Approximation of the Lerch zeta-function

Ramūnas Garunkštis*,
Department of Mathematics and Informatics,
Vilnius University, Naugarduko 24,
03225 Vilnius, Lithuania,
ramunas.garunkstis@maf.vu.lt

Abstract. We consider uniform in parameters approximations of the Lerch zeta-function by Dirichlet polynomials. It allows us to obtain uniform in parameters bounds in the critical strip.

Let $s = \sigma + it$ be a complex variable. For $\sigma > 1$, the Lerch zeta-function is given by the series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s},$$

with parameters $\lambda \in \mathbb{R}$ and $0 < \alpha \leq 1$. This function can be analytically continued to the whole complex plane with a possible exception of a pole at the point $s = 1$. Note that $L(\lambda, \alpha, s)$ is periodic in λ.

In [2], Chapter 3, we see that, for fixed $0 < \lambda < 1$ and for $\sigma \geq \sigma_0 > 0$, $|t| \leq \pi \lambda x$, the following approximation holds,

$$L(\lambda, \alpha, s) = \sum_{m=0}^{x} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s} + O\left(x^{-\sigma}\right).$$

In this paper we present a uniform in λ and α approximation of the above type. We also calculate an explicit constant in the error term. Let $\Theta(z)$ denote some complex number such that $|\Theta(z)| \leq |z|$.

*Partially supported by Grant from Lithuanian Foundation of Studies and Science.

Theorem 1 *Suppose that $-1/2 \le \lambda \le 1/2$, $\lambda \ne 0$, and let $\sigma \ge 0$, $|t| \le \pi|\lambda|x$. Then*

$$L(\lambda,\alpha,s) = \sum_{0\le m\le x} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s} + \Theta\left(\left(\frac{4}{\pi|\lambda|} + 10.5\right)\frac{1}{x^\sigma}\right).$$

We give the proof at the end of the paper.

By the estimate

$$\begin{aligned}
\sum_{1\le m\le \frac{t}{\pi|\lambda|}} \frac{1}{(m+\alpha)^\sigma} &\le \frac{1}{(1+\alpha)^\sigma} + \int_1^{\frac{t}{\pi|\lambda|}} \frac{dx}{(x+\alpha)^\sigma} \\
&\le \begin{cases} \frac{1}{(1-\sigma)}\left(\frac{t}{\pi|\lambda|}+\alpha\right)^{1-\sigma} & \text{if } 0\le\sigma\le 1-\frac{1}{\log\frac{t}{\pi|\lambda|}}, \\ e\log\left(\frac{t}{\pi|\lambda|}+\alpha\right) & \text{if } \sigma \ge 1-\frac{1}{\log\frac{t}{\pi|\lambda|}}, \end{cases}
\end{aligned}$$

we obtain the following:

Corollary 2 *Suppose that $-1/2 \le \lambda \le 1/2$, $\lambda \ne 0$, and let $\sigma \ge 0$. Then*

$$|L(\lambda,\alpha,s)| \le \frac{1}{\alpha^\sigma} + \frac{7}{|\lambda|t^\sigma} + \begin{cases} \frac{1}{(1-\sigma)}\left(\frac{t}{\pi|\lambda|}+\alpha\right)^{1-\sigma} & \textit{if } 0\le\sigma\le 1-\frac{1}{\log\frac{t}{\pi|\lambda|}}, \\ e\log\left(\frac{t}{\pi|\lambda|}+\alpha\right) & \textit{if } \sigma \ge 1-\frac{1}{\log\frac{t}{\pi|\lambda|}}. \end{cases}$$

To find an asymptotic dependence on the parameters λ and α, we need an approximate functional equation.

Theorem 3 *Let $0 < \lambda \le 1$, $0 < \alpha \le 1$, and $0 < \sigma \le 1$. Moreover, let $t \ge 1$, $y = (t/(2\pi))^{1/2}$, $q = [y]$, $k = [y-\alpha]$, and $\beta = q-k$.*

$$\begin{aligned}
L(\lambda,\alpha,s) &= \sum_{m=0}^{k} \frac{e(\lambda m)}{(m+\alpha)^s} \\
&+ \left(\frac{2\pi}{t}\right)^{\sigma-\frac12+it} e^{it+\frac{\pi i}{4}-2\pi i\lambda\alpha}\left(\sum_{m=0}^{q}\frac{e(-\alpha m)}{(m+\lambda)^{1-s}} - \frac{e^{-\pi t+\pi i\sigma+2\pi i\alpha}}{(1-\{\lambda\})^{1-s}}\right) \qquad (1) \\
&+ \left(\frac{2\pi}{t}\right)^{\frac{\sigma}{2}} e^{if(\lambda,\alpha,\sigma,t)}\psi(2y-2q-k-\{\lambda\}-\alpha) + O(t^{\frac{\sigma-2}{2}}),
\end{aligned}$$

where

$$\begin{aligned}
f(\lambda,\alpha,t) &= -\frac{t}{2\pi}\log\frac{t}{2\pi e} - \frac{7}{8} + \frac12(\alpha^2-\{\lambda\}^2) \\
&- \alpha\beta + 2y(\beta+\{\lambda\}-\alpha) - \frac12(q+k) - \{\lambda\}(\beta+\alpha)
\end{aligned}$$

and

$$\psi(a) = \frac{\cos(\pi(a^2/2 - a - 1/8))}{\cos(\pi a)}.$$

Formula (1) immediately follows from formula (1.11) of Chapter 4 of [2]. Though not mentioned in [2], one can see from the proof of (1.11) that formula (1) holds uniformly in λ and α.

From Theorem 3 we derive the following:

Corollary 4 *Let* $0 < \lambda, \alpha \leq 1$, $\sigma \geq 0$, *and* $t \geq 1$. *Then*

$$L(\lambda, \alpha, s) - \frac{1}{\alpha^s} - \left(\frac{2\pi}{t}\right)^{\sigma - \frac{1}{2} + it} e^{it + \frac{\pi i}{4} - 2\pi i \lambda\alpha} \left(\frac{1}{\lambda^{1-s}} - \frac{e^{-\pi t + \pi i\sigma + 2\pi i\alpha}}{(1 - \{\lambda\})^{1-s}}\right)$$
$$\ll \begin{cases} \frac{1}{(1-\sigma)} t^{\frac{1-\sigma}{2}} & \text{if } 0 < \sigma \leq 1 - \frac{1}{\log t}, \\ \log t & \text{if } \sigma \geq 1 - \frac{1}{\log t}, \end{cases}$$

uniformly in λ *and* α.

For negative t, we can use the formula $\overline{L(\lambda, \alpha, \sigma + it)} = L(1 - \lambda, \alpha, \sigma - it)$.

To prove Theorem 1 we need the following:

Lemma 5 *Let* $f(x)$ *be a real-valued function on* $[a, b]$ *such that* $f'(x)$ *is continuous and monotonic on* $[a, b]$ *and* $|f'(x)| \leq \delta < 1$. *Then*

$$\sum_{a<n\leq b} e(f(n)) = \int_a^b e(f(x))\, dx + \Theta\left(\frac{4\sqrt{2}\delta}{\pi(1-\delta)} + \frac{6\sqrt{2}\delta}{\pi} + 3\right).$$

Here $e(x) = e^{2\pi i x}$.

Proof of this lemma can be found, for example, in Ivić [3]; concerning the explicit constant, see Lemma 9 of [1].

Proof of Theorem 1. Let, for $u > 0$,

$$S(\lambda, u) = \sum_{m \leq u} e^{2\pi i\lambda m}.$$

Then

$$S(\lambda, u) = S(\lambda, [u]) = \frac{1 - e^{2\pi i\lambda([u]+1)}}{1 - e^{2\pi i\lambda}} = O\left(|\lambda|^{-1}\right). \tag{2}$$

Now summation by parts shows,

$$\sum_{0\leq m\leq x} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s} = S(\lambda, x)(x+\alpha)^{-s} + s\int_0^x S(\lambda, u)\frac{du}{(u+\alpha)^{s+1}}.$$

Thus, we have that, for $\sigma > 1$ and any positive integer N,

$$\begin{aligned} L(\lambda, \alpha, s) &= \sum_{m=0}^{N} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s} - S(\lambda, N)(N+\alpha)^{-s} \\ &\quad + s \int_N^\infty S(\lambda, u) \frac{du}{(u+\alpha)^{s+1}}. \end{aligned}$$

Since the latter integral converges uniformly on compact subsets of the half-plane $\sigma > 0$, the above expression also remains valid for $\sigma > 0$. In view of (2),

$$\int_N^\infty S(\lambda, u) \frac{du}{(u+\alpha)^{s+1}} = O\left(N^{-\sigma}\sigma^{-1}|\lambda|^{-1}\right),$$

and, therefore, we obtain the following approximation of the function $L(\lambda, \alpha, s)$ for $\sigma > 0$:

$$\begin{aligned} L(\lambda, \alpha, s) &= \sum_{m=0}^{N} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s} - S(\lambda, N)(N+\alpha)^{-s} \\ &\quad + O\left(|s| N^{-\sigma}\sigma^{-1}|\lambda|^{-1}\right). \end{aligned} \tag{3}$$

Now let us consider the sum

$$\sum_{x<m\le N} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s} = \sum_{x<m\le N} \frac{e^{2\pi i (\lambda m - (t\log(m+\alpha))/2\pi)}}{(m+\alpha)^\sigma}.$$

Let $f(u) = \lambda u - (t\log(u+\alpha))/2\pi$. Then $f'(u) = \lambda - t/2\pi(u+\alpha)$ and

$$|f'(u)| \le |\lambda| \frac{3}{2} \le \frac{3}{4} < 1$$

for $u \in [x, N]$. Consequently, by Lemma 5 and integration by parts,

$$\begin{aligned} A(N) := \sum_{x<m\le N} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^{it}} &= \int_x^N e^{2\pi i (\lambda u - (t\log(u+\alpha))/2\pi)} du + \Theta(10.5) \\ &= \Theta\left(\frac{4}{\pi|\lambda|} + 10.5\right). \end{aligned}$$

Hence, by partial summation we find

$$\sum_{x<m\le N} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s} = \frac{A(N)}{N+\alpha} + \sigma \int_x^N \frac{A(u)}{(u+\alpha)^{\sigma+1}} du.$$

Now, in view of (3), tending N to infinity, we obtain

$$L(\lambda, \alpha, s) = \sum_{0\le m\le x} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s} + \Theta\left(\left(\frac{4}{\pi|\lambda|} + 10.5\right)\frac{1}{x^\sigma}\right).$$

The last formula is proved for $\sigma > 0$. By the continuity of the Lerch zeta-function it also remains valid for $\sigma \ge 0$.

References

[1] R. GARUNKŠTIS, *The effective universality theorem for the Riemann zeta function*, in: Proceedings of the session in analytic number theory and Diophantine equations, MPI-Bonn, January - June 2002, Ed. by D. R. Heath-Brown, B. Z. Moroz, Bonner mathematische Schriften, **360** (2003), 21 pp.

[2] A. LAURINČIKAS AND R. GARUNKŠTIS, *The Lerch Zeta-function*, Dordrecht, Kluwer Academic Publishers (2002).

[3] A. IVIĆ, *The Riemann Zeta-function*, New York, John Wiley (1985).

Reziumė. Straipsnyje nagrinėjame tolygias parametrų atžvilgiu Lerch'o dzeta funkcijos aproksimacijas Dirichlet polinomais. Tai leidžia gauti tolygius parametrų atžvilgiu įverčius kritinėje juostoje.