ON MAHLER’S INEQUALITY FOR THE SUM OF POLYNOMIALS

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ABSTRACT. Suppose that \( m \geq 2 \) is a positive integer. We prove upper and lower bounds on the best possible constant \( \lambda_{m,n} \) for which the inequality \( M(f_1 + \cdots + f_m) \leq \lambda_{m,n}^{-1}(M(f_1) + \cdots + M(f_m)) \) holds for any polynomials \( f_1, \ldots, f_m \in \mathbb{C}[z] \) of degree at most \( n \) each. In particular, for \( n \geq 2m^2 \) and \( m \to \infty \), we prove that \( 2 - \pi^2/(12m^2) + o(m^{-2}) \leq \lambda_{m,n} \leq 2 - \pi/(2cm^2) + o(m^{-2}) \). For \( m = 5 \) and \( n \geq 43 \), our result yields the bounds \( 1.967 < \lambda_{5,n} < 1.977 \).

1. INTRODUCTION

Recall that, for any polynomial \( f(z) = a(z-z_1) \cdots (z-z_n) \in \mathbb{C}[z] \), its Mahler’s measure is defined as \( M(f) = |a| \prod_{j=1}^{n} \max \{1, |z_j| \} \). The Mahler measure of an algebraic number which is, by definition, the Mahler measure of its minimal (irreducible) integer polynomial is one of the subjects studied in number theory. It is one of the heights of an algebraic number and has applications in certain diophantine problems. By Jensen’s formula, Mahler’s measure may be also written as

\[
M(f) = \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(e^{it})| \, dt \right).
\]

This shows that the Mahler measure of \( f \) can be obtained as a limit of its \( L^p \) norm

\[
||f||_p = \left( \frac{1}{2\pi} \int_{0}^{2\pi} |f(e^{it})|^p \, dt \right)^{1/p},
\]

namely, \( M(f) = \lim_{p \to 0^+} ||f||_p \). This explains the notation \( M(f) = ||f||_0 \) which is sometimes more familiar to those working in analysis rather than in number theory. At the other end of the interval \((0, \infty)\), namely, as \( p \to \infty \), we have \( ||f||_{\infty} = \max_{|z|=1} |f(z)| \). Below, we shall use the fact that \( p \mapsto ||f||_p \) is a nondecreasing function in \( p \).

Let \( m \geq 2 \) be a fixed positive integer, and let \( f_1(z), \ldots, f_m(z) \) be polynomials with complex coefficients, of degree at most \( n \) each. With the above notation, using \( ||f+g||_{\infty} \leq ||f||_{\infty} + ||g||_{\infty} \), we get

\[
M(f_1 + \cdots + f_m) \leq ||f_1 + \cdots + f_m||_{\infty} \leq ||f_1||_{\infty} + \cdots + ||f_m||_{\infty}.
\]

Thus, for each \( j = 1, \ldots, m \), using \( ||f_j||_{\infty} \leq L(f_j) \leq 2^n M(f_j) \), where \( L(f_j) \) is the length of \( f_j \) (see, e.g., [11]), we deduce that

\[
M(f_1 + \cdots + f_m) \leq 2^n (M(f_1) + \cdots + M(f_m)).
\]

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In Section 2, by a standard method, we shall prove that

\[ M(f_1 + \cdots + f_m) \leq \frac{2^n \Gamma((n + 1)/2)}{\sqrt{\pi} \Gamma((n + 2)/2)} (M(f_1) + \cdots + M(f_m)). \]  

(1)

The constant \( v(n) = 2^n \Gamma((n + 1)/2)/\sqrt{\pi} \Gamma((n + 2)/2) \) is slightly smaller than \( (2^n)_n^{1/2} \). Observe that the inequality for the difference of two Mahler measures \( |M(f_1)^{1/n} - M(f_2)^{1/n}| \leq 2L(f_1 - f_2)^{1/n} \), where \( f_1, f_2 \) are complex polynomials of degree \( \leq n \), was recently obtained by Chern and Vaaler [7].

However, when \( m \) is fixed and \( n \) is large, for an improvement of the constant \( v(n) \) in (1), it is more convenient to search for the best constant \( c \) in \( M(f_1 + \cdots + f_m) \leq c^n m^{-1}(M(f_1) + \cdots + M(f_m)) \). We thus define \( \lambda_{m,n} \) as the infimum over all positive numbers \( c \) for which the inequality \( M(f_1 + \cdots + f_m) \leq c^n m^{-1}(M(f_1) + \cdots + M(f_m)) \) holds for any complex polynomials \( f_1, \ldots, f_m \) of degree at most \( n \) each, so that

\[ M(f_1 + \cdots + f_m) \leq \lambda_{m,n}^n m^{-1}(M(f_1) + \cdots + M(f_m)) \]

with the smallest possible positive number \( \lambda_{m,n} \).

Set

\[ c_m = \exp \left( \frac{m}{\pi} \int_0^{\pi/m} \log(2 \cos(t/2)) dt \right) \]  

(2)

and

\[ b_m = 2 \min_{y \in (0, \infty)} \left( \frac{m \Gamma(y + 1/2)}{\sqrt{\pi} \Gamma(y + 1)} \right)^{1/(2y)}. \]  

(3)

Let also \( y_m \) be the point, where the minimum of the function

\[ B_m(y) = 2 \left( \frac{m \Gamma(y + 1/2)}{\sqrt{\pi} \Gamma(y + 1)} \right)^{1/(2y)} \]

(4)
in \( (0, \infty) \) is attained, so that \( b_m = B_m(y_m) \). (We have \( b_m < 2 \), because \( B_m(y) \to \infty \) as \( y \to 0^+ \) and \( B_m(y) \to 2 \) as \( y \to \infty \), where \( B_m(y) \) is increasing for \( y \) large enough.)

In this note, we prove the following bounds on \( \lambda_{m,n} \).

**Theorem 1.** For each \( m \geq 2 \) and \( n \geq 2y_m + 1 \), we have

\[ M(f_1 + \cdots + f_m) \leq b_m^n m^{-1}(M(f_1) + \cdots + M(f_m)), \]

where \( f_1, \ldots, f_m \) are complex polynomials of degree at most \( n \) each, so that \( \lambda_{m,n} \leq b_m \).

Moreover, there exist \( f_1, \ldots, f_m \in \mathbb{C}[z] \) of even degree \( n \geq 2 \) each such that

\[ M(f_1 + \cdots + f_m) \geq c_m^n m^{-1}(M(f_1) + \cdots + M(f_m)), \]

so that \( \lambda_{m,n} \geq c_m \). (Here, \( c_m \) and \( b_m \) are given in (2) and (3).)
For $m \leq 10$, the values of $c_m$, $b_m$ and $2y_m$ truncated at the fifth decimal place are given in the following table.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$c_m$</th>
<th>$b_m$</th>
<th>$2y_m$</th>
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<td>1.84926</td>
<td>5.84239</td>
</tr>
<tr>
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<tr>
<td>10</td>
<td>1.99177</td>
<td>1.99421</td>
<td>172.04829</td>
</tr>
</tbody>
</table>

The gap between $c_m$ and $b_m$ tends to zero, because both $c_m$ and $b_m$ tend to 2 as $m \to \infty$. For $m \to \infty$, it is known [6] that

$$c_m = 2 - \pi^2/(2m^2) + o(m^{-2}).$$

In Section 4, we will show that

$$b_m = 2 - \pi/(2em^2) + o(m^{-2})$$

for $m \to \infty$. The numerical values $\pi^2/12 = 0.822467033\ldots$ and $\pi/(2e) = 0.577863675\ldots$ show the size of a gap between these two asymptotical bounds.

In the proof of the upper bound $\lambda_{m,n} \leq b_m$ we shall use the method of Arestov [3]. For $m = 2$, a small improvement from $\lambda_{2,n} \leq 40^{1/6} = 1.84931\ldots$ (see [3]) to $\lambda_{2,n} \leq b_2 = 1.84926\ldots$ comes simply by setting $y = y_2 = 2^{-1} \cdot 5.84239\ldots$ from the table into $B_m(y)$ rather than $y = 3$ as in [3]. The proof of the upper bound for $M(f_1 + \cdots + f_m)$ with the required constant $b_m^{n/m-1}$ will be given in the next section. The example producing the constant $c_m^{n/m-1}$ which will complete the proof of the theorem will be given in Section 3.

We conjecture that the constant $c_m$ is (asymptotically) best possible, namely, $\lambda_{m,n} \to c_m$ as $n \to \infty$.

The constants $c_m$ given in (2) have already appeared in the literature on several occasions. It is known that

$$c_2 = e^{2G/\pi} = M(1 + z_1 + z_2 - z_1z_2),$$

where $G = 1 - 1/3^2 + 1/5^2 - 1/7^2 + \ldots$ is Catalan’s constant and $M(1 + z_1 + z_2 - z_1z_2)$ is the Mahler measure of the polynomial in two variables $1 + z_1 + z_2 - z_1z_2$ (see, e.g., [5]). There is a relation between $c_m$ given in (2) and Clausen’s integral $Cl_2(\theta) = -\int_0^\theta \log(2\sin(t/2))dt$ (see, e.g., pp. 1005-1006 in [1])

$$\frac{\pi}{m} \log c_m = -\int_0^{(m-1)\pi/m} \log(2\sin(t/2))dt = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\sin(k\pi/m)}{k^2}. $$
In [6], Boyd showed that, for each \( m \geq 2 \), the number \( c_m \) is the best possible constant (as \( n \to \infty \)) for which the inequality

\[
\| f_1 \|_\infty \cdots \| f_m \|_\infty \leq c_m^n \| f_1 \cdots f_m \|_\infty
\]

holds for complex polynomials \( f_1, \ldots, f_m \) of degree at most \( n \). See also [4], where a similar inequality was obtained for polynomials in several variables.

2. Upper bound

We shall use the following inequality for two norms (\( L^p \) norm versus \( L^0 \) norm) of a polynomial \( f \) of degree \( \leq n \) due to Arestov [2], [3]

\[
\| f \|_p \leq N(p, n) M(f),
\]

where

\[
N(p, n) = \|(z + 1)^n\|_p = 2^n \left( \frac{\Gamma((np + 1)/2)}{\sqrt{\pi} \Gamma((np + 2)/2)} \right)^{1/p}.
\]

Taking \( p = 1 \) and using (5) and (6), we obtain

\[
M(f_1 + \cdots + f_m) \leq \| f_1 + \cdots + f_m \|_1 \leq \| f_1 \|_1 + \cdots + \| f_m \|_1
\]

\[
\leq N(1, n)(M(f_1) + \cdots + M(f_m)) = \frac{2^n \Gamma((n + 1)/2)}{\sqrt{\pi} \Gamma((n + 2)/2)}(M(f_1) + \cdots + M(f_m)),
\]

which proves (1).

To deduce an upper bound for \( M(f_1 + \cdots + f_m) \), let us start with the observation that this quantity is less than or equal to \( \| f_1 + \cdots + f_m \|_p \), where \( p = 2y_m/n \). For \( n \geq 2y_m \), we have \( p \in (0, 1] \). Thus, using \( |a + b|^p \leq |a|^p + |b|^p \), which holds for any real numbers \( a \) and \( b \), we derive that

\[
\| f_1 + \cdots + f_m \|_p \leq (\| f_1 \|^p_1 + \cdots + \| f_m \|^p_m)^{1/p}.
\]

Next, the inequality

\[
\left( \frac{a_1 + \cdots + a_m}{m} \right)^{1/p} \leq \frac{a_1^{1/p} + \cdots + a_m^{1/p}}{m},
\]

which holds for any non-negative numbers \( a_1, \ldots, a_m \), and \( p \in (0, 1] \), implies that \( (\| f_1 \|^p_1 + \cdots + \| f_m \|^p_m)^{1/p} \leq m^{1/p-1}(\| f_1 \|_p + \cdots + \| f_m \|_p) \). By Arestov’s inequality (5), this is further bounded from above by \( m^{1/p-1}N(p, n)(M(f_1) + \cdots + M(f_m)) \). Finally, from (6),

\[
m^{1/p-1}N(p, n) = n^{(2y_m)/2 - 1}N(2y_m/n, n) = 2^n m^{n/(2y_m) - 1} \left( \frac{\Gamma(2y_m + 1/2)}{\sqrt{\pi} \Gamma(2y_m + 1)} \right)^{n/(2y_m)}
\]

which is equal to \( b_m^n m^{-1} \), by (4) and (3). Hence

\[
M(f_1 + \cdots + f_m) \leq b_m^n m^{-1}(M(f_1) + \cdots + M(f_m)),
\]

as claimed.
3. Lower bound

For the lower bound, let us consider the following example. Set \( \mu_j = e^{2\pi i j/m} \) and, for each \( j = 1, \ldots, m \), put
\[
f_j(z) = e^{-\pi i (j-1)n/m}(z + \mu_{j-1})^n.
\]
Clearly, \( M(f_j) = 1 \) for each \( j = 1, \ldots, m \), so \( M(f_1) + \cdots + M(f_m) = m \).

It remains to evaluate the Mahler measure of the polynomial
\[
f(z) = f_1(z) + \cdots + f_m(z) = (z + 1)^n + \sum_{j=2}^{m} e^{-\pi i (j-1)n/m}(z + \mu_{j-1})^n.
\]
Setting \( z = e^{it} \), we obtain \( f_j(e^{it}) = e^{itn/2}(2 \cos(t/2 - \pi (j-1)/m))^n \), thus
\[
f(e^{it}) = e^{itn/2}((2 \cos(t/2))^n + \sum_{j=1}^{m-1} (2 \cos(t/2 - \pi j/m))^n). \tag{7}
\]
Recall that \( n \) is even. Then the function
\[
|f(e^{it})| = 2^n (| \cos(t/2)|^n + | \cos(t/2 - \pi/m)|^n + \cdots + | \cos(t/2 - \pi (m-1)/m)|^n) \tag{8}
\]
is periodic with period \( 2\pi/m \). It follows that
\[
\log M(f) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})|dt = \frac{1}{2\pi} \int_{-\pi/m}^{\pi/m} \log |f(e^{it})|dt.
\]
Clearly, (8) yields the inequality \( |f(e^{it})| \geq |2 \cos(t/2)|^n \). Thus, using (2), we obtain
\[
\log M(f) \geq \frac{m}{2\pi} \int_{-\pi/m}^{\pi/m} \log |2 \cos(t/2)|^n dt = \frac{mn}{2\pi} \int_{-\pi/m}^{\pi/m} \log |2 \cos(t/2)| dt
\]
\[
= \frac{mn}{\pi} \int_{0}^{\pi/m} \log(2 \cos(t/2))dt = n \log c_m.
\]
Hence \( M(f) \geq c_m^n \). From this, we conclude that
\[
M(f_1 + \cdots + f_m)/(M(f_1) + \cdots + M(f_m)) = M(f)m^{-1} \geq c_m^n m^{-1},
\]
as claimed. This proves the theorem.

It is easy to see that, by (7), for any (even or odd) \( n \), the same example with \( f = f_1 + \cdots + f_m \) gives \( \log M(f) = n \log c_m + o(n) \) as \( n \to \infty \). So, independently of the parity of \( n \), we deduce that \( (M(f_1) + \cdots + M(f_m))/(M(f_1) + \cdots + M(f_m))^{1/n} \to c_m \) as \( n \to \infty \).

4. Evaluation of the minimum

We shall use Stirling’s formula for the gamma function
\[
\log \Gamma(y) = y \log y - y + \frac{1}{2} \log(2\pi/y) + \varepsilon(y), \tag{9}
\]
where \( \varepsilon(y) \to 0 \) as \( y \to \infty \).
It follows from (9) that
\[
\log \Gamma(y + 1/2) - \log \Gamma(y + 1) = 1/2 + y \log(y + 1/2) - (y + 1/2) \log(y + 1) + \varepsilon_1(y),
\]
where \(\varepsilon_1(y) \to 0\) as \(y \to \infty\). Hence, by (4), we find that
\[
\log B_m(y) = \log 2 + (1/2y)\left(\log(m\Gamma(y + 1/2)) - \log(\sqrt{\pi} \Gamma(y + 1))\right)
= \log 2 + (1/4y)(\log(em^2/\pi) + 2y \log(y + 1/2) - (2y + 1) \log(y + 1) + 2\varepsilon_1(y)).
\]
The derivative of the function \(y^{-1}(\log(em^2/\pi) + 2y \log(y + 1/2) - (2y + 1) \log(y + 1))\) is equal to \(y^{-2}(\log((y + 1)\pi/em^2) - y/(y + 1)(2y + 1))\). Hence, for \(m\) large enough, \(B_m(y)\) attains its minimum when \(y + 1\) is “approximately” equal to \(em^2/\pi\). More precisely, \(y_m = em^2/\pi + o(m^2)\) for \(m \to \infty\). So in evaluating \(b_m = B_m(y_m)\) as \(m \to \infty\), we can simply select, for instance, \(y = em^2/\pi\). Putting this value into \(\log B_m(y)\), we find easily that
\[
\log B_m(em^2/\pi) = \log 2 - \pi/(4em^2) + o(m^{-2}).
\]
Hence, for \(m \to \infty\), we have \(\log b_m = \log B_m(y_m) = \log 2 - \pi/(4em^2) + o(m^{-2})\). It follows that \(b_m = 2 - \pi/(2em^2) + o(m^{-2})\), as claimed.

It is clear that the inequality \(n > 2y_m\) holds in the case when \(m\) is large enough and \(n > 2m^2 > 2em^2/\pi\), so the theorem can be applied for \(n > 2m^2\) and \(m\) large enough. This explains the condition \(n > 2m^2\) imposed on \(n\) in the abstract.

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References


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