Polynomials With High Multiplicity at Unity and Tarry's Problem

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Abstract. The existence of a polynomial with integer coefficients of relatively small length and sufficiently large multiplicity at unity is established. The proof of the corresponding statement is based on an estimate of the number of solutions of Tarry's system.

Key words: Tarry problem, polynomials with integer coefficients, height and length of a polynomial, Siegel lemma, Schur theorem, Stirling formula.

Let \( P(z) \) be a nonzero polynomial with integer coefficients of degree \( \deg P(z) \leq N - 1 \). We denote its multiplicity at the point \( z = 1 \) by \( r(P) = \text{ord}_{z=1} P(z) \).

Let \( H(P) \) and \( L(P) \) be the height and length of the polynomial \( P(z) \), respectively. It follows from the classic Siegel lemma that there exist polynomials of not very large height and high enough multiplicity at unity. Thus, for example, Bloch and Pólya [1] showed that there exists a polynomial with coefficients \( \{0, 1, -1\} \) such that

\[
\log 2 \log N - 1 \leq \frac{1}{2} \left( \frac{r(P)}{N} \right)^2 \log \left( \frac{N}{r(P)} \right) (1 + \varepsilon),
\]

where \( \varepsilon > 0 \) and \( N/r(P) \) is sufficiently large.

It is readily seen that (2) implies the existence of a polynomial with coefficients \( \{0, 1, -1\} \) such that

\[
r(P) \geq (1 - \varepsilon) \sqrt{4 \log 2 \frac{N}{\log N}},
\]

where \( \varepsilon > 0 \) and \( N \) is sufficiently large. Inequality (3) clearly sharpens (1).

Recently Amoroso [6], using the ideas worked out by Dobrowolski in his study of Lehmer's conjecture [7], showed that the constant 1/2 in (2) can be replaced by 1/4 under certain additional conditions. However, this is possible only for a sufficiently large height, i.e., this does not strengthen inequality (3). Finally, the author [8, 9] examined an explicitly constructed polynomial with high multiplicity at unity satisfying an inequality similar to (2).


It is well known that the multiplicity of a polynomial at unity cannot be too large if its height or length are not large. By the Schur theorem \[10\], the number of real roots of a polynomial \( P \) is at most \( 2\sqrt{N \log L(P)} \). Hence we have the inequality

\[
r(P) \leq 2\sqrt{N \log L(P)}.
\] (4)

Inequality (4) was also proved by Bombieri and Vaaler \[5\] in the case where \( H(P) \), and \( L(P) \) with it, is sufficiently large. Amoroso \[6\] showed that the constant 2 in (4) can be replaced by 1.21 if \( N \) and \( \log H(P) / \log N \) are sufficiently large. Note that in the case \( L(P) \leq N \), it follows from (4) that

\[
r(P) \leq 2\sqrt{N \log N}.
\]

Thus, the only way to sharpen (3), if any, is through the constant and \( \log N \).

In this paper we show that there exist polynomials of small, as compared to the degree, length that have a sufficiently high multiplicity at unity. The following theorem holds:

**Theorem.** For any \( \varepsilon > 0 \) there exists a \( \delta = \delta(\varepsilon) > 0 \) such that if \( 1/(\delta N) \leq a \leq 2^{N-1} \), then there exists a polynomial \( P \) with integer coefficients satisfying the conditions \( \deg P \leq N - 1 \), \( L(P) \leq a N \), and

\[
r(P) > (\sqrt{2} - \varepsilon) \sqrt{f(a)N / \log N},
\] (5)

where \( f(a) = (1 + a/2) \log(1 + a/2) - (a/2) \log(a/2) \). Furthermore, if \( N^{-\delta} \leq a \leq 2^{N-1} \) and \( N \) is sufficiently large, then

\[
r(P) > (2 - \varepsilon) \sqrt{f(a)N / \log N}.
\] (6)

Note that if \( a \) is a positive integer, then under the assumptions of the theorem, (2) yields the inequality

\[
r(P) \geq (2 - \varepsilon) \sqrt{N \log(1 + a) / \log N}.
\] (7)

As to the case \( a < 1 \), under the assumptions of the theorem, (3) yields the inequality

\[
r(P) \geq (1 - \varepsilon) \sqrt{4 \log 2 \cdot a N / \log(a N)}.
\] (8)

The lower bounds for \( r(P) \) in the theorem are stronger than (7) and (8). For instance, for \( a = 1 \), (6) is stronger than (7):

\[
f(1) = \frac{3}{2} \log 3 - \log 2 > \log 2,
\]

and for \( a = 1/\sqrt{N} \) inequality (5) excels (8) by the logarithmic factor \( \sqrt{\log N} \). If \( a \geq 2^{N-1}/N \), then the polynomial \( (1 - z)^{N-1} \) has the greatest possible multiplicity \( r(P) = N - 1 \).

The first version of the proof of the theorem, with a function slightly less than \( f(a) \), involved application of estimates obtained in the Tarry (or Prue-Tarry-Escott) problem. Consider the system of \( n \) equations

\[
z_1^j + z_2^j + \cdots + z_k^j - y_1^j - y_2^j - \cdots - y_k^j = 0, \quad j = 1, 2, \ldots, n,
\] (9)

where the variables \( x_1, x_2, \ldots, x_k \), \( y_1, y_2, \ldots, y_k \) take integer values from 1 to \( N \). A variety of questions concerning the number of solutions of this system is usually called the Tarry problem (see, for instance, \[11-13\] or \[14, Chap. III, Lemma 5\], where a lower bound for this number is given). A solution
(z₁, z₂, ..., zₖ, y₁, y₂, ..., yₖ) of the Tarry system is called trivial if the numbers z₁, z₂, ..., zₖ are obtained by a permutation of the numbers y₁, y₂, ..., yₖ.

Consider the polynomial

\[ F(z) = \sum_{j=1}^{k} z^{x_j} - \sum_{j=1}^{k} z^{y_j}, \]

where (z₁, z₂, ..., zₖ, y₁, y₂, ..., yₖ) is a nontrivial solution of the Tarry system. It is readily seen that

\[ F(1) = F'(1) = \cdots = F^{(n)}(1) = 0. \]

Set \( P(z) = f(z)/z \). Clearly, \( P(z) \) is a nonzero polynomial with integer coefficients of degree not exceeding \( N - 1 \). In addition, \( L(P) = L(F) \leq 2k \) and \( r(P) = r(F) \geq n + 1 \). To complete the proof of the theorem, it remains to verify that for \( k = \lfloor aN/2 \rfloor \) and sufficiently large \( r \) (corresponding to (5), (6)), the Tarry system (9) with \( n = r - 1 \) has a nontrivial solution.

The author thanks the reviewer who suggested to prove this fact directly, without using the familiar lower bounds for the number of all solutions of the Tarry system, which work well only if \( N \) is large as compared to \( n \) and \( k \).

**Proof.** It is clear that for \( n = r - 1 \) system (9) is equivalent to the following:

\[ \sum_{j=1}^{k} x_j - \sum_{j=1}^{k} y_j = 0, \quad j = 1, 2, \ldots, r - 1. \]

Let us consider all \((r - 1)\)-dimensional vectors

\[ \left( \sum_{i=1}^{k} z_i, \sum_{i=1}^{k} x_i, \ldots, \sum_{i=1}^{k} z_{r-1} \right), \]

where \( 1 \leq z_i \leq N \). Obviously, the \( j \)th component of these vectors takes nonnegative integer values not exceeding \( k(N) \). Moreover, the first component takes only positive values, and for \( r \leq N/2 \), the \( j \)th component \((1 \leq j \leq r - 1)\) is distinct from, say, \( k(N_j) - 1 \). Therefore, the total number of distinct vectors is not greater than

\[ \prod_{j=1}^{r-1} k(N_j) \leq k^{r-1} N^{r(r-1)/2} \prod_{j=1}^{r-1} (j!)^{-1}. \]

Using the inequality \( j! > (j/e)^j \) and induction on \( r \), we obtain

\[ \prod_{j=1}^{r-1} j! > \frac{r^2(\log r - 2)}{2}. \]

Consequently, the number of distinct vectors is not greater than

\[ k^{r-1} \left( \frac{8N}{r} \right)^{r^2/2} < \left( aN \right)^r \left( \frac{8N}{r} \right)^{r^2/2}. \]

If this is less than the number of unordered sets \( \{z_1, z_2, \ldots, z_k\} \), i.e., less than \( \binom{N+k-1}{k-1} \), then at least two vectors coincide for distinct sets and the Tarry system has a nontrivial solution.

To prove the theorem, it remains to check the inequality

\[ (aN)^r \left( \frac{8N}{r} \right)^{r^2/2} < \left( N + \lfloor aN/2 \rfloor - 1 \right), \quad (10) \]
where

\[ r = \left( \theta - \frac{\varepsilon}{2} \right) \sqrt{\frac{f(a)N}{\log N}} \]

and \( \theta \) is equal to \( \sqrt{2} \) or \( 2 \), respectively. In both cases of the theorem the numbers \( N \) and \( aN \) are sufficiently large; hence by the Stirling formula

\[ e^{-1/(12j)} < \frac{j!}{\sqrt{2\pi j(j/e)^j}} < e^{1/(12j)}, \]

the following lower bound for the binomial coefficient is readily derived:

\[
\binom{N + [aN/2] - 1}{N - 1} = \frac{N}{N + [aN/2]} \frac{(N + [aN/2])!}{N! [aN/2]!} > \frac{1}{(N + aN/2)^2} \left( \frac{(1 + a/2)^{1+a/2}}{a/2)^{a/2}} \right)^N = \frac{1}{(N + aN/2)^3} e^{f(a)N}.
\]

Let us take the logarithm of inequality (10) and check that

\[(r + 3) \log(N + aN) + \frac{r^2}{2} \log \left( \frac{8N}{r} \right) < f(a)N. \tag{11} \]

Let \( \delta > 0 \) be sufficiently small, and let \( N \) be sufficiently large. In both cases \( 1/(\delta N) \leq a \leq 2^{N-1} \) and \( N^{-\delta} \leq a \leq 2^{N-1} \) we have

\[(r + 3) \log(N + aN) = \left( \left( \theta - \frac{\varepsilon}{2} \right) \sqrt{\frac{f(a)N}{\log N}} + 3 \right) \log(N + aN) < \frac{\varepsilon}{8} f(a)N. \]

It is also clear that the number \( r \) is sufficiently large, since

\[ f(a) > \frac{a}{2} \log \left( 1 + \frac{2}{a} \right) > \frac{\log(\delta N)}{2\delta N}. \]

Therefore, if \( 1/(\delta N) \leq a \) and \( \varepsilon \) is sufficiently small, then

\[
\frac{r^2}{2} \log \left( \frac{8N}{r} \right) < \frac{r^2}{2} \log N = \frac{1}{2} (\sqrt{2} - \varepsilon)^2 f(a)N < (1 - \varepsilon) f(a)N.
\]

Consequently, inequality (11) is true, which proves (5).

If \( a \geq N^{-\delta} \) and \( N \) is sufficiently large, then

\[ f(a) > \frac{a}{2} \log \left( 1 + \frac{2}{a} \right) > \frac{\delta \log N}{2 N^\delta}. \]

This allows us to estimate \( r \) from below

\[ r = \left( 2 - \frac{\varepsilon}{2} \right) \sqrt{\frac{f(a)N}{\log N}} > N^{1/2 - \delta}. \]

Hence \( 8N/r < N^{1/2 - 2\delta} \), and

\[
\frac{r^2}{2} \log \left( \frac{8N}{r} \right) < \left( \frac{1}{4} - \delta \right) r^2 \log N < \left( \frac{1}{4} - \delta \right) \left( 4 - \frac{\varepsilon}{2} \right) f(a)N < \left( 1 - \frac{\varepsilon}{8} \right) f(a)N.
\]
This completes the proof of the theorem, because inequality (11) turns out to be true again.

Inequality (3) can be proved in a similar way. Let all $z_1, z_2, \ldots, z_k$ be distinct. The number of such sets is equal to $\binom{N}{k}$ and in place of (10) we now have the inequality

$$k^r \left( \frac{8N}{r} \right)^{r^2/2} < \binom{N}{k}.$$ 

For any $\varepsilon$, $0 < \varepsilon < 1/2$, and a sufficiently large $N$, this inequality holds with $k = \lfloor N/2 \rfloor$ and

$$r = \left( 1 - \frac{\varepsilon}{2} \right) \sqrt{\frac{4 \log 2 \cdot N}{\log N}}.$$

A somewhat more complicated combinatorial argument makes it possible to count the number of sets\footnote{\textcopyright 2023} \{\(z_1, \ldots, z_k\}\} in which at most $H$ of the elements $z_1, \ldots, z_k$ are identical. By the optimal choice of $k$, we obtain the inequality of Bombieri and Vaaler (2).

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References


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