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Prime and composite integers close to powers of a number

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Abstract We prove that, for any real numbers $\xi \neq 0$ and ν, the sequence of integer parts $[\xi 2^n + \nu]$, $n = 0, 1, 2, \dots$, contains infinitely many composite numbers. Moreover, if the number ξ is irrational, then the above sequence contains infinitely many elements divisible by 2 or 3. The same holds for the sequence $[\xi(-2)^n + \nu_n]$, $n = 0, 1, 2, \dots$, where $\nu_0, \nu_1, \nu_2, \dots$ all lie in a half open real interval of length $1/3$. For this, we show that if a sequence of integers $x_1, x_2, x_3, \dots$ satisfies the recurrence relation $x_{n+d} = cx_n + F(x_{n+1}, \dots, x_{n+d-1})$ for each $n \geqslant 1$, where $c \neq 0$ is an integer, $F(z_1, \dots, z_{d-1}) \in \mathbb{Z}[z_1, \dots, z_{d-1}]$, and $\lim_{n\to\infty} |x_n| = \infty$, then the number $|x_n|$ is composite for infinitely many positive integers n. The proofs involve techniques from number theory, linear algebra, combinatorics on words and some kind of symbolic computation modulo 3.

Keywords Prime numbers · Integer part · Power sequence · Sturmian sequence

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1 Introduction

Let throughout $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, real numbers, rational numbers, integers and positive integers, respectively. Let also $[x]$ and $\{x\}$ stand for the integral part and the fractional part of $x \in \mathbb{R}$, respectively.

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One of the classical number theoretic problems is to determine whether a sequence of positive integers contains infinitely many prime numbers or not. By Dirichlet's theorem, an arithmetic progression $an + b$, $n = 0, 1, 2, \dots$, where $a \in \mathbb{N}$, $b \in \mathbb{Z}$ and $\gcd(a, b) = 1$, contains infinitely many primes. However, similar questions concerning the sequences $n^2 + 1$ and $2^{2^n} + 1$, where $n = 0, 1, 2, \dots$, are far from being solved. In some cases, especially for a given sparse sequence, it is not even clear whether this sequence contains infinitely many composite numbers or not. For example, it is not known whether the sequence of Fermat numbers $2^{2^n} + 1$, where $n = 0, 1, 2, \dots$, contains infinitely composite numbers or not.

In 1947, Mills [14] proved that there is a constant $\alpha > 1$ such that all the integer parts $[\alpha^{3^n}]$, $n = 1, 2, 3, \dots$, are prime numbers. In fact, for any sequence of positive integers q_n satisfying $q_n > 2.1053 q_{n-1}$ for each integer $n \geqslant 2$, there exists an $\alpha > 1$ such that the numbers $[\alpha^{q_n}]$, $n = 1, 2, 3, \dots$, are all prime (see [2]). These results are applications of the estimates for the gaps between consecutive primes. The latter result is derived from [4]. Is there a more slowly, say, geometrically increasing prime producing function of integer parts?

Problem 1 *Is there a real number $\alpha > 1$ and an integer $n(\alpha)$ such that the integer part $[\alpha^n]$ is a prime number for each integer $n \geqslant n(\alpha)$?*

More generally, we ask the following:

Problem 2 *Do there exist three real numbers $\xi > 0$, $\alpha > 1$, $r > 0$ and an integer $n_0 = n_0(\xi, \alpha, r)$ such that, for each integer $n \geqslant n_0$, the interval $[\xi\alpha^n - r, \xi\alpha^n + r]$ contains a prime number?*

The second question gives more room for a possible prime to occur. However, it seems likely that, in both cases, the answer is "no." See Sect. 7, where we prove this for $\xi = 1$ and integer α. The solution of the first (easier) problem seems to be far beyond reach. Nevertheless, there is at least something done in this direction for some algebraic values α. Since a negative answer to Problem 1 is expected, one needs to prove that, for any $\alpha > 1$, the sequence $[\alpha^n]$, $n = 1, 2, 3, \dots$, contains infinitely many composite numbers. This is the case if $\alpha > 1$ is an integer, which immediately implies the negative answer for each number of the form $\alpha = a^{1/m}$, where $a \geqslant 2$ and $m \geqslant 1$ are integers.

The numbers $\alpha = 3/2$ and $\alpha = 4/3$ have been treated by Forman and Shapiro [10] (see the problem E19 on p. 220 in [11]). Both sequences $[\xi(3/2)^n]$, $n = 1, 2, 3, \dots$, and $[\xi(4/3)^n]$, $n = 1, 2, 3, \dots$, contain infinitely many composite numbers [9,10]. The fact that the sequence $[\alpha^n]$, $n = 1, 2, 3, \dots$, contains infinitely many composite numbers was also proved for some special algebraic integers α (see [6–8]). There is some intersection between some of the results in these papers and the results of the paper of Shapiro and Sparer [16] which was overlooked.

The case of $\alpha = 5/4$ turned out to be much more difficult [9]. In fact, it was proved in [9] that, for any $\xi > 0$, the sequence $[\xi(5/4)^n]$, $n = 1, 2, 3, \dots$, contains infinitely many numbers divisible by 2, 3, 7, 11 or 13. The next theorem solves a problem posed in [9]:

Theorem 3 *Let $\xi \neq 0$ and ν be real numbers. Then each of the sequences $[\xi 2^n + \nu]$, $n = 0, 1, 2, \dots$, and $[\xi(-2)^n + \nu]$, $n = 0, 1, 2, \dots$, contains infinitely many composite elements. Moreover, if the number ξ is irrational, then each of the sequences contains infinitely many elements divisible by 2 or 3.*

A negative integer a is said to be composite if $|a|$ is composite. For any finite set of primes $\mathcal{P}$, there are $\xi \in \mathbb{Q}$ and $\nu \in \mathbb{R}$ for which $[\xi 2^n + \nu]$ (and $[\xi(-2)^n + \nu]$) is not divisible by any $p \in \mathcal{P}$, so one cannot say more than the first part of Theorem 3 asserts. Indeed, if $\mathcal{P} = \{2\}$, we can simply take $\xi = \nu = 1$. If $\mathcal{P}$ contains at least one odd prime, setting $\mathcal{P} \setminus \{2\} = \{p_1, p_2, \dots, p_k\}$ and taking $\xi = 1$ and $\nu = p_1 p_2 \dots p_k$, we deduce that each $[2^n + p_1 p_2 \dots p_k] = 2^n + p_1 p_2 \dots p_k$, where $n \geqslant 0$, is not divisible by 2 or any odd $p \in \mathcal{P}$.

The most difficult part in the proof of Theorem 3 is to prove that, for $\xi \notin \mathbb{Q}$, the sequence $[\xi 2^n + \nu]$, $n = 0, 1, 2, \dots$, contains infinitely many elements divisible by 2 or 3. For this, we shall use the notion of Sturmian words (see Sect. 4) and some kind of symbolic computation modulo 3. The proof for the sequence $[\xi(-2)^n + \nu]$, $n = 0, 1, 2, \dots$, is simpler. In fact, in this case we shall prove a more general theorem (see Theorem 9 in Sect. 5).

The proof of Theorem 3 for $\xi \in \mathbb{Q}$ is quite simple. We shall derive the required result from the next theorem which is of independent interest.

Theorem 4 *Let d be a positive integer, and let $F(z_1, \dots, z_{d-1}) \in \mathbb{Z}[z_1, \dots, z_{d-1}]$. Suppose that $c \neq 0$ is an integer, and $x_1, x_2, x_3, \dots$ is a sequence of integers satisfying*

$$x_{n+d} = c x_n + F(x_{n+1}, \dots, x_{n+d-1}) \tag{1}$$

for each $n \in \mathbb{N}$. Then, for any $q \in \mathbb{N}$ satisfying $\gcd(c, q) = 1$, the sequence $x_1, x_2, x_3, \dots$ is purely periodic modulo q. In particular, if $x_1, x_2, x_3, \dots$ is a sequence of integers satisfying (1) and $\lim_{n\to\infty} |x_n| = \infty$, then it contains infinitely many composite numbers.

Recall that the sequence $u_1, u_2, u_3, \dots$ is called *ultimately periodic* if there is a $t \in \mathbb{N}$ such that $u_{n+t} = u_n$ for each $n \geqslant n_0$. If $n_0 = 1$, then the sequence $u_1, u_2, u_3, \dots$ is called *purely periodic*.

One should mention that a version of Theorem 3 for $[\xi 2^n + \nu]$ was obtained by Gerry Myerson in an unpublished manuscript, entitled "An arithmetic property of certain rational powers." His proof is different from ours and involves the reduction of $[\xi 2^n + \nu]$ modulo 2, 3 and 5 rather than just 2 and 3.

We remark that the problems for integer parts $[\xi \alpha^n + \nu]$ are related to similar problems for fractional parts $\{\xi \alpha^n + \nu\}$. For instance, an (unsolved) Mahler's question concerning the fractional parts $\{\xi(3/2)^n\}$, $n = 0, 1, 2, \dots$, [13] is equivalent to the question on whether, for each $\xi \neq 0$, the sequence $[\xi(3/2)^n]$, $n = 0, 1, 2, \dots$, contains infinitely many even and infinitely many odd numbers. See, e.g., [8] for some recent work related to the distribution of the fractional parts of power sequences and [1] for some applications of this kind of problems to so-called rational base number systems.

In the next section we give a proof of Theorem 4. Section 3 contains two auxiliary statements. Note that Lemma 5 is a bit more general than we need for Theorem 3. Section 4 describes Sturmian words. In Sect. 5 the proof of Theorem 3 is given for the sequence $[\xi(-2)^n + \nu]$, $n = 0, 1, 2, \dots$. As we already said above, the result given in Theorem 9 is more general, since ν is not necessarily a constant, but a sequence ν_n, $n = 0, 1, 2, \dots$, which depends on n. In Sect. 6 we complete the proof of Theorem 3. Finally, in Sect. 7, Problem 2 is solved for $\xi = 1$ and integer α.

2 On the periodicity of a recurrence sequence

Consider the vectors $v_n = (x_n, x_{n+1}, \dots, x_{n+d-1}) \in \mathbb{Z}^d$, where $n \in \mathbb{N}$. Such vectors take at most q^d distinct values modulo q. So there exist $n, t \in \mathbb{N}$ such that $v_n = v_{n+t}$ modulo q. The recurrence relation (1) implies that $x_{n+d} = x_{n+t+d}$ modulo q, hence $v_{n+1} = v_{n+t+1}$ modulo q. On applying this argument step by step, we deduce that $x_{n+j} = x_{n+t+j}$ for each integer $j \geqslant 0$. This means that the sequence $x_1, x_2, x_3, \dots$ is ultimately periodic modulo q. (Note that in this part the same implication holds if (1) is replaced by a more general recurrence relation $x_{n+d} = F_0(x_n, x_{n+1}, \dots, x_{n+d-1})$, where $F_0(z_0, z_1, \dots, z_{d-1}) \in \mathbb{Z}[z_0, z_1, \dots, z_{d-1}]$.)

Assume that there is an integer $\ell > 1$ such $x_{\ell+j} = x_{\ell+t+j}$ modulo q for each $j \geqslant 0$, but $x_{\ell-1} \neq x_{\ell-1+t}$ modulo q. Expressing $x_{\ell-1+d}$ by (1), then $x_{\ell-1+t+d}$ by (1), and subtracting the second expression from the first, we derive that $c(x_{\ell-1} - x_{\ell-1+t})$ is divisible by q. Since $\gcd(c, q) = 1$, this implies that $x_{\ell-1} = x_{\ell-1+t}$ modulo q, a contradiction. Thus such an $\ell > 1$ does not exist. It follows that $\ell = 1$, so $x_{1+j} = x_{1+t+j}$ modulo q for each integer $j \geqslant 0$. Hence the sequence $x_1, x_2, x_3, \dots$ is purely periodic modulo q.

Since $\lim_{n\to\infty} |x_n| = \infty$, there is a positive integer k_0 such that $|x_k| \geqslant |c| + 1$ for each $k \geqslant k_0$. There is nothing to prove if the numbers x_k, $k = k_0, k_0 + 1, \dots$, are all composite. Suppose at least one of them, say, $q = |x_m|$, where $m \geqslant k_0$, is prime. From $q = |x_m| > |c|$ we obtain $\gcd(c, q) = 1$. Since the sequence $x_m, x_{m+1}, x_{m+2}, \dots$ is purely periodic modulo q, it contains infinitely many elements divisible by q. Hence infinitely many of its elements are composite numbers.

3 Auxiliary results

Lemma 5 *Let $P(z) = a_d(z-\alpha_1)\dots(z-\alpha_d) = a_d z^d + \dots + a_0$ be a separable (not necessarily irreducible) polynomial with rational coefficients, and let $\xi_1, \dots, \xi_d \in \mathbb{C}$. Set $S_n = \sum_{j=1}^{d} \xi_j \alpha_j^n \in \mathbb{Q}$ for each integer $n \geqslant 0$. If the number S_n is rational for some d distinct nonnegative integer values of n, say, for $n = n_1, \dots, n_d$, where $n_j \geqslant 0$, then, for every $j = 1, \dots, d$, we have $\xi_i = G(\alpha_i)$, where $G(z) = g_m z^m + \dots + g_1 z + g_0$ is a polynomial with rational coefficients. Moreover, the number S_n is rational for each integer $n \geqslant 0$.*

Proof There is no loss of generality in assuming that $n_1 < \dots < n_d$. Consider the system of d linear equations

$$\xi_1\alpha_1^{n_j} + \xi_2\alpha_2^{n_j} + \dots + \xi_d\alpha_d^{n_j} = S_{n_j},$$

$j = 1, \dots, d$, in d unknowns $\xi_1, \dots, \xi_d$. The numbers $\alpha_1, \dots, \alpha_d$ are distinct, so $\xi_i = \det(A_i)/\det(A)$, $i = 1, \dots, d$, is the unique solution of the linear system. Here, $A = (\alpha_i^{n_j})_{i,j=1,\dots,d}$ and, for $i = 1, \dots, d$, A_i is the matrix A whose ith column is replaced by the column $(S_{n_1}, \dots, S_{n_d})^t$, where t stands for the transpose. Both $\det(A_i)$ and $\det(A)$ are divisible by the product $W_i = \prod_{1 \leqslant u < v \leqslant d,\ u,v \neq i}(\alpha_v - \alpha_u)$. Let $W = \prod_{1 \leqslant u < v \leqslant d}(\alpha_v - \alpha_u)$ be a corresponding Vandermonde determinant, and let

$$Q_i = (\alpha_i - \alpha_1) \dots (\alpha_i - \alpha_{i-1})(\alpha_{i+1} - \alpha_i) \dots (\alpha_d - \alpha_i).$$

Then $W = W_i Q_i$ for every $i = 1, \dots, d$, and $Q_i = (-1)^{d-i} P'(\alpha_i)/a_d$.

It is easy to see that $\det(A)$ is equal to $W\sigma$, where σ is a symmetric function in $\alpha_1, \dots, \alpha_d$, so $\sigma \in \mathbb{Q}$. Similarly, expanding $\det(A_1)$ by its first column, we deduce that

$$\det(A_1) = W_1 \sum_{j=1}^{d} S_{n_j} \sigma_j,$$

where each σ_j is a symmetric function in $\alpha_2, \dots, \alpha_d$. Hence $\sigma_j = G_j(\alpha_1)$ with some $G_j(z) \in \mathbb{Q}[z]$. It follows that

$$\xi_1 = \frac{\det(A_1)}{\det(A)} = \frac{W_1 \sum_{j=1}^{d} S_{n_j} \sigma_j}{W\sigma}$$

$$= \frac{\sum_{j=1}^{d} S_{n_j} \sigma_j}{\sigma Q_1} = \frac{a_d \sum_{j=1}^{d} S_{n_j} G_j(\alpha_1)}{(-1)^{d-1} P'(\alpha_1)\sigma} \in \mathbb{Q}(\alpha_1). \tag{2}$$

Hence there is a polynomial $G(z) = g_m z^m + \dots + g_0$ with rational coefficients such that $\xi_1 = G(\alpha_1)$. We will show that $\xi_i = G(\alpha_i)$ for every $i = 1, 2, \dots, d$.

Indeed, let A_i^* and A^* be the matrices obtained from A_i and A, respectively, by putting the ith column as the first column and moving the first $i - 1$ columns by one to the right. Then $\det(A_i^*) = (-1)^{i-1} \det(A_i)$ and $\det(A^*) = (-1)^{i-1} \det(A)$, so expanding $\det(A_i^*)$ by its first column and using $W = W_i Q_i$, $Q_i = (-1)^{d-i} P'(\alpha_i)/a_d$ we obtain

$$\xi_i = \frac{\det(A_i)}{\det(A)} = \frac{\det(A_i^*)}{\det(A^*)} = \frac{W_i \sum_{j=1}^{d} S_{n_j} \sigma_j^*}{(-1)^{i-1} W\sigma}$$

$$= \frac{\sum_{j=1}^{d} S_{n_j} \sigma_j^*}{(-1)^{i-1} \sigma Q_i} = \frac{a_d \sum_{j=1}^{d} S_{n_j} G_j(\alpha_i)}{(-1)^{d-1} P'(\alpha_i)\sigma}.$$

This is the same expression as obtained in (2) for ξ_1, except for α_1 is replaced by α_i. This proves our assertion claiming that $\xi_i = G(\alpha_i)$ for each $i = 1, 2, \dots, d$.

Clearly, for each integer $n \geqslant 0$, the sum

$$S_n = \sum_{j=1}^{d} \xi_j \alpha_j^n = \sum_{j=1}^{d} G(\alpha_j)\alpha_j^n$$
$$= \sum_{j=1}^{d} (g_m\alpha_j^m + \cdots + g_1\alpha_j + g_0)\alpha_j^n = \sum_{i=0}^{m} g_i \sum_{j=1}^{d} \alpha_j^{n+i}$$

is a rational number, because the sums $\alpha_1^n + \alpha_2^n + \cdots + \alpha_d^n$, $n = 0, 1, 2, \ldots$, are rational numbers. □

Lemma 6 *Let a be an integer satisfying $|a| \geqslant 2$, and let $\nu_0, \nu_1, \nu_2, \ldots \in I$ and ξ be real numbers, where I is a half open subinterval of $\mathbb{R}$ of length $1/(1 + |a|)$. Then the sequence*

$$[\xi a^{n+1} + \nu_{n+1}] - a[\xi a^n + \nu_n],$$

$n = 0, 1, 2, \ldots$, is ultimately periodic if and only if $\xi \in \mathbb{Q}$.

Proof Set

$$x_n = [\xi a^n + \nu_n] \quad \text{and} \quad y_n = \{\xi a^n + \nu_n\}.$$

Since $x_n + y_n = \xi a^n + \nu_n$, we derive that $a(x_n + y_n - \nu_n) = x_{n+1} + y_{n+1} - \nu_{n+1}$ for each $n \geqslant 0$. Hence

$$s_n = x_{n+1} - ax_n = ay_n - y_{n+1} + \nu_{n+1} - a\nu_n. \tag{3}$$

is an integer for each integer $n \geqslant 0$.

Suppose that $\xi = u/v$, where $u \in \mathbb{Z}$ and $v \in \mathbb{N}$. We will show that the sequence of fractional parts y_n, $n = 0, 1, 2, \ldots$, is ultimately periodic, which, by (3), implies that $s_n = x_{n+1} - ax_n$, $n = 0, 1, 2, \ldots$, is ultimately periodic. Indeed, if there is a $t \in \mathbb{N}$ such that $y_n = y_{n+t}$ for each sufficiently large n, then, by (3),

$$s_{n+t} - s_n = \nu_{n+t+1} - a\nu_{n+t} - \nu_{n+1} + a\nu_n.$$

Since all ν_n, $n \geqslant 0$, lie in the half open interval I of length $1/(1 + |a|)$, the right hand side is smaller than 1 and greater than -1. But $s_{n+t} - s_n \in \mathbb{Z}$, so $s_{n+t} = s_n$. Hence the sequence $s_0, s_1, s_2, \ldots$ is ultimately periodic.

Now, we shall prove that y_n, $n = 0, 1, 2, \ldots$, is ultimately periodic. Write $v = v_0 v_1$, where v_1 is the largest integer such that $v_1 | v$ and $\gcd(|a|, v_1) = 1$. Let also $\nu \in [k + k_1/v_1, k + (k_1 + 1)/v_1)$, where $k \in \mathbb{Z}$ and $k_1 \in \{0, 1, \ldots, v_1 - 1\}$. For a given $\nu \in \mathbb{R}$, such a pair k, k_1 is unique. Clearly, $v_0 | a^n$ for each sufficiently large $n \in \mathbb{N}$, say, for $n \geqslant n_0$. Hence $\{\xi a^n\} = \{ua^n/v\} = \{ua^n/v_0 v_1\}$ takes one of the values $0, 1/v_1, 2/v_1, \ldots, (v_1 - 1)/v_1$. Thus $\{\xi a^n\} = j_n/v_1$, where $j_n \in \{0, 1, \ldots, v_1 - 1\}$.

Since $\{\nu\} \in [k_1/v_1, (k_1+1)/v_1)$, we obtain $y_n = \{\xi a^n + \nu\} = \{(j_n + k_1)/v_1\}$. In order to show that $y_n = \{\xi a^n + \nu\}$, $n = 0, 1, 2, \dots$, is ultimately periodic, it suffices to prove that the sequence j_n, $n = 0, 1, 2, \dots$, is ultimately periodic. This is the case if the sequence of integers ua^n/v_0, $n = n_0, n_0 + 1, \dots$, modulo v_1 is ultimately periodic. Since $\gcd(|a|, v_1) = 1$, by Euler's theorem, we have $v_1 | (a^{\varphi(v_1)} - 1)$. (This holds for negative a too, because $\varphi(v_1)$ is even for each $v_1 \geqslant 3$. For $v_1 = 2$, the numbers $a - 1$ and $-a - 1$ have the same parity.) Consequently, the sequence of integers ua^n/v_0, $n = n_0, n_0 + 1, \dots$, modulo v_1 is purely periodic with period $\varphi(v_1)$. (This is not necessarily the smallest period.) This implies the assertion of the lemma for $\xi \in \mathbb{Q}$.

It remains to prove that the sequence s_n, $n = 0, 1, 2, \dots$, given in (3) is not ultimately periodic if $\xi \notin \mathbb{Q}$. Suppose that it is ultimately periodic with period t. Then $x_{n+1+t} - ax_{n+t} = x_{n+1} - ax_n$ for each $n \geqslant n_1$. Equivalently, we have

$$x_{n+1+t} = ax_{n+t} + x_{n+1} - ax_n \tag{4}$$

for $n \geqslant n_1$. The characteristic equation of (4) is $\lambda^{t+1} - a\lambda^t - \lambda + a = (\lambda - a)(\lambda^t - 1) = 0$. It has $t + 1$ distinct roots, so $x_n = c_0 a^n + c_1 w_1^n + \dots + c_t w_t^n$ for each $n \geqslant n_1$, where $w_j = e^{2\pi i(j-1)/t}$ for $j = 1, \dots, t$ and $c_j \in \mathbb{C}$ for $j = 0, 1, \dots, t$.

Similarly, from (3) and $s_{n+t} = s_n$, we find that $y_{n+t+1} - \nu_{n+t+1} - a(y_{n+t} - \nu_{n+t}) = y_{n+1} - \nu_{n+1} - a(y_n - \nu_n)$ for $n \geqslant n_1$. Consequently, as above, $y_n - \nu_n = f_0 a^n + f_1 w_1^n + \dots + f_t w_t^n$, where $f_0, f_1, \dots, f_t \in \mathbb{C}$. The quantity $f_0 a^n = y_n - \nu_n - f_1 w_1^n - \dots - f_t w_t^n$ is bounded by a constant independent of n. Letting $n \to \infty$, we deduce that $f_0 = 0$, because $|a| > 1$. Adding x_n and y_n, we obtain

$$x_n + y_n = c_0 a^n + (c_1 + f_1) w_1^n + \dots + (c_t + f_t) w_t^n + \nu_n = \xi a^n + \nu_n.$$

Since $w_1, \dots, w_t$ are roots of unity, the sum $(c_1 + f_1) w_1^n + \dots + (c_t + f_t) w_t^n - \nu_n$ is bounded by an absolute constant independent of n. Hence, as above, $c_0 = \xi$. Since $x_n \in \mathbb{Z}$ for every integer $n \geqslant 0$, by Lemma 5 applied to $d = t + 1$, $S_n = x_n$, any set of $t + 1$ distinct integers greater than n_1, and the polynomial $P(z) = (z - a)(z^t - 1)$, we deduce that $\xi = c_0 = G(a)$. As $a \in \mathbb{Z}$ and $G(z) \in \mathbb{Q}[z]$, we have $\xi = G(a) \in \mathbb{Q}$, contrary to our assumption $\xi \notin \mathbb{Q}$. □

4 Sturmian sequences

Let $\mathcal{A}$ be an alphabet consisting of two letters U and V. Any sequence (finite or infinite) of letters U and V of $\mathcal{A}$ is called *word*. Any string of consecutive letters of a word is called *factor* or *subword*. Let $p(\mathbf{w}, m)$ be the number of distinct blocks of length m occuring in a word $\mathbf{w}$. For instance, $p(UV^3U^2, 3) = 4$. Here and below, e.g., $V^3 = VVV$ and $(VU)^\infty = VUVUVU\dots$. By an old result of Morse and Hedlund [15], every infinite word $\mathbf{w}$ is either ultimately periodic or $p(\mathbf{w}, m) \geqslant m + 1$ for each $m \in \mathbb{N}$. An infinite word $\mathbf{w}$ is called *Sturmian* if $p(\mathbf{w}, m) = m + 1$ for each $m \in \mathbb{N}$.

Lemma 7 *Suppose that* $\mathbf{w}$ *is an infinite word of the alphabet* $\{U, V\}$ *which is not ultimately periodic. Then* $\mathbf{w}$ *is Sturmian, if and only if, for any finite word* $\mathbf{v}$ *of the* *alphabet* $\{U, V\}$, *either* $U\mathbf{v}U$ *or* $V\mathbf{v}V$ *is not a subword of* $\mathbf{w}$. *Furthermore, every Sturmian word* $\mathbf{w}$ *of the alphabet* $\{U, V\}$, *starting from some place, consists either of the blocks* UV^k *and* UV^{k+1} *only or of the blocks* VU^k *and* VU^{k+1} *only, where* k *is a positive integer.*

Proof The first statement is well known. See, e.g., Chap. 2 in [12] or combine Proposition 10.5.7 and Theorem 10.5.8 in [3]. Let us prove the second statement directly (without using the first part). If $\mathbf{w}$ is a Sturmian word, then it is not ultimately periodic, so contains infinitely many subwords UV and VU. Since $p(\mathbf{w}, 2) = 3$, the word $\mathbf{w}$ contains either V^2 or U^2. Suppose it contains V^2 and no U^2. Then the word $\mathbf{w}$, starting from some place, has the form $UV^{k_1}UV^{k_2}UV^{k_3}\dots$, where $k_1, k_2, k_3, \dots \in \mathbb{N}$. Let $k = \min(k_1, k_2, k_3, \dots)$. If $k_j \geqslant k+2$ for at least one $j \in \mathbb{N}$, then $\mathbf{w}$ contains the blocks UV^kU, V^kUV, $\dots$, UV^{k+1}, V^{k+2}, $V^{k+1}U$. Then $p(\mathbf{w}, k+2) \geqslant k+4$, so the word $\mathbf{w}$ is not Sturmian, a contradiction. The other case, when $\mathbf{w}$ contains U^2 and no V^2, is symmetric. □

Lemma 8 *Let* ξ *be an irrational real number and* $\nu \in \mathbb{R}$. *If the sequence* $[\xi 2^n + \nu]$, $n = 0, 1, 2, \dots$, *contains only finitely many even elements, then there exist* $m \in \mathbb{N}$ *and* $k \in \mathbb{Z}$ *such that, for each* $n \geqslant m$,

$$y_n = \{\xi 2^n + \nu\} = \nu + 2k - 1 + \sum_{j=0}^{\infty} \ell_{n+j} 2^{-j}, \tag{5}$$

where $\ell_m, \ell_{m+1}, \ell_{m+2}, \dots \in \{0, 1\}$, *and* $\ell_m \ell_{m+1} \ell_{m+2} \dots$ *is a Sturmian word on the alphabet* $\{0, 1\}$.

Proof The proof is similar to that of Theorem 2.1 in [5]. By (3) with $a = 2$ and $\nu_0 = \nu_1 = \nu_2 = \dots = \nu$, we have $s_n = x_{n+1} - 2x_n = 2y_n - y_{n+1} - \nu \in (-1-\nu, 2-\nu)$, where $x_n = [\xi 2^n + \nu]$. The interval $(-1-\nu, 2-\nu)$ contains at most three (consecutive) integers, say, $b-1, b, b+1$, so $x_{n+1} - 2x_n = s_n \in \{b-1, b, b+1\}$ for each $n \geqslant 0$. Since x_{n+1}, $n = 0, 1, 2, \dots$, is odd for each sufficiently large n, the corresponding $s_n \in \{b-1, b, b+1\}$ must be odd. By Lemma 6, the sequence $s_0, s_1, s_2, \dots$ is not ultimately periodic. Hence b must be even, say, $b = 2k$, and, moreover, there is an $m \in \mathbb{N}$ such that $s_n \in \{b-1, b+1\} = \{2k-1, 2k+1\}$ for each $n \geqslant m$.

Thus, for any $n \geqslant m$, we can write

$$s_n = 2k - 1 + 2\ell_n \tag{6}$$

with some $\ell_n \in \{0, 1\}$. From $s_n = 2y_n - y_{n+1} - \nu = 2(y_n - \nu) - (y_{n+1} - \nu)$, we deduce that

$$\begin{aligned} y_n - \nu &= 2^{-1}(s_n + y_{n+1} - \nu) \\ &= 2^{-1}s_n + 2^{-2}(s_{n+1} + y_{n+2} - \nu) = \dots = \sum_{j=1}^{\infty} s_{n+j-1} 2^{-j}, \end{aligned}$$

because $\lim_{j\to\infty}(y_{n+j}-\nu)2^{-j}=0$. Setting (6) into the last infinite sum, we get that $y_n-\nu=2k-1+\sum_{j=0}^{\infty}\ell_{n+j}2^{-j}$ for each $n\geqslant m$. This proves (5).

We already know that the word $\mathbf{w}=\ell_m\ell_{m+1}\ell_{m+2}\ldots$ is not ultimately periodic [see (6)]. Suppose that $\mathbf{w}$ is not Sturmian. Then, by Lemma 7, for some word $\mathbf{u}$ of the alphabet $\{0,1\}$, the word $\mathbf{w}$ contains two subwords $1\mathbf{u}1$ and $0\mathbf{u}0$. Suppose they start at kth and lth positions, respectively. If $\mathbf{u}$ is of length $L\geqslant 0$, then, selecting $n=m+k-1$ and $n=m+l-1$ in (5), we obtain

$$\begin{aligned} y_{m+k-1}-y_{m+l-1} &= \sum_{j=0}^{\infty}(\ell_{m+k-1+j}-\ell_{m+l-1+j})2^{-j} \\ &= 1+2^{-L-1}+\sum_{j=L+2}^{\infty}(\ell_{m+k-1+j}-\ell_{m+l-1+j})2^{-j}. \end{aligned}$$

Since the word $\ell_m\ell_{m+1}\ell_{m+2}\ldots$ is not ultimately periodic, $\ell_{m+k-1+j}-\ell_{m+l-1+j}\neq -1$ for some $j\geqslant L+2$. This implies that the sum $\sum_{j=L+2}^{\infty}(\ell_{m+k-1+j}-\ell_{m+l-1+j})2^{-j}$ is strictly greater than -2^{-L-1}. It follows that $y_{m+k-1}-y_{m+l-1}>1+2^{-L-1}-2^{-L-1}=1$. This is a contradiction with the trivial inequality $y_{m+k-1}-y_{m+l-1}\leqslant y_{m+k-1}<1$. □

5 Powers of −2

In this section, we prove the following:

Theorem 9 *Let $\xi\neq 0$ be a real number, and let $I\subset\mathbb{R}$ be a half open interval of length $1/3$. Suppose that $\nu_0,\nu_1,\nu_2,\ldots\in I$. Then the sequence $[\xi(-2)^n+\nu_n]$, $n=0,1,2,\ldots$, contains infinitely many composite elements. Moreover, if the number ξ is irrational, then the sequence contains infinitely many elements divisible by 2 or 3.*

Proof By (3), we obtain $s_n=x_{n+1}+2x_n=2\nu_n+\nu_{n+1}-2y_n-y_{n+1}$. Write $I=[\nu,\nu+1/3)$ or $I=(\nu,\nu+1/3]$. In both cases, all s_n belong to the interval $(3\nu-3,3\nu+1]$ of length 4. This interval contains exactly 4 consecutive integers, say, $b-1,b,b+1,b+2$. Hence $s_n\in\{b-1,b,b+1,b+2\}$. If the sequence s_n, $n=0,1,2,\ldots$, is ultimately periodic, then $x_{n+t+1}+2x_{n+t}=x_{n+1}+2x_n$ for some $t\in\mathbb{N}$ and each $n\geqslant n_0$. By Lemma 6, this can only happen if $\xi\in\mathbb{Q}$. Then, by Theorem 4, where $d=t+1$, $c=2$ and $F(z_1,z_2,\ldots,z_{d-1})=z_1-2z_{d-1}$, the sequence $x_1,x_2,x_3,\ldots$ contains infinitely many composite numbers.

Suppose that ξ is irrational, but the sequence $x_n=[\xi(-2)^n+\nu_n]$ contains only finitely many elements divisible by 2 and 3. By Lemma 6, the sequence s_n, $n=0,1,2,\ldots$, is not ultimately periodic. Note that if s_n is even, then $x_{n+1}=-2x_n+s_n$ is even too. Hence s_n must be odd for each sufficiently large n. The set $\{b-1,b,b+1,b+2\}$ contains two even and two odd numbers, say, $2k-1$ and $2k+1$. It follows that $s_n\in\{2k-1,2k+1\}$ for each sufficiently large n, say, for $n\geqslant n_0$. Suppose that n_1 is so large that, for each $n\geqslant n_1$, x_n is not divisible by 3. So, for $n\geqslant N=\max(n_0,n_1)$, the number x_n modulo 3 is either 1 or 2. The pair $2k-1,2k+1$ modulo 3 is either

0, 2 or 1, 0 or 2, 1. We shall refer to these cases as the first case, the second case and the third case, respectively.

Consider the sequence $s_N, s_{N+1}, s_{N+2}, \dots$ modulo 3. Instead of s_n we shall write the letter U if $s_n \pmod 3 = 0$, the letter V if $s_n \pmod 3 = 1$, and the letter T if $s_n \pmod 3 = 2$, and consider a corresponding word. For instance, if $s_N = 9$, $s_{N+1} = 11$, $s_{N+2} = s_{N+3} = 9, \dots$, then the word is $UTUU\dots$. In each of the three cases, since $s_n \in \{2k-1, 2k+1\}$, where k is a fixed integer, our word consists of some two letters of the alphabet $\{U, V, T\}$. Obviously, the word is ultimately periodic if and only if the sequence $s_N, s_{N+1}, s_{N+2}, \dots \in \{2k-1, 2k+1\}$ itself is ultimately periodic. By Lemma 6, the sequence $s_N, s_{N+1}, s_{N+2}, \dots$ is not ultimately periodic, so the word of the alphabet $\{U, V, T\}$ obtained from $s_N, s_{N+1}, s_{N+2}, \dots \pmod 3$ is also not ultimately periodic. To obtain a contradiction, we shall consider the sequence $x_N, x_{N+1}, x_{N+2}, \dots \pmod 3 \in \{1, 2\}$.

Suppose, for instance, that the letter U stands at the jth place of the word, i.e., the jth element of the sequence $s_N, s_{N+1}, s_{N+2}, \dots \pmod 3$ is 0, namely, $s_{N+j-1} \pmod 3 = 0$. Setting $n = N + j - 1$ and using $s_n = x_{n+1} + 2x_n$, where x_n, x_{n+1} modulo 3 are both in $\{1, 2\}$, we shall also use this letter U to denote the function which acts as a map $\{1, 2\} \mapsto \{1, 2\}$ and is defined by

$$U : x_n \pmod 3 \mapsto x_{n+1} \pmod 3.$$

Clearly, U maps 1 to 1, because

$$x_{n+1} \pmod 3 = (-2x_n + s_n) \pmod 3 = -2x_n \pmod 3 = 1,$$

if $x_n \pmod 3 = 1$, and 2 to 2, because

$$x_{n+1} \pmod 3 = (-2x_n + s_n) \pmod 3 = -2x_n \pmod 3 = 2,$$

if $x_n \pmod 3 = 2$. So $U : \{1, 2\} \mapsto \{1, 2\}$ can be written as the permutation $U = (1)(2)$.

Similarly, let $V : x_n \pmod 3 \mapsto x_{n+1} \pmod 3$. Now, $s_n \pmod 3 = 1$, so $x_{n+1} = -2x_n + s_n$ modulo 3 is 2, if $x_n \pmod 3 = 1$, and 0, if $x_n \pmod 3 = 2$. This is impossible, because $x_{n+1} \pmod 3 \in \{1, 2\}$. So the letter V acts as a permutation, which maps 1 to 2 and 2 to "nowhere." We shall write it as $V = (12|$, where $|$ means that 2 maps to "nowhere." More precisely, this means that V cannot occupy the jth place of $s_N, s_{N+1}, s_{N+2}, \dots \pmod 3$ provided that $n = N + j - 1$ and $x_n \pmod 3 = 2$. By the same argument as above, the letter T acts as $T = (21|$.

Summarizing, we conclude that U, V, T are the permutations of $\{1, 2\}$, given by

$$U = (1)(2), \quad V = (12|, \quad T = (21|.$$

Two consecutive symbols work as $UV = (12|$, $VU = (12|$, $U^2 = (1)(2)$, $V^2 = (1|(2|$, and so on. We multiply them as permutations reading from the left to the right. In particular, this means that the subword V^2 cannot occur in our word. (Indeed, if our word contains somewhere the subword $V^2 = VV$ then, for some $n \geqslant N$, we have

$s_n = s_{n+1} = 3s + 2$, where $s \in \mathbb{Z}$, so at least one of the integers x_n, x_{n+1}, x_{n+2} is divisible by 3, because $x_{n+1} + 2x_n = x_{n+2} + 2x_{n+1} = 3s + 2$.) Furthermore, our word cannot contain any subword consisting of U, V, T, which acts as $(1|(2|$.

In the first case, when $(2k-1) \pmod 3 = 0$ and $(2k+1) \pmod 3 = 2$, the word consists of U and T. Since $T^2 = (1|(2|$ cannot occur, the word (which is not ultimately periodic), starting from a certain place, is $TU^{k_1}TU^{k_2}\dots$, where $k_1, k_2, \dots, \in \mathbb{N}$. However, if $k \in \mathbb{N}$, then $TU^kT = (1|(2|$, so this subword cannot occur, a contradiction.

In the second case, when $2k-1, 2k+1$ modulo 3 is 1, 0, the word consists of U and V. Now, $V^2 = (1|(2|$ cannot occur, so the word, starting from a certain place, consists of $VU^{k_1}VU^{k_2}\dots$, where $k_1, k_2, \dots, \in \mathbb{N}$. However, $VU^kV = (1|(2|$ cannot occur, a contradiction.

In the third case, $2k-1, 2k+1$ modulo 3 is 2, 1. Now, we have the word consisting of V and T, which is not ultimately periodic. Neither $V^2 = (1|(2|$ nor $T^2 = (1|(2|$ can occur, so the word, starting from a certain place, must be $(VT)^\infty = VTVTVT\dots$, so it is ultimately periodic, a contradiction.

This proves that, for each $\xi \notin \mathbb{Q}$, the sequence $x_n = [\xi(-2)^n + \nu_n]$, $n = 0, 1, 2, \dots,$ contains infinitely many elements divisible by 2 or 3. □

Note that in the above proof we did not use the results on Sturmian sequences given in Sect. 3.

6 Proof of Theorem 3

Now, let $a = 2$. This time, by (3), where $\nu_0 = \nu_1 = \nu_2 = \dots = \nu$, we have $s_n = x_{n+1} - 2x_n = 2y_n - y_{n+1} - \nu$, so $-1 - \nu < s_n < 2 - \nu$. The interval $(-1-\nu, 2-\nu)$ of length 3 contains at most 3 integers. Hence $s_n \in \{b-1, b, b+1\}$ for some $b \in \mathbb{Z}$. If the sequence s_n, $n = 0, 1, 2, \dots,$ is ultimately periodic, then $x_{n+t+1} - 2x_{n+t} = x_{n+1} - 2x_n$ for some $t \in \mathbb{N}$ and each $n \geqslant n_0$. As above, by Lemma 6, this can only happen if $\xi \in \mathbb{Q}$. So, by Theorem 4, where $d = t+1$, $c = -2$ and $F(z_1, z_2, \dots, z_{d-1}) = z_1 + 2z_{d-1}$, the sequence $x_1, x_2, x_3, \dots$ contains infinitely many composite numbers.

Suppose that ξ is irrational, but the sequence $x_n = [\xi 2^n + \nu]$ contains only finitely many elements divisible by 2. By Lemma 8, there exist $m \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that, for each $n \geqslant m$, y_n is expressible by (5), where $\ell_m, \ell_{m+1}, \ell_{m+2}, \dots \in \{0, 1\}$, and $\ell_m\ell_{m+1}\ell_{m+2}\dots$ is a Sturmian word of the alphabet $\{0, 1\}$. Moreover, $b = 2k$ and $s_n = 2k - 1 + 2\ell_n$, by (6). Hence $s_n \in \{2k-1, 2k+1\}$ for each $n \geqslant m$.

Assume that the sequence x_n, $n = 1, 2, 3, \dots,$ contains only finitely many elements divisible by 3. Let n_2 be so large that, for each $n \geqslant n_2$, x_n is not divisible by 3. Now, for $n \geqslant N = \max(m, n_2)$, the sequence $x_N, x_{N+1}, x_{N+2}, \dots$ modulo 3 consists of 1 and 2 only. As above, the pair $2k-1, 2k+1$ modulo 3 is either 0, 2 (the first case) or 1, 0 (the second case) or 2, 1 (the third case).

Once again consider the sequence $s_N, s_{N+1}, s_{N+2}, \dots$ modulo 3. Instead of s_n we shall write the letter U if $s_n \pmod 3 = 0$, the letter V if $s_n \pmod 3 = 1$, and the letter T if $s_n \pmod 3 = 2$. As above, in each of the three cases, our word consists of some two letters of the alphabet $\{U, V, T\}$. In addition, this time, by Lemma 8, we know that this word is Sturmian.

As in Sect. 5, we may look at U, V, T as some permutations acting on the sequence $x_N, x_{N+1}, x_{N+2}, \dots \pmod 3 \in \{1, 2\}$ as some functions $\{1, 2\} \mapsto \{1, 2\}$, given by $x_n \pmod 3 \mapsto x_{n+1} \pmod 3$. Now, assume that U stands at the jth place of the word, i.e., the jth element of the sequence $s_N, s_{N+1}, s_{N+2}, \dots \pmod 3$ is 0, namely, $s_{N+j-1} \pmod 3 = 0$. Set $n = N + j - 1$. Using $x_{n+1} = 2x_n + s_n$, where $s_n \pmod 3 = 0$ and x_n, x_{n+1} modulo 3 are both in $\{1, 2\}$, we have $U : x_n \pmod 3 \mapsto 2x_n \pmod 3$. Therefore, U maps 1 to 2 and 2 to 1, giving $U = (12)$. Similarly, $V : x_n \pmod 3 \mapsto (2x_n + 1) \pmod 3$ maps 2 to 2, but cannot be applied to 1, i.e., V cannot occupy the jth place of $s_N, s_{N+1}, s_{N+2}, \dots \pmod 3$ provided that $n = N + j - 1$ and $x_n \pmod 3 = 2$. Thus V can be written in the form $V = (1|(2)$, where the meaning of | is the same as that in Sect. 5. Analogously, $T : x_n \pmod 3 \mapsto (2x_n + 2) \pmod 3$ maps 1 to 1, but cannot be applied to 2. It follows that

$$U = (12), \quad V = (1|(2), \quad T = (1)(2|.$$

As above, we can multiply these permutations left to right, so, e.g., $UV = (12|$. We will show, however, that every Sturmian word of the alphabet $\{U, V, T\}$ contains a subword, which acts as $(1|(2|$, so it cannot occur!

In the first case, when $2k - 1, 2k + 1$ modulo 3 is $0, 2$, respectively, we have a Sturmian word on the alphabet $\{U, T\}$. By the second part of Lemma 7, from a certain place, the word is either $UT^{k_1}UT^{k_2}\dots$ or $TU^{k_1}TU^{k_2}\dots$, where $k_1, k_2, \dots \in \{k, k+1\}$ with $k \in \mathbb{N}$. Note that $UT^kUT = (1|(2|$ cannot occur, so the word cannot be $UT^{k_1}UT^{k_2}\dots$. However, every Sturmian word $TU^{k_1}TU^{k_2}\dots$, where $k_1, k_2, \dots \in \{k, k+1\}$, must contain the subword $TU^kTU^{k+1}T$. Since $TU^rT = (1)(2|$ for r even, and $TU^rT = (1|(2|$ for r odd, either TU^kT or $TU^{k+1}T$ acts as $(1|(2|$. Hence $TU^kTU^{k+1}T$ always acts as $(1|(2|$ independent of the parity of k, a contradiction. This completes the proof in the first case.

In the second case, when $2k - 1, 2k + 1$ modulo 3 is $1, 0$, we have a Sturmian word on the alphabet $\{U, V\}$. By the second part of Lemma 7, from a certain place, the word is either $UV^{k_1}UV^{k_2}\dots$ or $VU^{k_1}VU^{k_2}\dots$, where $k_1, k_2, \dots \in \{k, k+1\}$ with $k \in \mathbb{N}$. Now, the subword $UV^kUV = (1|(2|$ cannot occur, which rules out the possibility $UT^{k_1}UT^{k_2}\dots$. It is easy to see that the word VU^rV may only occur for r even, when $VU^rV = (1|(2)$, because $VU^rV = (1|(2|$ for r odd. But every Sturmian word $VU^{k_1}VU^{k_2}\dots$, where $k_1, k_2, \dots \in \{k, k+1\}$, must contain the subword $VU^kVU^{k+1}V$. By the above, at least one of the words VU^kV, $VU^{k+1}V$ is $(1|(2|$, so $VU^kVU^{k+1}V = (1|(2|$ independent of the parity of k. Hence the subword $VU^kVU^{k+1}V$ cannot occur, a contradiction. This finishes the proof in the second case.

In the third case, $2k - 1, 2k + 1$ modulo 3 is $2, 1$. Now, our word is a Sturmian word on the alphabet $\{V, T\}$. In particular, it is not ultimately periodic. Neither $VT = (1|(2|$ nor $TV = (1|(2|$ can occur. Hence the word, starting from a certain place, must be V^∞ or T^∞, so it is ultimately periodic, a contradiction. This completes the proof of Theorem 3 for $a = 2$. □

7 Consecutive composite numbers close to powers of an integer

In this section, we show that the answer to Problem 2 is negative if $\xi = 1$ and α is an integer. The proof is elementary. It can be generalized to any rational $\xi > 0$.

Theorem 10 *Suppose that $a \geqslant 2$ and $r \geqslant 0$ are integers. Then, for infinitely many positive integers n, every integer $\ell \in [a^n - r, a^n + r]$ is a composite number.*

Proof Suppose that $j_1 < j_2 < \dots < j_k$ are all integers of the set $\{1, 2, \dots, r\}$ which are relatively prime to a. Evidently, if $j \in [-r, r]$ is an integer which is not of the form $\pm j_t$ for some $t \in \{1, 2, \dots, k\}$, then $a^n + j$ is composite for each sufficiently large n.

Alternatively, suppose that $j = \pm j_t$ for some $t \in \{1, 2, \dots, k\}$. Set

$$S = \varphi(a^u - j_1)\varphi(a^u + j_1)\dots\varphi(a^u - j_k)\varphi(a^u + j_k), \tag{7}$$

where φ is Euler's function, and u is a fixed positive integer satisfying $a^u - j_k \geqslant 2$. We claim that the numbers $a^{nS+u} \pm j_t$ are composite for each sufficiently large $n \in \mathbb{N}$.

Indeed, by Euler's theorem, $a^{\varphi(a^u+j_t)}$ modulo a^u+j_t is 1, because $\gcd(a^u+j_t, a) = 1$. Hence, by (7), $a^{nS+u} + j_t$ modulo $a^u + j_t$ is zero. Thus $a^{nS+u} + j_t$ is divisible by $a^u + j_t$ and greater than $a^u + j_t$, if n is large enough. Hence $a^{nS+u} + j_t$ is a composite number. Similarly, by the choice of S in (7), the number $a^{nS+u} - j_t$ is composite for each sufficiently large $n \in \mathbb{N}$, because it is divisible by $a^u - j_t$ and greater than $a^u - j_t$.

It follows that all the integers $a^{nS+u} - r, a^{nS+u} - r + 1, \dots, a^{nS+u} + r$ are composite, provided that $n \in \mathbb{N}$ is large enough. This completes the proof. □

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