Note on the zeros of the Hurwitz zeta-function
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Abstract. It is known that for rational (≠ 1/2, 1) and transcendental values of the parameter the Hurwitz zeta-function has an infinity of zeros in the critical strip off the critical line \( \sigma = 1/2 \). Here we investigate the case then the parameter is an algebraic irrational number.

1. Introduction

Let \( s = \sigma + it \) be a complex variable, and let \( 0 < \alpha \leq 1 \). For \( \sigma > 1 \) the Hurwitz zeta-function is defined by

\[
\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},
\]

and by analytic continuation elsewhere, except for a simple pole at \( s = 1 \).

Suppose further that \( \alpha \neq 1/2 \) and \( \alpha \neq 1 \). Davenport and Heillbronn [3] proved that \( \zeta(s, \alpha) \) has an infinity of zeros for \( \sigma > 1 \) for transcendental and rational values of \( \alpha \). Cassels [2] showed that the same is also true for algebraic irrational values of \( \alpha \). In the critical strip the situation is similar. Let \( \sigma_1, \sigma_2 \) be fixed with \( 1/2 < \sigma_1 < \sigma_2 < 1 \). Then \( \zeta(s, \alpha) \) has an infinity of zeros in the strip \( \sigma_1 < \sigma < \sigma_2 \) when \( \alpha \) is rational (Voronin [7], see also Gonek [4]) or transcendental (Gonek [4]). It is expected that algebraic irrational values of \( \alpha \) are not exceptional. The next theorem supports this expectation.

Theorem 1. Let \( 0 < \alpha < 1 \), \( \alpha \neq 1/2 \) be rational or transcendental number. Let \( A \) be any complex number and \( 1/2 < \sigma_1 < \sigma_2 < 1 \). Then for any number \( N \) there is an interval \( I \) of a positive length containing \( \alpha \) such that for \( \alpha' \in I \) the function \( \zeta(s, \alpha') - A \) has more than \( N \) zeros in the strip \( \sigma_1 < \sigma < \sigma_2 \).

Every nonempty interval contains an infinity of algebraic irrational numbers (for example numbers of the type \( \beta \sqrt{2} \), where \( \beta \) is a rational number). Rational numbers (also transcendental) forms a dense set in the interval \((0, 1)\). Thus for any fixed \( N \) the set of algebraic irrational numbers \( \alpha \), for which \( \zeta(s, \alpha) - A \) has more than \( N \) zeros in the strip \( \sigma_1 < \sigma < \sigma_2 \), are dense in the interval \((0, 1)\).

For the proof of Theorem 1 we use some continuity arguments and Rouché theorem.

2. Proof

By works of Voronin [7], Gonek [4] and Bagchi [1] we have the following universality theorem for the Hurwitz zeta-function.
Lemma 1. Let $0 < \alpha < 1$, $\alpha \neq 1/2$ be rational or transcendental. Let $K$ be a compact subset of the strip $1/2 < \sigma < 1$ with a connected complement and let $f(s)$ be a continuous function on $K$ analytic in the interior of $K$. Then for every $\varepsilon > 0$

$$\lim_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} > 0.$$ 

We remind Rouché theorem.

Lemma 2. (Rouché theorem) Suppose that $f(s)$ and $g(s)$ are analytic functions inside and on a regular closed curve $\gamma$ and that $|f(s)| > |g(s)|$ for all $s \in \gamma$. Then $f(s) + g(s)$ and $f(s)$ have the same number of zeros inside $\gamma$.

Proof of Theorem 1. In the first chapter of Karatsuba and Voronin [5] it is shown that for $\sigma > 0$

$$\zeta(s, \alpha) = \frac{1}{\alpha^s} + \frac{1}{(1 + \alpha)^s} + \frac{1}{s - 1} \left(\frac{3}{2} + \alpha\right)^{1-s} + s \int_{3/2}^{\infty} \frac{(1/2 - \{z\})dz}{(z + \alpha)^{s+1}},$$

where $\{z\}$ means a fractional part of $z$. Easy to see, that the last integral converges uniformly for $s$ is in a compact set and $\alpha \in [0, 1]$. Thus $\zeta(s, \alpha)$ is a continuous function in $s \neq 1$ and $\alpha \neq 0$.

Let us choose

$$f(s) = s - \frac{\sigma_1 + \sigma_2}{2}.$$

Clearly

$$\min_{|s - \sigma_1 - \sigma_2| = \sigma_2 - \sigma_1} |f(s)| = \frac{\sigma_2 - \sigma_1}{2}.$$

By Lemma 1, for any given number $N$ we can find $T$ and more than $N$ distinct numbers $\tau$, such that $0 \leq \tau \leq T$, a distance between any of distinct $\tau$ is greater than $\frac{\sigma_2 - \sigma_1}{2}$ and

$$\max_{|s - \sigma_1 + \sigma_2| \leq \sigma_2 - \sigma_1} |\zeta(s + i\tau, \alpha) - (f(s) + A)| < \frac{\sigma_2 - \sigma_1}{10}.$$ 

By the continuity of the Hurwitz zeta-function we have that for any $\alpha$ there is a nonempty interval $I$ containing $\alpha$ such that for $\alpha' \in I$ and for $\sigma_1 \leq u \leq \sigma_2$, $0 \leq v \leq T + 1$,

$$\max |\zeta(u + iv, \alpha') - \zeta(u + iv, \alpha)| < \frac{\sigma_2 - \sigma_1}{10}.$$ 

Now by Rouché theorem, for each $\tau$ the functions

$$\zeta(s + i\tau, \alpha') - A$$
Note on the zeros of the Hurwitz zeta-function

and

$$\zeta(s + i\tau, \alpha') - A + \left( f(s) - (\zeta(s, \alpha') - A) \right) = f(s)$$

have the same number of zeros in the disc $|s - \frac{\sigma_1 + \sigma_2}{2}| \leq \frac{\sigma_2 - \sigma_1}{2}$. The theorem is proved.

REFERENCES


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