SUM OF THE LERCH ZETA FUNCTION OVER NONTRIVIAL ZEROS
OF THE DIRICHLET L-FUNCTION

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Dedicated to the memory of Professor Wolfgang Schwarz

Abstract. For $0 < \alpha \leq 1$ and $0 < \lambda \leq 1$, $\lambda$ is rational, we consider the sum of values of
the Lerch zeta-function $L(\lambda, \alpha, s)$ taken at the nontrivial zeros of the Dirichlet $L$-function
$L(s, \chi)$, where $\chi \mod Q, Q \geq 1$, is a primitive Dirichlet character.

1. Introduction

Let $s = \sigma + it$ denote a complex variable. We use the notation $e(x) = \exp(2\pi ix)$. By $\{x\}$,
$(a, b)$, and $[a, b]$ we denote the fractional part of the real number $x$, the greatest common
divisor of integers $a, b$, and the least common multiple of integers $a, b$ respectively. In this
paper $T$ always tends to plus infinity and $\varepsilon$ is any positive number.

The Lerch zeta-function is defined by

$$L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e(\lambda n)}{(n+\alpha)^s} \quad (\sigma > 1),$$

where $0 < \lambda \leq 1$ and $0 < \alpha \leq 1$. The Dirichlet $L$-function is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (\sigma > 1),$$

where $\chi(n)$ is a Dirichlet character modulo some positive integer $Q$. For $\chi \mod 1$ we get the
Riemann zeta-function $L(s, \chi) = \zeta(s)$. The yet unsolved Generalized Riemann Hypothesis
(GRH) states that inside the critical strip $0 < \sigma < 1$ every Dirichlet $L$-function has zeros
only on the critical line $\sigma = \frac{1}{2}$. Zeros in the critical strip are called nontrivial and we denote
them by $\rho_\chi = \beta_\chi + i\gamma_\chi$. In view of the functional equation (see formula (6) below) the
nontrivial zeros are symmetrically distributed with respect to the critical line. A Dirichlet
character $\chi \mod Q$ is said to be primitive if it is not induced by any other character of
modulus strictly less than $Q$. The unique principal character modulo $Q$ is denoted by $\chi_0$.
The character $\chi_0 \mod 1$ is the only one principal and primitive character. For $T > 0$, let
$N(T, \chi)$ denote the number of the nontrivial zeros with $0 \leq \gamma_\chi \leq T$. For the primitive
character $\chi \mod Q$, we have (Montgomery and Vaughan [17, Corollary 14.7])

$$N(T, \chi) = \frac{T}{2\pi} \log \frac{QT}{2\pi e} + O(\log QT),$$

where $T \geq 4$.

For special values of $\alpha$ and $\lambda$ the Lerch zeta-function reduces to the Riemann zeta function
$L(1, 1, s) = \zeta(s)$, $L(1, 1/2, s) = (2^s - 1)\zeta(s)$, $L(1/2, 1, s) = (1 - 2^{1-s})\zeta(s)$, the Dirichlet $L$-
function $L(1/2, 1/2, s) = 2^sL(s, \psi)$, where $\psi \mod 4$ is an odd Dirichlet character. If we fix
\( \lambda = 1 \) we get the Hurwitz zeta-function \( L(1, \alpha, s) = \zeta(s, \alpha) \) and if we fix \( \alpha = 1 \) we get the periodic zeta-function \( e(\lambda)L(\lambda, 1, s) = F(s, \lambda) \). Nontrivial zeros \( \rho(\lambda, \alpha) = \beta(\lambda, \alpha) + i\gamma(\lambda, \alpha) \) of the Lerch zeta-function \( L(\lambda, \alpha, s) \) are located in the strip \(-1 < \sigma < 1 + \alpha \) (10). In view of the formula (9, Corollary 2)

\[
\sum_{|\gamma(\lambda, \alpha)| \leq T} \left( \beta(\lambda, \alpha) - \frac{1}{2} \right) = \frac{T}{2\pi} \log \frac{\alpha}{\sqrt{\lambda(1 - \{\lambda\})}} + O(\log T),
\]

we see that the nontrivial zeros are not always symmetrically distributed with respect to the critical line. It is pleasant to recall that the paper [9] was written when the first author was visiting W. Schwarz at Frankfurt university.

Let

\[
\Lambda(x) = \begin{cases} 
\log p & \text{if } x = p^k \text{ for some prime } p \text{ and integer } k \geq 1, \\
0 & \text{otherwise}
\end{cases}
\]

be the von Mangoldt function. For the integer \( n \) and the character \( \chi \mod Q \) the Gauss sum is defined by

\[
G(n, \chi) = \sum_{a=1}^{Q} \chi(a) \exp\left(2\pi i \frac{an}{Q}\right).
\]

If \( (n, Q) = 1 \) then, for a primitive character \( \chi \mod Q \), we have \( |G(n, \chi)| = \sqrt{Q} \) and, for principle character \( \chi_0 \mod Q \), it is known that \( G(n, \chi_0) = \mu(n) \), where \( \mu(n) \) is the Möbius function. We shall prove the following result.

**Theorem 1.** Let \( \chi \mod Q, Q \geq 1, \) be a primitive Dirichlet character.

We have, for \( 0 < \lambda = \frac{k}{q} \leq 1, \) \( (k, q) = 1, \) and \( 0 < \alpha < 1, \)

\[
\sum_{0 < \gamma \chi \leq T} L(\lambda, \alpha, \rho(\chi)) = -\left( \Lambda\left( \frac{1}{\alpha} \right) \chi\left( \frac{1}{\alpha} \right) + \delta(Q, q)e(-\alpha \lambda) \frac{\mu(q)}{\phi(q)} L(1 - \alpha, \lambda, 1) \right) \frac{T}{2\pi} \tag{1}
\]

\[
+ O\left(T \exp(-c \log^{\frac{3}{4} - \epsilon} T)\right),
\]

where \( \delta(Q, q) = 1 \) if \( Q|q, \delta(Q, q) = 0 \) otherwise, and \( c \) is a positive absolute constant.

Further, for \( 0 < \lambda = \frac{k}{q} < 1, \) \( (k, q) = 1, \)

\[
\sum_{0 < \gamma \chi \leq T} L(\lambda, 1, \rho(\chi)) = \frac{T}{2\pi} \log \frac{TQ}{2\pi e} - \delta(Q, q)e(-\lambda) \frac{\mu(q)}{\phi(q)} T \log \frac{T}{2\pi e} + C(\chi, \lambda) \frac{T}{2\pi} \tag{2}
\]

\[
+ O\left(T \exp(-c \log^{\frac{3}{4} - \epsilon} T)\right),
\]

where the constant \( C(\chi, \lambda) \) is defined by the equality (19) below.

Let \( A \) be a positive constant. Both asymptotic formulas of this theorem are valid uniformly for \( Q \ll \log^A T. \)

Next we discuss the asymptotic formula (1) of Theorem 1. From the proof of Theorem 1.2 in Nakamura [18] we know that, for \( 0 < \lambda < 1 \) and \( 0 < \alpha \leq 1, \)

\[ L(\lambda, \alpha, 1) \neq 0. \]
The proof of the last formula is nice and short. By the integral representation (Lerch [16] formula (2)) or [15] formula (2.6)) we have
\[ \Gamma(\sigma)L(\lambda, \alpha, \sigma) = \int_0^\infty \frac{x^{\sigma-1}e^{(1-\alpha)x} - e^{-2\pi i \lambda}}{e^x - e^{2\pi i \lambda}} \frac{dx}{x^{\sigma-1}} \quad (\sigma > 0). \]

Now \( RL(\lambda, \alpha, 1) > 0 \) since \( R(e^{(1-\alpha)x} - e^{-2\pi i \lambda}) > 0 \) for \( x > 0 \).

The formula (1) extends results obtained in Fujii [4], Steuding [19], [11], where the case \( \lambda = 1, 0 < \alpha < 1, \) and \( Q = 1 \) was investigated. In [6] and [14] the formula (2) for \( 0 < \lambda = k/q < 1, (k, q) = 1, \alpha = 1, \) and \( Q = 1 \) was considered, see also Steuding [20].

In the paper [8] the sum of \( L(s, \chi) \) and in the paper [7] the sum of \( L(s, \chi)L(1 - s, \chi) \) over nontrivial zeros of another Dirichlet \( L \)-function were studied.

The next section is devoted to the proof of Theorem 1.

2. Proof of Theorem 1

We will use contour integration. The proof of the theorem relies on the method of Conrey, Ghosh, and Gonek [2]. We divide the proof to following subsections: 2.1 Beginning of the proof. Functional equations; 2.2 Gonek’s lemma; and 2.3 Peron’s formula.

2.1. Beginning of the proof. Functional equations. Similarly as in [8], without loss of generality, we consider \( T \) which satisfies the inequality
\[ \min_{\gamma_\chi} |T - \gamma_\chi| \gg \frac{1}{\log(QT)}. \]

For \( q \geq 1, \chi \mod q \) and \( t \geq 0 \) we have (Prachar [22] Theorem 3.3 in Chapter 7)
\[ N_\chi(t + 1) - N_\chi(t) := \#\{\rho_\chi = \beta_\chi + i\gamma_\chi : t < \gamma_\chi \leq t + 1\} \ll \log Q(t + 2). \]

Thus
\[ \sum_{0 < \gamma_\chi \leq T} L(\lambda, \alpha, \rho_\chi) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{L'(s, \chi)L(\lambda, \alpha, s)}{L(s, \chi)L(\lambda, \alpha, s)} ds + O(\log Q), \]

where the contour \( \mathcal{C} \) is a rectangle with vertices \( a + ib, a + iT, 1 - a + iT, \) and \( 1 - a + ib, \)
where \( a = 1 + 1/\log(QT), 2 \leq b \leq 3, \) min_{\gamma_\chi} |b - \gamma_\chi| \gg 1/\log Q. Further
\[ \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{L'(s, \chi)L(\lambda, \alpha, s)}{L(s, \chi)L(\lambda, \alpha, s)} ds \]
\[ = \frac{1}{2\pi i} \left\{ \int_{a + iT}^{a + iT} + \int_{1 - a + iT}^{1 - a + iT} + \int_{1 - a + ib}^{a + ib} \right\} \frac{L'(s, \chi)L(\lambda, \alpha, s)}{L(s, \chi)L(\lambda, \alpha, s)} ds \]
\[ =: \sum_{j=1}^{4} J_j. \]
First we obtain
\[
J_1 = \frac{1}{2 \pi} \int_{a+ib}^{a+iT} \frac{L'(s, \chi)L(\lambda, \alpha, s)}{L(s, \chi)} ds
\]
\[
= \frac{1}{2 \pi} \int_b^T \frac{L'(a + it, \chi)L(\lambda, \alpha, a + it)}{L(a + it, \chi)} dt
\]
\[
= - \sum_{m=2}^{\infty} \sum_{n=0}^{\infty} \frac{\chi(m)\Lambda(m)e(\lambda n)}{(m(n + \alpha))^a} \frac{1}{2 \pi} \int_b^T \frac{1}{(m(n + \alpha))^{it}} dt
\]
\[
= - \Lambda \left( \frac{1}{\alpha} \right) \chi \left( \frac{1}{\alpha} \right) \frac{T}{2 \pi} + O(\log^2(QT)).
\]

Second we consider \( J_2 \). Formula (6) from [8] gives (recall that \( T \) satisfies (3))
\[
\frac{L'}{L} (\sigma + iT, \chi) \ll \log^2(QT) \quad \text{for} \quad -1 \leq \sigma \leq 2, T \geq 2.
\]
By Corollary 4 from [3] we have
\[
L(\lambda, \alpha, \sigma + iT) \ll T^{\frac{1}{2}} \log T \quad \text{for} \quad -\frac{1}{\log(QT)} \leq \sigma \leq 1 + \frac{1}{\log(QT)}.
\]
Thus we get
\[
J_2 = \frac{1}{2 \pi} \int_{a+iT}^{1-a+iT} \frac{L'(s, \chi)L(\lambda, \alpha, s)}{L(s, \chi)} ds = O \left( T^{\frac{1}{2}} \log^3(QT) \right).
\]
Similarly we see that
\[
J_4 = O(\log^3 Q).
\]
A change of variables \( s \mapsto 1 - \overline{s} \) gives
\[
J_3 = - \frac{1}{2 \pi} \int_{a+iT}^{a+iT} \frac{L'(1 - \overline{s}, \chi)L(\lambda, \alpha, 1 - \overline{s})}{L(s, \chi)} ds.
\]
After conjunction we get
\[
\overline{J_3} = - \frac{1}{2 \pi} \int_{a+iT}^{a+iT} \frac{L'(1 - s, \overline{\chi})L(\lambda, \alpha, 1 - \overline{s})}{L(s, \overline{\chi})} ds.
\]
To evaluate this integral we will use functional equations. Dirichlet \( L \)-function to a primitive character \( \chi \mod Q \) satisfies the functional equation (Apostol [1, Theorem 12.11])
\[
L(1 - s, \chi) = \Delta(1 - s, \chi)L(s, \overline{\chi}),
\]
where
\[
\Delta(1 - s, \chi) = \tau(\psi) \frac{1}{Q} \left( \frac{Q}{2 \pi} \right)^s \Gamma(s) \left( \exp \left( -\frac{\pi is}{2} \right) + \psi(-1) \exp \left( \frac{\pi is}{2} \right) \right).
\]
Taking the logarithmic derivative of the functional equation we get
\[
\frac{L'}{L} (1 - s, \chi) = \Delta' \frac{1}{\Delta}(1 - s, \chi) - \frac{L'}{L}(s, \overline{\chi}),
\]
where

\begin{equation}
\frac{\Delta'(s,\chi)}{\Delta(1-s,\overline{\chi})} = \frac{\Delta'(1-s,\overline{\chi})}{\Delta} = -\log \frac{tQ}{2\pi} + O\left(\frac{1}{t}\right),
\end{equation}

for \( t > 1 \). The Lerch zeta-function satisfies the functional equation (Lerch [16] or [15])

\begin{equation}
L(\lambda,\alpha,1-s) = (2\pi)^{-s}\Gamma(s) \left( e \left( \frac{s}{4} - \alpha\lambda \right) L(1-\alpha,\lambda,s) + e \left( -\frac{s}{4} + \alpha(1-\{\lambda\}) \right) L(\alpha,1-\{\lambda\},s) \right).
\end{equation}

Therefore

\begin{equation*}
\frac{L'(1-s,\overline{\chi})}{L}(1-s,\overline{\chi})L(\lambda,\alpha,1-s) = \left( \frac{\Delta'(s,\chi)}{\Delta} - \frac{L'(s,\chi)}{L} \right) (2\pi)^{-s}\Gamma(s) \left( e \left( -\frac{s}{4} + \alpha\lambda \right) L(\alpha,\lambda,s) + e \left( \frac{s}{4} - \alpha(1-\{\lambda\}) \right) L(1-\alpha,1-\{\lambda\},s) \right).
\end{equation*}

Then

\begin{align*}
\mathcal{J}_3 &= -e(\alpha\lambda) \frac{1}{2\pi i} \int_{a+ib}^{a+iT} \frac{\Delta'(s,\chi)(2\pi)^{-s}\Gamma(s)}{\Delta(s,\chi)} e \left( -\frac{s}{4} \right) L(\alpha,\lambda,s) ds \\
&\quad - e(-\alpha(1-\{\lambda\})) \frac{1}{2\pi i} \int_{a+ib}^{a+iT} \frac{\Delta'(s,\chi)(2\pi)^{-s}\Gamma(s)}{\Delta(s,\chi)} e \left( \frac{s}{4} \right) L(1-\alpha,1-\{\lambda\},s) ds \\
&\quad + e(\alpha\lambda) \frac{1}{2\pi i} \int_{a+ib}^{a+iT} (2\pi)^{-s}\Gamma(s) e \left( -\frac{s}{4} \right) \frac{L'(s,\chi)}{L} L(\alpha,\lambda,s) ds \\
&\quad + e(-\alpha(1-\{\lambda\})) \frac{1}{2\pi i} \int_{a+ib}^{a+iT} (2\pi)^{-s}\Gamma(s) e \left( \frac{s}{4} \right) \frac{L'(s,\chi)}{L} L(1-\alpha,1-\{\lambda\},s) ds \\
&= \sum_{j=1}^{4} \mathcal{F}_j,
\end{align*}

say. By the bound (7), Stirling’s formula, and by the bound ([5, Corollary 2])

\begin{equation}
L(\lambda,\alpha,a+it) \ll \log t
\end{equation}

we have

\[ \mathcal{F}_2, \mathcal{F}_4 \ll 1. \]

Summarising obtained results we see that, for \( 0 < \lambda \leq 1, 0 < \alpha \leq 1, \) and \( Q \geq 1, \)

\begin{equation}
\sum_{0 < \gamma \leq T} L(\lambda,\alpha,\rho_{\chi}) = -\Lambda \left( \frac{1}{\alpha} \right) \chi \left( \frac{1}{\alpha} \right) \frac{T}{2\pi} + \mathcal{F}_1 + \mathcal{F}_3 + O(T^{1/2+\epsilon}).
\end{equation}

2.2. **Gonek’s lemma.** For \( \mathcal{F}_1 \) and \( \mathcal{F}_3 \) we will use the following version of Gonek’s Lemma (c.f. Gonek [12, Lemma 5] and [3, Lemma 2]).
Lemma 2. Assume that \( \sum_{m \leq x} |a_m| \ll x \) and \( b_n \ll 1 \). Let \( 1 < c \leq 1 + 1/\log \tau \) and \( 0 < \delta < 1 \), then

\[
\frac{1}{2\pi i} \int_{c-i\tau}^{c+i\tau} \Delta(1 - s) \sum_{m=1}^{\infty} \frac{a_m}{m^s} \sum_{n=0}^{\infty} \frac{b_n}{(n + \alpha)^s} ds
\]

\[=
\sum_{m \geq 1, n \geq 0 \atop m(n + \alpha) \leq \frac{\tau}{\pi}} a_m b_n e(-m(n + \alpha)) + O\left(\tau^{\frac{1}{2}}(c - 1)^{-2}\right)
\]

uniformly in \( \alpha \in [\delta, 1] \). Here

\[
\Delta(1 - s) = 2(2\pi)^{-s} \Gamma(s) \cos \frac{\pi s}{2}
\]

is a factor from the functional equation \( \zeta(1 - s) = \Delta(1 - s)\zeta(s) \).

Proof. This is Lemma 5 in [11]. \(\square\)

Let \( \delta = \pm 1 \). To apply Lemma 2 for our purposes note that

\[
e\left(\alpha\right) \frac{\delta}{2} \cos \frac{s\pi}{2} = \begin{cases} O\left| \exp\left(-\pi |t|\right) \right| & \text{if } \delta t \geq 0, \\
1 + O\left| \exp\left(-\pi |t|\right) \right| & \text{otherwise}. \end{cases}
\]

We turn to the integral \( F_1 \). We rewrite \( F_1 \) in the following form

\[
F_1 = -e(\alpha \lambda) \int_b^T \frac{\Delta'}{\Delta}(a + i\tau, \chi) d\left(\frac{1}{2\pi i} \int_{a+i}^{\alpha+i\tau} (2\pi)^{-s} \Gamma(s) e\left(-\frac{s}{4}\right) L(\alpha, \lambda, s) ds\right).
\]

In view of (11) and the bound (9) Lemma 2 gives that

\[
\frac{1}{2\pi i} \int_{a+i}^{\alpha+i\tau} (2\pi)^{-s} \Gamma(s) e\left(-\frac{s}{4}\right) L(\alpha, \lambda, s) ds = \sum_{0 \leq n \leq \frac{T}{\pi} - \lambda} e(\alpha n) e(-n + \lambda) + O(\tau^{\frac{1}{2} + \epsilon})
\]

\[= e(-\lambda) \sum_{0 \leq n \leq \frac{T}{\pi} - \lambda} e(\alpha n) + O(\tau^{\frac{1}{2} + \epsilon}) = O(\tau^{\frac{1}{2} + \epsilon}).
\]

Then using the expression (7) we have, for \( \alpha \neq 1 \),

\[
F_1 = -e(\alpha \lambda) \int_b^T \frac{\Delta'}{\Delta}(a + i\tau, \chi) d\left(O(\tau^{\frac{1}{2} + \epsilon})\right) = O(T^{1/2 + \epsilon})
\]

and, for \( \alpha = 1 \),

\[
F_1 = \frac{T}{2\pi} \log \frac{TQ}{2\pi e} + O(T^{1/2 + \epsilon}).
\]

Next we consider the integral \( F_3 \). Again by formula (11) and Lemma 2 we get

\[
F_3 = -e(\alpha \lambda) \sum_{m \geq 1, n \geq 0 \atop m(n + \alpha) \leq \frac{\tau}{\pi}} \Lambda(m) \chi(m) e(\alpha n) e(-m(n + \lambda)) + O(T^{1/2 + \epsilon}).
\]
We split the last sum into two sums with \((m, q) = 1\) and \((m, q) > 1\) accordingly. For the case \((m, q) > 1\) we obtain, for \(\alpha \neq 1\),

\[
\sum_{m \geq 1, n \geq 0, \frac{m}{n} \leq \frac{T}{n + \lambda}} \Lambda(m)\chi(m)e(\alpha n)e(-m\lambda) = O\left(\sum_{m \leq \frac{T}{\pi}, (m, q) > 1} \Lambda(m)\right) = O(\log T)
\]

and, for \(\alpha = 1\),

\[
\sum_{m \geq 1, n \geq 0, \frac{m}{n} \leq \frac{T}{\pi}} \Lambda(m)\chi(m)e(\alpha n)e(-m\lambda) = \frac{T}{2\pi} \sum_{m \leq \frac{T}{\pi}, (m, q) > 1} \Lambda(m)\chi(m)e(-m\lambda) \frac{1}{m} + O(\log T)
\]

\[
= \frac{T}{2\pi} \sum_{p | q} \log p \sum_{j=1}^{\infty} \frac{\chi(p^j)e(-p^j\lambda)}{p^j} + O(\log T).
\]

Now we use that \(\lambda = \frac{k}{q}\). If \((m, q) = 1\) then the orthogonality relation for Dirichlet characters gives

\[
\Lambda(m)\chi(m)e\left(-\frac{mk}{q}\right) = \frac{1}{\phi(q)} \sum_{\psi \mod q} \psi(a) e\left(-\frac{k}{q}\right) \Lambda(m)\chi(m)\psi(m)
\]

\[
= \frac{1}{\phi(q)} \sum_{\psi \mod q} G(-k, \overline{\psi}) \Lambda(m)\chi(m)\psi(m),
\]

where \(G(-k, \overline{\psi})\) is the Gauss sum defined before Theorem 1. Then, for \(0 < \alpha \leq 1\),

\[
\mathcal{F}_3 = -\frac{e(\alpha\lambda)}{\phi(q)} \sum_{\psi \mod q} G(-k, \overline{\psi}) \sum_{m \geq 1, n \geq 0, \frac{m}{n} \leq \frac{T}{\pi}} \Lambda(m)\chi(m)\psi(m)e(\alpha n)
\]

\[
- (\alpha - \{\alpha\})e(\lambda) \frac{T}{2\pi} \sum_{p | q} \log p \sum_{j=1}^{\infty} \frac{\chi(p^j)e(-p^j\lambda)}{p^j} + O(T^{1/2+\varepsilon}).
\]

Again, we summarize obtained results. In view of formula \([10]\) Subsection 2.2 gives, for \(0 < \lambda \leq 1, 0 < \alpha \leq 1, \) and \(Q \geq 1\),

\[
\sum_{0 < \gamma \leq T} L(\lambda, \alpha, \rho_\gamma) = (\alpha - \{\alpha\}) \frac{T}{\pi} \log \frac{TQ}{2\pi e} - \Lambda\left(\frac{1}{\alpha}\right) \chi\left(\frac{1}{\alpha}\right) \frac{T}{2\pi} + \mathcal{F}_3 + O(T^{1/2+\varepsilon}).
\]

2.3. Perron’s formula. In Titchmarsh \([23]\), Lemma 3.12 Perron’s formula for ordinary Dirichlet series is proved. Easy to check that the analogous lemma is true for Dirichlet series \(L'(s, \chi\psi)L(\alpha, \lambda, s)\). Note that Perron \([21]\) proved his formula (without error term) for general Dirichlet series. Therefore, for the second sum of the formula \([12]\), Perron’s formula
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\[ \sum_{m \geq 1, n \geq 0, m(n+\lambda) \leq \frac{T}{2\pi}} \Lambda(m)\chi(m)\psi(m)e(\alpha n) = -\frac{1}{2\pi i} \int_{a-iU}^{a+iU} \frac{L'(s, \chi\psi)}{L(s, \chi\psi)} L(\alpha, \lambda, s) \left( \frac{T}{2\pi} \right)^s \frac{ds}{s} + O\left( \frac{T \log^2 T}{U} \right). \]

In [22, Chapter 8, Theorem 6.2] considering \([q, Q] \ll \log^B T\) with \(B\) being any positive constant we find

\[ L(s, \chi\psi) \neq 0 \quad \text{for} \quad \sigma > 1 - \frac{c}{\log \frac{4}{3} + \epsilon} T, \]

where \(c\) is an absolute positive constant. With regard to this zero-free region for \(L(s, \chi\psi)\) let \(b_1 = 1 - c/\log(\frac{4}{3} + \epsilon) T\). Shifting the line of integration we get

\[
\begin{align*}
\sum_{m \geq 1, n \geq 0, m(n+\lambda) \leq \frac{T}{2\pi}} & \Lambda(m)\chi(m)\psi(m)e(\alpha n) = -\text{Res}_{s=1} \frac{L'(s, \chi\psi)}{L(s, \chi\psi)} L(\alpha, \lambda, s) \left( \frac{T}{2\pi} \right)^s \frac{1}{s} \\
&+ \frac{1}{2\pi i} \left\{ \int_{a+iU}^{b_1+iU} + \int_{b_1+iU}^{b_1-iU} + \int_{b_1-iU}^{a-iU} \right\} \frac{L'(s, \chi\psi)}{L(s, \chi\psi)} L(\alpha, \lambda, s) \left( \frac{T}{2\pi} \right)^s \frac{ds}{s} \\
&+ O\left( \frac{T \log^2 T}{U} \right) \\
=: & -\text{Res}_{s=1} \frac{L'(s, \chi\psi)}{L(s, \chi\psi)} L(\alpha, \lambda, s) \left( \frac{T}{2\pi} \right)^s \frac{1}{s} + \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 + O\left( \frac{T \log^2 T}{U} \right).
\end{align*}
\]

Recall that the character \(\chi\) is primitive. Thus if \(\chi\psi\) is a principal character mod\([q, Q]\), then \(Q|q\) and

\[ \psi = \chi\psi_0, \]

where \(\psi_0\) is a principle character mod\(q\). In view of this we introduce the notation

\[
\delta(Q, q) = \begin{cases} 
1 & \text{if } Q|q, \\
0 & \text{otherwise}.
\end{cases}
\]

The Dirichlet \(L\)-function attached to the principal character has a simple pole at \(s = 1\), otherwise it is an entire function. In view of \(L(s, \psi_0) = \zeta(s) \prod_{p|q} (1 - p^{-s})\) we have

\[ \frac{L'(s, \psi_0)}{L(s, \psi_0)} = -\frac{1}{s-1} + \gamma + \sum_{p|q} \frac{\log p}{p^s - 1} + O(s-1) \quad (s \to 1), \]

here \(\gamma\) is the Euler-Mascheroni constant. For \(0 < \alpha < 1\), the function \(L(\alpha, \lambda, s)\) is entire and, for \(\alpha = 1\), the function \(L(\alpha, \lambda, s) = \zeta(s, \lambda)\) has a simple pole at \(s = 1\). It is known (Ivić [13, formula (1.122)]) that

\[ \zeta(s, \lambda) = \frac{1}{s-1} + \gamma(\lambda) + O(s-1) \quad (s \to 1), \]
where
\[
\gamma(\lambda) = \lim_{N \to \infty} \left( \sum_{m=0}^{N} \frac{1}{m + \lambda} - \log(N + \lambda) \right) = - \frac{\Gamma'(\lambda)}{\Gamma(\lambda)}.
\]
The last equality can be found in Wilton [24]. Note that \(\gamma(1)\) is the Euler-Mascheroni constant. Thus, for \(\alpha \neq 1\),
\[
- \text{Res}_{s=1} \frac{L'(s, \psi_0)}{L(s, \lambda)} L(\alpha, \lambda, s) \left( \frac{T}{2\pi} \right)^s \frac{1}{s} = L(\alpha, \lambda, 1) \frac{T}{2\pi}.
\]
and, for \(\alpha = 1\),
\[
- \text{Res}_{s=1} \frac{L'(s, \chi \psi)}{L(s, \lambda)} L(\alpha, \lambda, s) \left( \frac{T}{2\pi} \right)^s \frac{1}{s} = - \frac{L'(1, \chi \psi)}{L(1, \chi \psi)} \frac{T}{2\pi}.
\]
Integrals \(K_1, K_2,\) and \(K_3\) are analogous to the integrals which appear in the formula (9) of [3]. Reasoning similarly as in [3], using bounds (4), (5), and choosing \(U = T^{1-b_1}\), we get that
\[
K_j \ll T^{1 - \frac{1}{2\pi j + \frac{1}{4}}} T^c,
\]
for \(j = 1, 2, 3\). Then in view of formulas (12) – (15) the residue (16) gives the expression (1) of Theorem 1. Finally, residues (17) and (18) give the expression (2) with the constant
\[
C(\lambda) = - \delta(Q, q) e(-\lambda) \frac{\mu(q)}{\phi(q)} \left( \frac{\Gamma'(\lambda)}{\Gamma(\lambda)} + \gamma + \sum_{\nu|q \nu_0} \log \nu \right) + \frac{e(-\lambda)}{\phi(q)} \sum_{\psi \bmod q \psi \neq \psi_0} G(k, \psi) \frac{L'(1, \chi \psi)}{L(1, \chi \psi)}
\]
\[
- e(-\lambda) \sum_{\nu|q} \log \nu \sum_{j=1}^{\infty} \frac{\chi(p^j)}{p^j} e(p^j \lambda).
\]
By this Theorem 1 is proved.

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**References**


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