FORMALISING LEGAL RELATIONS

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A. A VARIETY OF RELATIONS IN LAW

This paper concerns arbitrary relations in law and extends beyond concrete types of legal relations. There is no established ontology of relations in the legal domain, although ontologies of legal concepts are discussed in literature. Different legal terms, such as duty, contract, debt, husband, etc. can be viewed as relations. However, on the meta-level, there is no model of relation types in law. This paper tackles the following types of relations: (a) direct relation; (b) an indirect relation through tertium comparationis; (c) subsumption, i.e. a relation between fact, which is in Is, and the normative condition, which is in Ought; and (d) amplitude relation. We follow Arthur Kaufmann’s assertion that relations in law can be managed similarly to substances.
The theme of formalising legal relations is strongly related with formalizations by Hajime Yoshino. His Legal Jurisprudence\(^1\) deals not with legal norms, but with propositions that are called Legal Sentences. Predicate logic formulae, called Compound Predicate Formulae, are used to represent Legal Sentences. In this way, Legal Sentences represent the meaning of legal relations.

To explore a notation for different types of legal relations, we would first mention Ulrich Klug\(^2\) who presented relation calculus in the context of legal logic. Hence, we articulate binary relations \(R(A,B)\). They can also be written \(A R B\) – in infix notation.

A categorical distinction between Is and Ought\(^3\) implies two wide types of relations: causality relations \(A\) causes \(B\) and imputation relations \(A\) imputation \(B\). Binary relations can be represented in the form of pairs and also graphically (Figure 1):

![Figure 1. A binary relation \(R(A,B)\): without the name of a relation and with the name](image)

A variety of causality relations appears in the nature and is studied in natural sciences. A central theme in law is legal relation between different entities. The following three cases can be distinguished:

1. A vinculum juris between persons (“bonds of law”).
3. The relation between the factual (Lebenssachverhalt) and the normative (gesetzlicher Tatbestand, Tatbestand der Norm, Normhypothese). This relation is involved when making subsumption. It should be noted that the factual appears in Is, whereas the

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The normative appears in Ought. The normative condition is expressed in legal terms, which constitute a modally indifferent substrate.4

There is a difference between a relation $A R B$ and a statement that $A$ and $B$ are in a relation $R$. This serves to express that a teller says, ‘$A$ and $B$ are in a relation $R$’. There is a difference in who the teller is – for example, a judge or a layperson.

Creatively new is to name a relation $R$, for example, $\text{counts\_as}$. Next, a pointing direction from the referent to the relatum is also important because they are in different roles. This holds for asymmetric relations, such as “inheritor”, “self-defence” and is of no importance for symmetric relations, such as “be-in-contract-partnership”, “sibling”, etc. Hence, different notations can be used: $A R B$, $A \rightarrow R B$ and $A \leftarrow R B$, for example: $A \text{ legal\_relationship } B$, parent care $\rightarrow$ child, parent $\leftarrow$ obey child, $A \text{ right } \rightarrow B$, $A \leftarrow \text{duty } B$.

There may be inverse relations.5 For example, the inverse of $\text{employee}(A,B)$ is $\text{employer}(B,A)$. A relation, which is transitive and symmetric, is called equivalence relation ($\text{Gleichheitrelation}$).6 It is important from a mathematical point of view, because this allows it to divide tuples into a partition.7

A huge variety of relations is immanent to law. A reason is that people are highly interrelated. The importance of this variety can be compared with knowledge-based systems in artificial intelligence, where a huge variety of rules and facts outweighs an inference engine. The knowledge principle8 sums up the importance of taking into account a huge variety of the world: “God is in the details.”

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5 See Klug, Juristische Logik, Note 2, p. 79.
6 Ibid., p. 83.
7 In mathematics, a partition of a set $X$ is a set of nonempty subsets of $X$ such that every element $x$ in $X$ is in exactly one of these subsets (i.e., $X$ is a disjoint union of the subsets); see Halmos, Paul R., Naive Set Theory, 1960, Springer Nature Switzerland, p. 28.
Following are the themes to discuss in this paper: different types of relations, distinguishing between relations and statements about the relations, relations in JURIX papers, and Peter Chen’s entity–relationship model.

B. DEFINITION OF RELATION IN MATHEMATICS

While attending the classes of law and while one can hear the term “legal relations”. However, a formal definition of a relation is provided not always. Therefore a listener can raise a question what legal relation is. Deep studies can lead to philosophical problems, such as the problem of universals, meaning and reference (Sinn und Bedeutung). In ontologies (in the sense of computer science), this leads to defining the terms extensional relational structure and intensional relational structure.\(^9\)

A relation \(R\) over the sets \(X_1, \ldots, X_n\) is defined as a subset of their Cartesian product, written \(R \subseteq X_1 \times \ldots \times X_n\). A notation for an \(n\)-ary relation \((n \geq 1)\) is \(R(a_1, \ldots a_n)\). We may also write \((a_1, \ldots a_n) \in R\). A binary relation \((n = 2)\) can be written in prefix notation \(Ra_1,a_2\), infix notation \(a_1Ra_2\) and postfix notation \(a_1a_2R\).

A relation \(R\) corresponds to the \(n\)-ary predicate, which is usually given the same name:

\[
P(a_1, \ldots a_n) = \text{true} \iff (a_1, \ldots a_n) \in R, \quad \text{false otherwise}, \text{ i.e. } (a_1, \ldots a_n) \notin R
\]

Unary relations \((n = 1)\) are usually called predicates. Predicates for \(n = 0\) denote atoms. In propositional logic, atomic formulas are called propositional variables. In a sense, these are nullary (i.e. \(0\)-arity) predicates. In mathematics, a predicate is commonly understood to be a Boolean-valued function \(P: X \to \{\text{true}, \text{false}\}\), called the predicate on \(X\). An extreme case \(n = 0\) is obtained when the domain set \(X\) is empty.

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Figure 2. A ternary relation \( R(A, B, C) \) and a quaternary relation \( R(A, B, C, D) \)

Graphical representation of binary relations can be extended to \( n \)-ary relations that consist on \( n \)-tuples. Ternary relations consist of triples \( R(A, B, C) \), 4-ary relations consist of quadruplets \( R(A, B, C, D) \), etc. They can be represented as bipartite graphs (Figure 2), where the node for the relation name differs from tuple element nodes.

Figure 3. A situation, which explains the right of succession in inheritance law\(^{10}\)

**Reification.** In artificial intelligence, a branch of computing, *reification* is defined as the process of turning a predicate or function into an object of the language.\(^{11}\) As it is

\(^{10}\) ZANKL, WOLFGANG, Bürgerliches Recht, 2008, Facultas Verlag, Wien, p. 269.

\(^{11}\) RUSSELL/STUART write: “The term “reification” comes from the Latin word *res*, or thing. ...There are two choices for representing categories in first-order logic: predicates and objects. That is, we can use the predicate *Basketball*(\(b\)), or we can *reify* the category as an object, *Basketballs*. 

shown above, a relation implies a predicate and *vice versa*. Hence, a relation can be processed analogous to an object. This complies with Arthur Kaufmann’s assertion that a relation can be processed analogous to a substance.

C. REPRESENTING SITUATIONS GRAPHICALLY

Representing a situation may involve a set of objects and relations and can also be represented as a graph. An example in Figure 3 illustrates a situation, which explains the right of succession in inheritance law. Parent–child is a relation called *ancestor*. This is an elementary, directed relation. Transitive closure of ancestor constitutes the descendent relationship. Descendants are children, grandchildren, etc.

![Figure 4](image)

*Figure 4. A hint on elements to be included in a notation for situations*

**Notation for Situations.** A precise notation to represent situations is a separate theme which extends beyond the scope of this paper. We only note what elements are important. A situation appears on the horizontal Is stage and is described by the following entities (Figure 4):

We could then say *Member*(b, Basketballs) (which we will abbreviate as *b*∈Basketballs) to say that *b* is a member of the category of basketballs. We say *Subset*(Basketballs, Balls) (abbreviated as *Basketballs* ⊆ Balls) to say that *Basketballs* is a subcategory, or subset, of *Balls*. So you can think of a category as being the set of its members, or you can think of it as more complex object that just happens to have the *Member* and *Subset* relations defined for it." RUSSELL, STUART/NORVIG, PETER, Artificial Intelligence: A Modern Approach, 2003, Prentice Hall, pp. 323, 341.
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- **Situational elements.** These elements are the constituents of the situation and exist in time and space. They are denoted by small letters, e.g., \( a, b, c, \) driver, pedestrian, etc.

- **Relations.** These are the relations between the situational elements. There are many kinds of relations: causal (\( c \rightarrow \)), teleological (\( te \rightarrow \)), instrumental (\( instr \rightarrow \)), contextual (\( contx \rightarrow \)), etc. These relations are comprised by both legal relations, such as debt, but also by empirical non-legal relations. The relations represent different perspectives.

In a situation, the alternatives of behaviour are of essential importance. Suppose you do not have money and, therefore, you cannot take a tram; but if you have you can take a transport. A predicate language can serve to represent situations; cf. block worlds in early artificial intelligence. Situative elements are represented by constants (0-arity predicates) and relations by predicates \( P(a_1, \ldots, a_n) \).

D. **RELATIONS IN LAW AND THE LAYERS OF LAW**

We use a visualization pattern that is composed of the vertical stage and the horizontal one (Figure 5). They depict Kelsen’s categorical distinction between Is and Ought.

![Figure 5. The horizontal and vertical stages correspond to Is and Ought, respectively](image)

We decompose the Is realm into two layers: everyday life and subcultures (Figure 6). The Ought consists of other layers, such as a constitution, general norms and individual norms. Other elements form meta-layers, such as legal theory, meta-theories (e.g., Kelsen’s Pure Theory of Law) and a system of legal terms.
We differentiate between the following types of relations:

1. **Horizontal relations.** A relation connects elements of one layer.
2. **Vertical next layer relations.** A relation connects elements of different layers.
3. **Vertical legal dogmatics relations.** A relation connects elements from the legal dogmatics meta-level layer with elements from Is and Ought layers.
4. **Vertical legal theory relations.** A relation connects elements from the legal theory meta-level layer with elements of the individual norms layer and also with elements of the Is layers.

![Diagram](image)

**Figure 6. The layers of law and relations: (1) horizontal relations, (2) vertical next layer relations, (3) vertical legal dogmatics relations, and (4) vertical legal theory relations**

Vertically drawn relations include the subsumption relation, which is important in law and is worth a separate discussion. The factual–normative subsumption relation can be viewed as matching a fact $a$ to a normative circumstance $A$. This formalization leads to a congruence relation – a token’s congruence with the type. Another
formalization is instance–class *instance_of* relationship in computer science. The primary semantics of *instance_of A* is *a*∈*A*, i.e. the element–set membership relation “*a* is an element of *A*.”

Ideology is also a meta-level layer. Therefore, the whole scheme, which is shown in Figure 6, appears in ideological context. Hence, relations with ideological concepts are involved (Figure 7).

![Ideological context diagram](image)

*Figure 7. The context of relations with ideological concepts*
E. TERTIUM COMPARATIONIS IN LAW: VARIATIONS ON ARTHUR KAUFMANN’S THEME

Arthur Kaufmann replaces ontologies of substances by ontologies of relations. There is a distinction between causality relations \( A \) causal \( B \) and finality relations \( A \) telos \( B \), as they have different legal meanings. Similarly, the notions of correspondence (Entsprechung) and contradiction (Widerspruch) have also different meaning. Next, we differentiate between explicit and implicit relations and also between direct and indirect relations. This section is inspired by Lachmayer’s variations on ontology of legal relations by Arthur Kaufmann.

An Indirect Relation. Tertium comparisonis (Latin – the third [part] of the comparison) is the quality that two things that are being compared have in common. “The third of comparison denotes a point of commonality without which no comparison seems possible”; see Figure 8.

![Figure 8. An indirect relation between A and B through tertium comparisonis, the common property](image)


However, a course through *tertium comparationis* modifies the relation. With *tertium comparationis*, one deals not with a direct relation between two elements, but, rather, with an indirect relation between them that is mediated over a third element. In order to explain the nature of tertium comparationis, we provide the following example. Suppose four apples being brought into relation with four pears. This is about the number, in this case about the number four, which occurs as *tertium comparationis*. It does not compare apples with pears, but four elements with four other elements. This indirect relation is a reflected relation and can also be characterised as a broken relation. A broken relation, a direct one, is replaced by two relations. For instance, a translation from Portugal into Lithuanian would be performed – not directly, but through English. Another example is making two information systems interoperable. Interoperability requires a bridge between the systems.

**Overcoming Barriers with Tertium Comparationis.** The response of the broken or reflected *tertium comparationis* is able through walls and other barriers to make a connection. The situation is similar to a mirror, which can also allow a view into areas that are not accessible to direct view. In this way, one can look not only in the distance of today, but also into the past and into the future. *Tertium* is a perfectly suitable technique to make connections in the unconscious that cannot be drawn directly.

**Tertium Communicationis.** An indirect relation can be made more dynamic and personal. In this way, *tertium comparationis* can be converted into *tertium communicationis* and lead further to *tertium identificationis* and *tertium socialisationis*.

**Projecting a Relation.** Legal relations are generally not simple matters. In most cases, a relation is not like a bridge between two banks because it is not even observable in the outside world. Often, relations are projected and a relation becomes the result of projecting. Here projection is the content of a thought act, a speech act, or a legal act.

**Comparison.** A comparison also concerns relations. Various elements can be compared, and hence, brought into a relation. If a relationship is projected, the elements that are connected in the relation are also projected. Hence, (a) Is can be compared with Ought (i.e. meaning), (b) Ought with Is, and (c) Ought with Ought.
**Interpretation and Comparison.** A classical usage is a relation between the factual and the normative. It is meaningful to examine this relation because it usually appears in judgements, i.e., legal acts. We hold that interpretation precedes comparison. The factual and the normative are compared – not directly, but through their meaning. This is then projected onto the factual and the normative, respectively (Figure 9). In legal language, it is not the case that a fact (which appears in Is) is compared directly with the content of a norm, but the interpreted fact is compared with the meaning of the norm (which appears in Ought). The interpretation (*Deutung*) is a prerequisite. The comparison compares the meaning-structure of the fact with the meaning-structure of the normative hypothesis. Legal terms serve as *tertium comparationis*.

![Diagram](image)

Figure 9. An indirect relation between A and B through a common quality *tertium comparationis*: (a) pattern and (b) explanation

**Pretextual Universals.** A textual culture dominates in law and, therefore, there is little that is pretextual or non-verbal. However, there are also normative approaches that are centrally non-verbal. Examples are the simulated measurement units of the body, such as the radius or the cubit, the foot or the step. Hence, there are archetypes that are non-linguistic and have a social normative effect.

**Subject-Internal Tertium Comparationis.** We spoke above about the abstract structural background that lies behind universal interpretation schemas, such as language, types, and terms, and that thus, lies behind supposed objectivity. However, another course can be followed to facilitate *tertium comparationis*, specifically
through the subject. Universals can also be derived from the subject. There are *
universalia in rem* that are internally in the subject (Figure 10); they differ from *
universalia ante rem* that are in the objective area before the subject and the thing. 
These universals can, but need not be formulated verbally. Such indirect relationships 
can be produced in the subject for a preliminary understanding. Since we hold that 
language is a distinct human competence, the pre-verbal ability may be associated 
with the development stages before humans. A comparison is also possible, to a 
certain extent, and thus, a thought. The big advantage of language is less in the 
standardization in the projected meaning, but, rather, in the inter-subjectivity.

![Diagram](image)

**Figure 10. An indirect relation between A and B following the course universalia in rem**

**Two Poles of *Tertium Comparationis*.** There are, thus, two poles of *tertium 
comparationis* – namely, *universalia ante rem* that are assigned to the objective and 
*universalia post rem* that are attributed subjectively. Although you can find such 
comparison measures in different areas, they are still functionally lifted from the 
things whose conceptual link they make possible.

**Relations and Personality.** Relations are assigned to the meaning level. There are 
many different types of relations, especially in the area of law. If a case is brought into 
a relation with a norm, the projecting onto the relation of correspondence is 
performed. However, it is different with complementary roles. Here there is 
something like a vinculum juris between people. The personal relation of the 
complementary roles of two or more persons is probably what Arthur Kaufmann had 
in mind when he developed his theory of the person.
Substance of Tertium Comparationis. The question "What is the substance of tertium comparationis?" is not trivial. A tertium comparationis, such as the meter or the kilogram (of the International Bureau of Weights and Measures), can be assigned a concrete substance. However, the substance of tertium comparationis can be weakened; think, for example, of merely projected units of measurement. Here, the substance is not as clear as in the case of concrete universalia in rem examples, like the meter or a cardboard/computer model of a house that is built by an architect.

Amplitude Relations. We call an amplitude relation a relation which involves unconsciousness and cognitive emotions. A goal of our investigations is to model indirect relations through tertium comparationis (A indirect B) and amplitude relations (A amplitude B). They both involve a third element (tertium and unconsciousness correspondingly) while relating two elements, A and B. This third element can be represented explicitly: indirect(A, B, tertium) and amplitude(A, B, unconsciousness). We would model this in our dialect of the Chen notation, as shown in Figure 11 (we add line ends •).

Towards an Ontology of Relations. Stressing the ontology of relations is a radical step that is interesting from a linguistic viewpoint. However, the practical consequence of this step has not been sufficiently considered. Is it, in fact, the case that only relations, and not the substances that are associated with them, are real?
Through the elimination of the substances, one falls into a bottomless abyss, and the relations alone are not able to slow down this fall. An attempt to visualise the ontology of relations is shown in Figure 12. This ontology can be treated as a classification of relations, which are grouped according to ‘Is’–‘Is’, ‘Is’–‘Ought’ and ‘Ought’–‘Ought’ combinations. The proposed concept of the ontology of relations is at a very abstract level, and does not conform entirely to the treatment of ontologies in computer science.\textsuperscript{15}

![Figure 12. Towards an ontology of relations in law](image)

**F. STATEMENTS ABOUT RELATIONS**

There is a difference between a relation $A R B$ and a statement that $A$ and $B$ are in a relation $R$. Therefore, we introduce a predicate $\text{STM}(A, R, B, \text{statement}, \text{teller})$ or $\text{STM}[A R B]$.

This serves to express that a teller says: ‘$A$ and $B$ are in a certain relation $R’$. There is a distinction between two different categories: speech acts and statements. A speech act appears in Is, but its meaning appears in Ought. In contrast, statements appear as mathematical objects, such as propositions and predicates in the world of science. There is an essential difference if a speech act is pronounced by a judge or a layman.

\textsuperscript{15} GUARINO/OBERLE/STAAB, What is an Ontology?, Note 9.
In the first case, the speech act constitutes a legal act and, in the second case, does not. A speech act can be treated as an institutional fact and, hence, lead to a legal act whose meaning appears in Ought, for example, as a norm. This is shown in Figure 13.

![Diagram of Norm, Statement, and Relation]

Figure 13. A speech act, statement and a relation

G. THE THEME OF RELATIONS IN JURIX PROCEEDINGS

The mainstream of formalizations of legal relations can be determined with reference to JURIX, the Dutch Foundation for Legal Knowledge-Based Systems and its annual conference proceedings. It is creative at JURIX to give a name to a relationship $A R B$. In other words, when writing $A <nnn> B$, it is explicitly explained what $nnn$ is. Relation type, such as right, duty, etc., can be written inside.

As an example, let us take the paper by Pace/Schapachnik. They claim that “[d]eontic modalities, such as permission and obligation, have been debated exhaustively” and “[t]his is not the case with more intricate concepts, such as

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16 http://jurix.nl/proceedings/.

Hohfeld’s *claim right, power, freedom and immunity.*\(^{18}\) Pace/Schapachnik\(^{19}\) recall Kanger/Kanger\(^{20}\) who identify eight simple types of rights:

(a) Party \(p\) has versus party \(p'\) a *claim* that \(S(p, p')\).
(b) Party \(p\) has versus party \(p'\) a *freedom* that \(S(p, p')\).
(c) Party \(p\) has versus party \(p'\) a *power* that \(S(p, p')\).
(d) Party \(p\) has versus party \(p'\) an *immunity* that \(S(p, p')\).

(a’) Party \(p\) has versus party \(p'\) a *counter-claim* that \(S(p, p')\).
(b’) Party \(p\) has versus party \(p'\) a *counter-freedom* that \(S(p, p')\).
(c’) Party \(p\) has versus party \(p'\) a *counter-power* that \(S(p, p')\).
(d’) Party \(p\) has versus party \(p'\) a *counter-immunity* that \(S(p, p')\).

We propose the following notation:

(a) \(p <\text{claim~}S\rangle \rightarrow p'\)
(b) \(p <\text{freedom~}S\rangle \rightarrow p'\)
(c) \(p <\text{power~}S\rangle \rightarrow p'\)
(d) \(p <\text{immunity~}S\rangle \rightarrow p'\)

(a’) \(p <\text{claim not-S}\rangle \rightarrow p'\)
(b’) \(p <\text{freedom not-S}\rangle \rightarrow p'\)
(c’) \(p <\text{power not-S}\rangle \rightarrow p'\)
(d’) \(p <\text{immunity not-S}\rangle \rightarrow p'\)

*H. ON PETER CHEN’S ENTITY–RELATIONSHIP MODEL*

We hold that a relation can be managed analogously as atomic entities. An entity–relationship model (ER model) refers to the techniques proposed in Peter Chen’s paper (1976).\(^{21}\) However, variants of the idea existed previously. Chen holds that “The

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\(^{19}\) Ibid., p. 109.


entity-relationship model adopts the more natural view that the real world consists of entities and relationships. It incorporates some of the important semantic information about the real world."

Chen’s notation for entity–relationship modelling uses rectangles to represent entity sets, and diamonds to represent relationships appropriate for first-class objects: they can have attributes and relationships of their own. If an entity set participates in a relationship set, they are connected with a line. There are related diagramming convention techniques, such as UML class diagrams. When speaking about the problem of visual dialects, Daniel Moody notes that ER modelling exists in a variety of dialects, such as the Chen notation, the most commonly used in academic context; the Information Engineering (IE) notation, the most commonly used in practice; the Bachman notation, IDEFiX notation, etc.

I. CONCLUSIONS

Relations in law is a theme that extends beyond concrete relations which are examined in legal theory. Investigations within this theme can extend to ontological theory of law. The goal of this paper is to identify cornerstones of investigations. The concepts of indirect and amplitude relations are introduced.

We tackle the concept of tertium communicationis. We aim to use tertium communicationis as a conceptual definition that improves communication between human beings or machines. A classification of legal relations based on ‘Is’–‘Ought’ combinations is provided.

...Diagrams created to design these entities and relationships are called entity–relationship diagrams; see https://en.wikipedia.org/wiki/Entity%E2%80%93relationship_model.
