EJ^y(\Omega_{m_2})Q_{xy}^2(B_4) \ll \sigma_x^2\sigma_y EJ^y(\Omega_{m_2}) \ll \frac{\sigma_x^2}{(1 + s_2)^{10}}.
\end{align*}

Here we used bounds (133). Collecting inequalities (90) and (91) in (89) we obtain \( t^2 R \prec R \) thus proving the second formula of (81). We arrive at (65).

The proof of (66) is almost the same. The proof of (50) is complete.

Let us prove (49). Using the notation (67) write
\begin{align*}
F_3 &= H_1^* + tQ_x(D_2^r), \quad H_1^* := H_1^* + tQ_y(D_2^r), \\
H_2^* &= H_3^* + tQ_{xy}(B_2), \quad H_3^* := tQ_{xy}(B_3)
\end{align*}
and denote, for \( k = 1, 2, 3, \)
\[
f_k^* = E J_k, \quad J_k = J_0 \exp\{i H_k^*\}, \quad \text{and} \quad f_4^* = E J_0. \quad (92)
\]
In order to prove (49) we shall show that
\[
f_2 \sim f_1^* + f_5^*, \quad f_5^* := it E \exp\{i L\} Q_x(D_2^x), \quad (93)
\]
\[
f_1^* \sim f_2^* + f_6^*, \quad f_6^* := it E \exp\{i L\} Q_y(D_2^y), \quad (94)
\]
\[
f_2^* \sim f_3^* + f_7^*, \quad f_7^* := it E \exp\{i L\} Q_{xy}(B_2), \quad (95)
\]
\[
f_3^* \sim f_4^* + f_8^*, \quad f_8^* := it E \exp\{i L\} Q_{xy}(B_3), \quad (96)
\]
\[
f_4^* \sim E \exp\{i L\} + it E \exp\{i L\} F_2. \quad (97)
\]
Let us prove (93). Expanding the exponent in powers of \( it Q_x(D_2^x) \) we obtain
\[
f_2 = f_1^* + it E J_1 Q_x(D_2^x) + t^2 R, \quad |R| \leq E J_{xy}^0(\Omega_{m_2}) Q_x^2(D_2^x).
\]
We shall show that
\[
l^2 R \prec \mathcal{R} \quad \text{and} \quad it E J_1 Q_x(D_2^x) \sim f_5^*.
\]
In order to prove the first bound write \( R = R_1 + R_2 \), where we denote
\[
R_1 = E J_{xy}^0(\Omega_{m_2}) \mathbb{I}_y Q_x^2(D_2^x), \quad R_2 = E J_{xy}^0(\Omega_{m_2}) \mathbb{I}_y Q_x^2(D_2^x),
\]
\[
\mathbb{I}_y = \mathbb{I}_{(\alpha \leq \gamma_y(m_2) < \sigma^2)}, \quad \mathbb{I}_y = 1 - \mathbb{I}_y.
\]
It follows from (133) and (119) by symmetry, that
\[
R_1 \ll (3 + s_2^2)^{-10} \mathbb{E} Q_x^2(D_2^x) \ll (3 + s_2^2)^{-10}(m_1/N_1) \sigma^2(Q_x). \quad (99)
\]
Therefore, we have \( l^2 R_1 \prec \mathcal{R} \). In order to show that \( l^2 R_2 \prec \mathcal{R} \) we apply Chebyshev’s inequality and symmetry and use the inequality \( J_{xy}^0(\Omega_{m_2}) \leq 1 \). We have
\[
R_2 \ll \sigma_y^{-2} \mathbb{E} Q_x^2(D_2^x) \mathbb{E} J_{xy}^0(\Omega_{m_2}) = R_{21} + R_{22},
\]
\[
R_{21} = \sigma_y^{-2} \mathbb{E} Q_x^2(D_2^x)(v_1^y)^2 = \sigma_y^{-2} \tau_2^{-2} m_1 N_1 \sigma^2(Q_x) \sigma^2(Q_y),
\]
\[
R_{22} = \sigma_y^{-2} \mathbb{E} Q_x^2(D_2^x)(v_1^y)^2 = \sigma_y^{-2} m_1 q_x q_y \mathbb{E}(v_1^y)^2 (u_1^y)^2.
\]
Invoking the bound (121) we obtain
\[
R_{22} \ll (m_1/N_1) \tau_2^{-2} \sigma_y^{-2}(\tau_1^{-2}(\gamma_2 + \sigma^2(Q_x) \sigma^2(Q_{xy})).
\]
Since \( t^2 R_{21} \prec \mathcal{R} \) and \( t^2 R_{22} \prec \mathcal{R} \) we obtain \( t^2 R_2 \prec \mathcal{R} \). This completes the proof of the first bound of (98).

Let us prove the second part of (98). By symmetry, it suffices to show that

\[
E J_1 Q_x(D^x_\delta) \sim \tilde{J}_0^*, \quad \tilde{J}_0^* := E \exp\{it\tilde{L}\}Q_x(D^x_\delta),
\]

where \( D^x_\delta = (\Omega_{m_1} \setminus \Omega_{m_2}) \times (\Omega_{N_1} \setminus \Omega_{m_1}) \). We shall prove (100). Expanding in powers of \( t\tilde{Q}_y(D^y_2) \) we obtain

\[
it E J_1 Q_x(D^x_\delta) = \tilde{J}_0^* + (it)^2 R_1 + t^3 R_2, \quad \tilde{J}_0^* := it E J_2 Q_x(D^x_\delta),
\]

where

\[
R_1 := E J_2 Q_x(D^x_\delta) Q_y(D^y_2), \quad |R_2| \leq E J_2(\Omega_{m_3}) Q_x(D^x_\delta) Q_y(D^y_2).
\]

Let us show that \( t^3 R_2 \prec \mathcal{R} \). Using inequalities \( a \leq 1 + a^2 \) and \( J_y^*(\Omega_{m_3}) \leq 1 \) we write

\[
|R_2| \leq R_{21} + R_{22}, \quad R_{21} := E J_y^*(\Omega_{m_3}) Q_y(D^y_2), \quad R_{22} := E Q_y^2(D^x_\delta) Q_y(D^y_2).
\]

We have \( R_{22} \ll (m_1/N_1)(m_2/N_2)\sigma^2(Q_x)\sigma^2(Q_y) \). Therefore, \( t^3 R_2 \ll \mathcal{R} \). Furthermore, write

\[
R_{21} = R'_{21} + R''_{21}, \quad R'_{21} := E J_y^*(\Omega_{m_3}) I_y Q_y^2(D^y_2), \quad R''_{21} := E J_y^*(\Omega_{m_3}) I_y Q_y^2(D^y_2),
\]

where we denote \( I_y = I_{(\alpha \chi_y^*(\Omega_{m_3}) < \sigma^2)} \) and \( I_y = 1 - I_y \). Invoking (132) we obtain

\[
R'_{21} \ll (3 + s_\delta^2)^{-10} E Q_y^2(D^y_2) \ll (3 + s_\delta^2)^{-10} (m_2/N_2)\sigma^2(Q_y).
\]

Therefore, we have \( t^3 R'_{21} \ll \mathcal{R} \). Furthermore, by Chebyshev’s inequality and symmetry we write

\[
R''_{21} \ll \sigma_x^2 E Q_y^2(D^y_2) \chi_y^*(\Omega_{m_3}) = \sigma_x^2 E Q_y^2(D^y_2) (u_1^*)^2
\]

\[
= \sigma_x^2 m_2 p_y q_y E (u_1^*)^2 (u_1^*)^2
\]

\[
\ll \sigma_x^2 (m_2/N_2) \tilde{\tau}_1^{-2} (\tilde{\tau}_2^{-2} \gamma_y + \sigma^2(Q_y)\sigma^2(Q_{xy})).
\]

In the last step we used (121). It follows that \( t^3 R''_{21} \ll \mathcal{R} \). Finally, we have

\[
|t|^3 |R_2| \leq |t|^3 (R'_{21} + R''_{21} + R_{22}) \ll \mathcal{R}.
\]

Let us show that \( t^2 R_1 \ll \mathcal{R} \) for \( R_1 \) in (101). By symmetry, it suffices to show that

\[
t^2 R_1' \ll \mathcal{R} \quad \text{for} \quad R_1' := E J_2 Q_x(D^x_\delta) Q_y(D^y_2),
\]

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where $D^y = (\Omega_{m_2} \setminus \Omega_{m_4}) \times (\Omega_{N_2} \setminus \Omega_{m_2})$. We have

\[
|R_1'| \leq EJ^y_\frac{1}{x}(\Omega_{m_3})J^y_\frac{1}{x}(\Omega_{m_4})|Q_\frac{1}{x}(D^x_*)Q_\frac{1}{y}(D^y_*)| \leq R_{11} + R_{12},
\]

\[
R_{11} : = EJ^y_\frac{1}{x}(\Omega_{m_3})Q^2_\frac{1}{x}(D^x_*) , \quad R_{12} := EJ^y_\frac{1}{x}(\Omega_{m_4})Q^2_\frac{1}{x}(D^x_*).
\]

Proceeding as in the proof of the bound $t^3R_{21} \prec \mathcal{R}$ above, we obtain $t^2R_{11} \prec \mathcal{R}$ and $t^2R_{12} \prec \mathcal{R}$, thus proving (101).

It follows from (101), (102), (103) that

\[
itEJ_1Q_x(D^x_*) \sim \tilde{f}^*_6.
\]

Let us show that

\[
\tilde{f}^*_6 \sim \tilde{f}^*_7, \quad \tilde{f}^*_7 := itEJ_3Q_x(D^x_*) .
\]

Expanding the exponent in powers of $itQ_{xy}(B_2)$ we obtain

\[
\tilde{f}^*_6 = \tilde{f}^*_7 + t^2R, \quad |R| \leq EJ^y_\frac{1}{y}(\Omega_{m_2})|Q_\frac{1}{x}(D^x_*)Q_\frac{1}{y}(B_2)|.
\]

We have

\[
|R| \leq R_1 + R_2, \quad R_1 := EJ^y_\frac{1}{y}(\Omega_{m_2})Q^2_\frac{1}{y}(D^x_*), \quad R_2 := EJ^y_\frac{1}{y}(\Omega_{m_2})Q^2_\frac{1}{y}(B_2).
\]

Proceeding as in the proof of the bound $t^3R_{21} \prec \mathcal{R}$ above we obtain the bound $t^2R_1 \prec \mathcal{R}$. In order to prove the bound $t^2R_2 \prec \mathcal{R}$ write, by symmetry,

\[
R_2 = m_1(N_2-m_2)p_xq_xp_yq_yE \leq (m_1/N_1)t_1^2\tau_2 E, \quad V := EJ^y_\frac{1}{y}(\Omega_{m_2})\psi^2_{xy}(X_1,Y_{N_2}).
\]

Split

\[
V = V_1 + V_2, \quad V_1 := EJ^y_\frac{1}{y}(\Omega_{m_2})\tilde{\mathbb{I}}_y \psi^2_{xy}(X_1,Y_{N_2}), \quad V_2 := EJ^y_\frac{1}{y}(\Omega_{m_2})\mathbb{I}_y \psi^2_{xy}(X_1,Y_{N_2}),
\]

where we denote $\tilde{\mathbb{I}}_y = \mathbb{I}_{\{\alpha \psi^2_{xy}(\Omega_{m_2}) < \sigma^2_y\}}$ and $\mathbb{I}_y = 1 - \mathbb{I}_y$. It follows from (133) that

\[
V_1 \ll (3 + s_2)^{-10}E\psi^2_{xy}(X_1,Y_{N_2}) \ll (3 + s_2)^{-10}\tau_1^{-2}\tau_2^{-2}\sigma^2(\mathcal{Q}_{xy}).
\]

By Chebyshev’s inequality and symmetry,

\[
V_2 \ll \sigma_y^{-2}E\psi^2_{xy}(X_1,Y_{N_2})\sigma_y^2(\Omega_{m_2}) = \sigma_y^{-2}E\psi^2_{xy}(X_1,Y_{N_2})(u_1^y)^2 \\
\ll \sigma_y^{-2}\tau_1^{-2}\sigma^4_{xy} \ll \sigma_y^{-2}\tau_1^{-2}\tau_2^{-4}\sigma^4(\mathcal{Q}_{xy}).
\]

Since $t^2(m_1/N_1)t_1^2\tau_2 V_k \prec \mathcal{R}$ for $k = 1, 2$, we obtain $t^2R_2 \prec \mathcal{R}$, thus completing the proof of (104).
Let us show that
\[ \tilde{f}_8^* \sim \tilde{f}_8^*, \quad \tilde{f}_8^* := itEJ_0Q_x(D_x^e). \]  
(105)

Expanding the exponent in powers of \( itQ_{xy}(B_3) \) we obtain
\[
\tilde{f}_7^* = \tilde{f}_8^* + (it)^2R_1 + t^3R_2, \\
R_1 := EJ_0Q_x(D_x^e)Q_{xy}(B_3), \quad |R_2| \leq EJ^x(\Omega_{m_3})|Q_x(D_x^e)|Q_{xy}^2(B_3).
\]

In order to prove (105) we shall show that
\[ t^2R_1 \prec \mathcal{R} \quad \text{and} \quad t^3R_2 \prec \mathcal{R}. \]  
(106)

Let us prove the second bound. Using the inequality \( a \leq 1 + a^2 \) write
\[ |R_2| \leq R_{21} + R_{22}, \quad R_{21} := EJ^x(\Omega_{m_3})Q_{xy}^2(B_3), \quad R_{22} := EQ_x^2(D_x^e)Q_{xy}^2(B_3). \]

By symmetry and (132), we have
\[ R_{21} \leq (m_2/N_2)\tau_1^2\tau_2^2EJ^x(\Omega_{m_3})\psi_{xy}(X_{N_1}, Y_1) \ll (m_2/N_2)(3 + s_1^2)^{-10}\sigma^2(Q_{xy}). \]

Therefore, \( t^3R_{21} \prec \mathcal{R} \). Furthermore, by symmetry and (121), we have
\[ R_{22} = (m_1-m_3)m_2p_xp_yQ_x(xy)(\tau_1^2\tau_x + \sigma^2(Q_x)\sigma^2(Q_{xy})). \]

Therefore, \( t^3R_{22} \prec \mathcal{R} \). We obtain the second bound of (106). In order to prove the first bound it suffices to show that
\[ t^2R_1' \prec \mathcal{R} \quad \text{for} \quad R_1' := EJ_0Q_x(D_x^e)Q_{xy}(B_*) \]
where \( B_* = (\Omega_{N_1} \setminus \Omega_{m_1}) \times (\Omega_{m_2} \setminus \Omega_{m_4}) \). We have
\[
|R_1'| \leq EJ^x(\Omega_{m_3})J^y(\Omega_{m_4})|Q_x(D_x^e)Q_{xy}(B_*)| \leq R_{11} + R_{12}, \\
R_{11} := EJ^x(\Omega_{m_3})Q_{xy}^2(B_*), \quad R_{12} := EJ^y(\Omega_{m_4})Q_{xy}^2(D_x^e).
\]

Invoking (132) and (133) and using symmetry we obtain \( t^2R_{11} \prec \mathcal{R} \) and \( t^2R_{12} \prec \mathcal{R} \), thus completing the proof of (106).

Finally expanding the exponent in \( \tilde{f}_8^* \) in powers of \( iF_2 \) and using symmetry we obtain
\[ \tilde{f}_8^* = \tilde{f}_8^* + tR, \quad |R| \leq EJ^x(\Omega_{m_1})J^y(\Omega_{m_2})|Q_x(D_x^e)F_2|. \]  
(107)

We have \( |R| \ll |t|(R_1 + R_2 + R_3 + R_4) \), where
\[
R_1 := EJ^y(\Omega_{m_2})Q_{xy}^2(D_x^e), \quad R_2 := EJ^x(\Omega_{m_3})J^y(\Omega_{m_2})Q_{xy}^2(D_x^e), \\
R_3 := EJ^x(\Omega_{m_3})J^y(\Omega_{m_2})Q_{xy}^2(D_x^e), \quad R_4 := EJ^x(\Omega_{m_3})J^y(\Omega_{m_2})Q_{xy}^2(B_4).
\]
Invoking (132) and (133) and using symmetry we obtain \( t^2 R_k \prec \mathcal{R} \) for \( k = 1, 2, 3, 4 \). This implies \( \tilde{f}_s^* \sim \tilde{f}_t^* \) and completes the proof of (100). We arrive to (93). The proof of (94) is almost the same.

Let us prove (95). Expanding the exponent in powers of \( itQ_{xy}(B_2) \) we obtain

\[
f_s^* = f_t^* + itEJ_3Q_{xy}(B_2) + t^2 R, \quad |R| \leq EJ_x^\mu(\Omega_{m_2})Q_{xy}^2(B_2).
\]

In view of the bound (which is shown in the proof of (104)) \( t^2|R| \prec \mathcal{R} \), we need to prove that \( itEJ_3Q_{xy}(B_2) \sim f_t^* \). By symmetry, it suffices to show that

\[
\begin{align*}
\text{it}EJ_3Q_{xy}(B_*) & \sim \tilde{f}_0^* \quad \text{and} \quad \tilde{f}_0^* := \text{it}E\exp\{i\tilde{L}\}Q_{xy}(B_*),
\end{align*}
\]

where \( B_* = (\Omega_{m_1} \setminus \Omega_{m_3}) \times (\Omega_{N_2} \setminus \Omega_{m_4}) \). We shall show that

\[
\begin{align*}
\text{it}EJ_3Q_{xy}(B_*) & \sim \text{it}EJ_0Q_{xy}(B_*) \quad \text{and} \quad \text{it}EJ_0Q_{xy}(B_*) \sim \tilde{f}_0^*. \quad (109)
\end{align*}
\]

In order to prove the first relation of (109) we expand the exponent in powers of \( itQ_{xy}(B_3) \),

\[
\begin{align*}
itEJ_3Q_{xy}(B_*) & = \text{it}EJ_0Q_{xy}(B_*) + (it)^2 R_1 + it^3 R_2, \\
R_1 & = EJ_0Q_{xy}(B_*)Q_{xy}(B_3), \\
|R_2| & \leq EJ_x^\mu(\Omega_{m_3})Q_{xy}(B_*)Q_{xy}^2(B_3)
\end{align*}
\]

and show that \( t^2R_1 \prec \mathcal{R} \) and \( t^3|R_2| \prec \mathcal{R} \). The proof of these bounds is similar to that of (106). The proof of the second relation of (109) is similar to that of the relation \( \tilde{f}_s^* \sim \tilde{f}_t^* \), see (107) above. We obtain (108) thus completing the proof of (95). The proof of (96) and (97) is similar. The proof of (45) is complete.

8.2.2. Proof of (46). We shall show that

\[
\begin{align*}
\text{it}E\exp\{i\tilde{L}\}Q_{xy} & \sim \tau_1^2 \tau_2^2 \kappa_{xy}(it)^3 E\exp\{i\tilde{L}\}, \quad (110) \\
\text{it}E\exp\{i\tilde{L}\}Q_x & \sim 2^{-1} \tau_1^4 \kappa_x(it)^3 E\exp\{i\tilde{L}\}, \quad (111) \\
\text{it}E\exp\{i\tilde{L}\}Q_y & \sim 2^{-1} \tau_2^4 \kappa_y(it)^3 E\exp\{i\tilde{L}\}. \quad (112)
\end{align*}
\]

Write

\[
\begin{align*}
\tilde{L} & = T + Z, \quad T = \sum_{k=1}^{N_1} t_k w_k, \quad Z = \sum_{j=1}^{N_2} z_j \xi_j, \\
t_k & = tg_x(X_k) + s_1/\tau_1, \quad z_j = tg_y(Y_j) + s_2/\tau_2.
\end{align*}
\]
Let us prove (110). By symmetry,
\[ it \mathbf{E} \exp \{ i \tilde{L} \} Q_{xy} = N_1 N_2 h_1, \quad h_1 = it \mathbf{E} \exp \{ i \tilde{L} \} \psi_{xy}(X_1, Y_1) \omega_1 \xi_1. \]

Split \( \tilde{L} = \tilde{L}' + t_1 \omega_1 + z_1 \xi_1. \) Expanding the exponent \( \exp \{ i \tilde{L}' + it_1 \omega_1 + iz_1 \xi_1 \} \) in powers of \( it_1 \omega_1 \) and \( iz_1 \xi_1 \) we obtain
\[
\begin{align*}
  h_1 &= h_2 + R_1 + R_2, \quad (114) \\
  h_2 &= it \mathbf{E} \exp \{ i \tilde{L}' \} w_1, \quad w_1 := -\psi_{xy}(X_1, Y_1)t_1 z_1 \omega_1^2 \xi_1^2, \\
  |R_1| &\leq |t| \mathbf{E} J^x(\Omega_{N_1}) J^y(\Omega_{N_2}) |w_1 t_1 \omega_1|, \\
  |R_2| &\leq |t| \mathbf{E} J^x(\Omega_{N_1}) J^y(\Omega_{N_2}) |w_1 z_1 \xi_1|.
\end{align*}
\]

Here \( \Omega_{N_k} = \Omega_{N_k} \setminus \{1\} \). We have
\[
h_2 = it \mathbf{E} \exp \{ i \tilde{L}' \} w_2, \quad \text{where} \quad w_2 := -p_x^2 q_x^2 p_y^2 q_y^2 \psi_{xy}(X_1, Y_1) t_1 z_1.
\]

Proceeding as in (114) we obtain
\[
\begin{align*}
  h_2 &= h_3 + R_3 + R_4, \quad h_3 = it \mathbf{E} \exp \{ i \tilde{L} \} w_2, \quad (115) \\
  |R_3| &\leq |t| \mathbf{E} J^x(\Omega'_{N_1}) J^y(\Omega'_{N_2}) |w_2 t_1 \omega_1|, \\
  |R_4| &\leq |t| \mathbf{E} J^x(\Omega'_{N_1}) J^y(\Omega'_{N_2}) |w_2 z_1 \xi_1|.
\end{align*}
\]

Let \( \mathbf{E}_{\nu, \xi} \) denote the conditional expectation given all the random variables, but \( \{ \nu_k, k = 1, \ldots, N_1 \} \) and \( \{ \xi_r, r = 1, \ldots, N_2 \} \). Note that \( (t, s_1, s_2) \to \mathbf{E}_{\nu, \xi} \exp \{ i \tilde{L} \} \) is a non-random function. Therefore, we have
\[
\begin{align*}
  h_3(t) &= \mathbf{E} \mathbf{E}_{\nu, \xi} \exp \{ i \tilde{L} \} w_2 = \mathbf{E}_{\nu, \xi} \exp \{ i \tilde{L} \} \mathbf{E} w_2 \quad (116) \\
  &= \mathbf{E} \exp \{ i \tilde{L} \} w_3,
\end{align*}
\]

where \( w_3 := \mathbf{E} w_2 = (it)^2 p_x q_x p_y q_y \kappa_{xy}. \) The last identity uses (25). From (114), (115) and (116) we obtain
\[ it \mathbf{E} \exp \{ i \tilde{L} \} Q_{xy} = \tau_1^2 \tau_2^2 \kappa_{xy} (it)^3 \mathbf{E} \exp \{ i \tilde{L} \} + N_1 N_2 R, \quad R := R_1 + R_2 + R_3 + R_4. \]

We complete the proof of (110) by showing that \( R \ll \mathcal{R}. \)

Invoking (132) and (133) we obtain
\[
|R| \ll |t| (3 + s_1^2)^{-10} (3 + s_2^2)^{-10} \mathbf{E} (|w_1| + |w_2|)(|t_1| + |z_1|). \quad (117)
\]

Using Cauchy-Schwartz we obtain for \( k = 1, 2, \)
\[
\begin{align*}
  \mathbf{E} |w_k t_1| &\ll (|t|^3 + |s_1|^3 + |s_2|^3) N_1^{-1} N_2^{-1} \tau_1^{-1} \sigma(Q_{xy}) B_{t_1}^{1/2}, \quad (118) \\
  \mathbf{E} |w_k z_1| &\ll (|t|^3 + |s_1|^3 + |s_2|^3) N_1^{-1} N_2^{-1} \tau_2^{-1} \sigma(Q_{xy}) B_{z_1}^{1/2}.
\end{align*}
\]

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These bounds in combination with (117) imply $R < R$. We arrived to (110).

The proof of (111) and (112) is almost the same. We show that

$$|t\mathbb{E}\exp\{i\mathcal{L}\}Q_x - 2^{-1} \tau_1^4 \kappa_x (it)^3 \mathbb{E}\exp\{i\mathcal{L}\}| < |t|(1 + |t|^3) \tau_1^{-1} \sigma(Q_x) B^{1/4}_{x},$$

$$|t\mathbb{E}\exp\{i\mathcal{L}\}Q_y - 2^{-1} \tau_2^4 \kappa_y (it)^3 \mathbb{E}\exp\{i\mathcal{L}\}| < |t|(1 + |t|^3) \tau_2^{-1} \sigma(Q_y) B^{1/4}_{y}. $$

The proof of (46) is complete. Lemma 8.1 is proved.

9 Moment inequalities

It follows from (27) that

$$\sigma_x^2 \ll \frac{\sigma^2(L_x)}{\tau_1^2}, \quad \sigma_y^2 \ll \frac{\sigma^2(L_y)}{\tau_2^2}, \quad \sigma_{x} \ll \frac{\sigma^2(Q_x)}{\tau_1^4}, \quad (119)$$

$$\sigma_{yy}^2 \ll \frac{\sigma^2(Q_y)}{\tau_2^2}, \quad \sigma_{xy}^2 \ll \frac{\sigma^2(Q_{xy})}{\tau_1^2 \tau_2^2}.$$  

Lemma 9.1. We have

$$\mathbb{E}Q^2_{xy}(B_1) \ll \frac{m_1 m_2}{N_1 N_2} \sigma^2(Q_{xy}),$$  

$$\mathbb{E}Q^2_{x}(D^y_{1}) \ll \frac{m_1^2}{N_1} \sigma^2(Q_x), \quad \mathbb{E}Q^2_{y}(D^y_{1}) \ll \frac{m_2^2}{N_2} \sigma^2(Q_y),$$  

$$\mathbb{E}(v^2_i)(u^2_j) \ll \tau_1^{-2} \tau_2^{-2} \left(\tau_1^{-2} \gamma_x + \sigma^2(Q_x) \sigma^2(Q_{xy})\right), \quad (121)$$

$$\mathbb{E}(u^2_i)(u^2_j) \ll \tau_1^{-2} \tau_2^{-2} \left(\tau_2^{-2} \gamma_y + \sigma^2(Q_y) \sigma^2(Q_{xy})\right).$$

Proof of Lemma 9.1. Inequalities (120) follow from (119) and the identities

$$\mathbb{E}Q^2_{xy}(B_1) = m_1 m_2 \mathbb{E} \omega^2_1 \xi^2_2 \sigma^2_{xy}, \quad \mathbb{E}Q^2_{x}(D^y_{1}) = 2^{-1} m_1 (m_1 - 1) \mathbb{E} \omega^2_1 \omega^2_2 \sigma^2_{xx},$$

$$\mathbb{E}Q^2_{y}(D^y_{1}) = 2^{-1} m_2 (m_2 - 1) \mathbb{E} \xi^2_1 \xi^2_2 \sigma^2_{yy}.$$  

Let us prove the first bound of (121). The proof of the second bound is almost the same. Denote for brevity $a_i = \psi_x(X_1, X_i) \omega_i$ and $b_j = \psi_{xy}(X_j, Y_1) \omega_j$. By independence, we have

$$\mathbb{E}\left(\sum_i a_i\right)^2 \left(\sum_j b_j\right)^2 = \mathbb{E}S_1 + \mathbb{E}S_2 + 4 \mathbb{E}S_3,$$  

$$S_1 = \sum_i a_i^2 b_i^2, \quad S_2 = \sum_{i<j} (a_i^2 b_j^2 + a_j^2 b_i^2), \quad S_3 = \sum_{i<j} a_i a_j b_i b_j.$$  

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Here $\sum_i$ denotes the sum over $m_1 < i \leq N_1$ and $\sum_{i<j}$ denotes the sum over $m_1 < i < j \leq N_1$. By symmetry we have

$$E S_1 = (N_1 - m_1) E a_{N_1}^2 b_{N_1}^2 \ll \tau_1^2 E \psi_x^2(X_1, X_{N_1}) \psi_{xy}^2(X_{N_1}, Y_1) \leq \tau_1^{-4} \tau_2^{-2} \gamma_x.$$  

Furthermore,

$$E S_2 = \left(\frac{N_1 - m_1}{2}\right) E a_{N_1}^2 b_{N_1-1} \ll \tau_1^4 E \psi_x^2(X_1, X_{N_1}) \psi_{xy}^2(X_{N_1+1}, Y_1) \ll \tau_1^{-4} \tau_2^{-2} \sigma_x^2 \sigma_{xy}^2 \ll \tau_1^{-2} \sigma_x^2 \sigma_{xy}^{-2} (Q_x) \sigma_{xy}.$$

Finally, by Cauchy-Schwartz,

$$E S_3 \leq \sum_{i<j} (E a_i^2 b_j^2 + E a_j^2 b_i^2) = E S_2.$$  

Collecting the bounds for $E S_1$ and $E S_2 = E S_3$ in (122) we obtain the first bound of (121).

### 10 Auxiliary inequalities

Denote

$$\begin{align*}
\beta_x(t) &= E e^{i \omega_1 t}, \quad \beta_y(t) = E e^{i \xi_1 t}; \\
\Theta(t) &= \left(\frac{2 \pi - t}{\pi + t}\right)^2, \quad \alpha = \frac{32 \pi}{\Theta(1)} (1 + \frac{4}{\Theta(1)}), \\
u_{\{d,x\}}(t) &= 1 - p_x \Theta(d) 2 t^2 \mathbb{1}_{t < d + \pi}, \\
u_{\{d,y\}}(t) &= 1 + p_x \frac{2 \pi d}{\Theta(d)} (\frac{4}{\Theta(d)} + 1) t^2,
\end{align*}$$

(123)

$$\begin{align*}
u_{\{d,y\}}(t) &= 1 - p_y \Theta(d) 2 t^2 \mathbb{1}_{t < d + \pi}, \\
v_{\{d,y\}}(t) &= 1 + p_y \frac{2 \pi d}{\Theta(d)} (\frac{4}{\Theta(d)} + 1) t^2.
\end{align*}$$

In proofs we use the following result, see Lemma 5.2 in Bloznelis and Götze (2000), Bernoulli, 6, 729-760.

**Lemma 10.1.** For each $d \in (0, \pi)$ and $t, s \in \mathbb{R}$ we have

$$|\beta_x(s+t)|^2 \leq u_{\{d,x\}}(t) v_{\{d,x\}}(s), \quad |\beta_y(s+t)|^2 \leq u_{\{d,y\}}(t) v_{\{d,y\}}(s).$$

Denote

$$\begin{align*}
\varepsilon_x &= 8 (N_1^{2/3} D_x^{2/3} + 1)/(N_1 - 4), \\
\varepsilon_y &= 8 (N_2^{2/3} D_y^{2/3} + 1)/(N_2 - 4).
\end{align*}$$

(125)
Lemma 10.2. Let $c_T > 2$ and $d \in (0, \pi)$. Assume that $|t| \leq T$, where $T^{-1} = c_T B_{x3} \sigma_x$. For $c' = \min\{c_T d, c_T^2 d^2\}$ we have

$$
E(tg_x(X_1) + \frac{s_1}{\tau_1})^2 \mathbb{I}_{|tg_x(X_1)| < d} \geq (\sigma_x^2 t^2 + \frac{s_1^2}{\tau_1^2})(1 - 2/c'),
$$

(126)

and

$$
E[tg_x(X_1) + \frac{s_1}{\tau_1}]^2 \mathbb{I}_{|tg_x(X_1)| < d} \geq (\sigma_x^2 t^2 + \frac{s_1^2}{\tau_1^2})(1 - \varepsilon - 2/c').
$$

(127)

Here $E[\cdot]$ denotes the conditional expectation given $X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}$, where $1 \notin \{i_1, i_2, i_3, i_4\}$.

Proof of lemma 10.2. The proof is similar to the proof of Lemma 5.3 ibidem. Let us prove (126). Denote

$$
\mathbb{I} = \mathbb{I}_{|tg_x(X_1)| < d}, \quad \mathbb{I} = 1 - \mathbb{I}, \quad Z_1 = tg_x(X_1) + s_1/\tau_1, \quad A = EZ_1^2 \mathbb{I}.
$$

For $|t| \leq T$ we have

$$
EZ_1^2 \mathbb{I}_{|tg_x(X_1)| < d} \geq EZ_1^2 \mathbb{I} = EZ_1^2 - A = \sigma_x^2 t^2 + \tau_1^{-2} s_1^2 - A.
$$

In the last identity we use $Eg_x(X_1) = 0$. Therefore, in order to prove (126) it suffices to show that

$$
A \leq (2/c')(\sigma_x^2 t^2 + \tau_1^{-2} s_1^2).
$$

(128)

Introduce the set $K \subset X$, $K = \{x_j : T|g_x(x_j)| \geq d\}$. Write

$$
A = N_1^{-1} \sum_{x_j \in K} (tg_x(x_j) + \tau_1^{-1} s_1)^2 \leq \frac{2}{N_1} t^2 W + \frac{2}{N_1} \frac{s_1^2}{\tau_1^2} |K|,
$$

(129)

where $W = \sum_{x_j \in K} g_x^2(x_j)$. By Chebyshev’s inequality, for $r > 0$,

$$
|K| \leq \sum_{x_j \in K} (T/d)^r |g_x(x_j)|^r \leq (T/d)^r N_1 E|g_x(X_1)|^r = (T/d)^r N_1 B_{xr} \sigma_x^r.
$$

(130)

In particular, we have

$$
|K| \leq (T/d)^3 N_1 B_{x3} \sigma_x^3, \quad |K| \leq (T/d)^2 N_1 \sigma_x^2.
$$

(131)

Furthermore, by Minkovski’s inequality,

$$
W \leq \left( \sum_{x_j \in K} |g_x(x_j)|^3 \right)^{2/3} |K|^{1/3} \leq (N_1 E|g_x(X_1)|^3)^{2/3} |K|^{1/3}.
$$

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We have almost surely \(\subset D\) in (47) is larger than \(E\). The random variables but

Proof of Lemma 10.2.

The last inequality follows by Minkowskii inequality,

\[
\sum_{j=2}^{5} g(x_j)^2 \leq 4^{1/3} N_1^{2/3} B_{x_3}^{2/3} \sigma_x^2.
\]

The last inequality follows by Minkowski inequality,

\[
\sum_{j=2}^{5} g(x_j)^2 \leq 4^{1/3} \left( \sum_{j=2}^{5} |g(x_j)|^{3/2} \right)^{2/3} \leq 4^{1/3} N_1^{2/3} \left( \frac{1}{N_1} \sum_{j=1}^{N_1} |g(x_j)|^{3/2} \right)^{2/3}.
\]

Proof of Lemma 10.2 is complete.

Let \(E_X\) (respectively \(E_Y\)) denote the conditional expectation given all the random variables but \(X^*\) (respectively \(Y^*\)). By \(E_X^{i_1 \ldots i_k}\) (respectively \(E_Y^{i_1 \ldots i_k}\)), we denote the conditional expectation given all the random variables but \(\{X_1, \ldots, X_N\} \setminus \{X_{i_1}, \ldots, X_{i_k}\}\) (respectively \(\{Y_1, \ldots, Y_{N_2}\} \setminus \{Y_{i_1}, \ldots, Y_{i_k}\}\).

Lemma 10.3

Assume that \(|t| \leq T\), where the number \(T\) is defined in (43) and the constant \(c_T\) in (43) is larger than 8. Assume that numbers \(\varepsilon_x, \varepsilon_y\) (defined in (125)) are smaller than 1/4. Assume that the constant \(C'\) in (47) is larger than \(10^3/\Theta(1)\). Then for every \(B \subseteq \Omega_{m_1}, |B| \geq m_1/4\) and \(D \subseteq \Omega_{m_2}, |D| \geq m_2/4\), and for every \(\{ij\} \subseteq \Omega_{N_1} \setminus B\), and \(\{k,l\} \subseteq \Omega_{N_2} \setminus D\) we have almost surely

\[
E^{ij}_{X} J_{xy}^{*}(B) \ll (3 + s_1^{2})^{-10}, \quad E^{ij}_{X} J_{x}^{*}(B) \ll (3 + s_1^{2})^{-10}, \quad (132)
\]

\[
E^{ij}_{Y} J_{xy}^{*}(B) \ll (3 + s_2^{2})^{-10}, \quad E^{ij}_{Y} J_{x}^{*}(B) \ll (3 + s_2^{2})^{-10},
\]

\[
E^{kl}_{Y} J_{xy}^{*}(D) \ll (3 + s_2^{2})^{-10}, \quad E^{kl}_{Y} J_{y}^{*}(D) \ll (3 + s_2^{2})^{-10}, \quad (133)
\]

\[
E^{kl}_{Y} J_{xy}^{*}(D) \ll (3 + s_2^{2})^{-10}, \quad E^{kl}_{Y} J_{y}^{*}(D) \ll (3 + s_2^{2})^{-10}.
\]
The integer valued functions \( m_1 = m_1(s_1) \) and \( m_2 = m_2(s_2) \) are defined in (47).

**Remark.** Inequalities (132) and (133) remain valid if we replace the expectation \( E^y \) (respectively \( E^y_x \)) by \( E^y_X \) or \( E^y_{X,x} \) (respectively \( E^y_X \) or \( E^y_{X,x} \)).

**Proof of Lemma 10.3.** We shall prove the first inequality of (132) only. Denote \( Z_k = t g_x(X_k) + \tau_1^{-1}s_1 \). It follows from (124) that
\[
J^x_{xy}(B) \leq W_1 W_2, \quad W_1 := \prod_{k \in B} u_{\{1,x\}}(Z_k), \quad W_2 := \prod_{k \in B} v_{\{1,x\}}(t v_k + t u_k).
\]
(134)
The inequality \( 1 + z \leq e^z \) implies almost surely
\[
W_2 I_{\{\alpha x y(B) < \sigma_x^2\}} \leq \prod_{\{\alpha x y(B) < \sigma_x^2\}} \exp\{\exp\{2 \pi (1 + 4/\Theta(1))|B| p_x q_x \alpha x y(B)|^2\}\}
\leq \prod_{\{\alpha x y(B) < \sigma_x^2\}} \exp\{16^{-1}\Theta(1)\alpha x y(B)|B|/N_1 \tau_1^2 t^2\}
\leq \exp\{16^{-1}\Theta(1)|B|/N_1 \sigma_x^2 \tau_1^2 t^2\} =: r.
\]
(135)
Therefore, we have almost surely
\[
J^x_{xy}(B)I_{\{\alpha x y(B) < \sigma_x^2\}} \leq W_1 r.
\]
(136)
Hoeffding’s (1963) Theorem 4 implies \( E^j x W^2_i \leq (E^j x u_{\{1,x\}}(Z_1)) |B| \). Therefore,
\[
E^j x W^2_i \leq (E^j x u_{\{1,x\}}(Z_1)) |B|/2.
\]
(137)
For \( |s_1/\tau_1| \leq \pi \) and \( |t| \leq T \) we obtain from (127) that
\[
E^j x Z^2_{\{t g_x(X_1) + s_1/\tau_1 \}} \geq \exp\{16^{-1}(1 - \epsilon_x - 2/c_T)\}(1 - \epsilon_x - 2/c_T),
\]
where \( \epsilon_x \) is defined in (125). Therefore, we have
\[
E^j x u_{\{1,x\}}(Z_1) \leq 1 - 2^{-1}p_x q_x \Theta(1)(\sigma_x^2 \tau_1^2 + \tau_1^2/\tau_1^2)(1 - \epsilon_x - 2/c_T).
\]
(138)
The inequality \( 1 + z \leq e^z \) combined with (137) and (138) implies
\[
E^j x W_1 \leq \exp\{-4^{-1}(1 - \epsilon_x - 2/c_T)\}(1 - \epsilon_x - 2/c_T)\Theta(1)p_x q_x |B|(\sigma_x^2 \tau_1^2 + \tau_1^2/\tau_1^2)
\leq \exp\{-4^{-1}(1 - \epsilon_x - 2/c_T)\}(1 - \epsilon_x - 2/c_T)\Theta(1)(|B|/N_1)(\sigma_x^2 \tau_1^2 t^2 + \tau_1^2)
\leq \exp\{-8^{-1}\Theta(1)(|B|/N_1)(\sigma_x^2 \tau_1^2 t^2 + \tau_1^2)\}.
\]
(139)
Combining this bound and (136) we obtain
\[
E^j x J^x_{xy}(B)I_{\{\alpha x y(B) < \sigma_x^2\}} \leq \exp\{-16^{-1}\Theta(1)(|B|/N_1)(\sigma_x^2 \tau_1^2 t^2 + \tau_1^2)\}
\leq \exp\{-64^{-1}\Theta(1)(m_1/N_1)(\sigma_x^2 \tau_1^2 t^2 + \tau_1^2)\}
\leq \exp\{-\Theta(1)m_1 N_1 \sigma^2/64 \leq \frac{1}{(3 + \tau_1^2)^{10}}.
\]
The proof of the first inequality of (132) is complete. The proof of the remaining inequalities of the lemma is much the same.

11 References


