THE RIEMANN HYPOTHESIS AND UNIVERSALITY OF THE RIEMANN
ZETA-FUNCTION

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Abstract. We prove that, under the Riemann hypothesis, a wide class of analytic functions
can be approximated by shifts \( \zeta(s + i\gamma_k) \), \( k \in \mathbb{N} \), of the Riemann zeta-function, where \( \gamma_k \) are
imaginary parts of nontrivial zeros of \( \zeta(s) \).

1. Introduction

The Riemann zeta-function \( \zeta(s) \), \( s = \sigma + i\tau \), is defined, for \( \sigma > 1 \), by the series
\[
\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},
\]
and can be meromorphically continued to the whole complex plane with the unique simple pole
at the point \( s = 1 \). The distribution of zeros occupies the central place in the theory of \( \zeta(s) \), and
plays an important role in various applications. The zeros \( s = -2m, m \in \mathbb{N} \), are called trivial,
they come from the functional equation
\[
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s),
\]
where \( \Gamma(s) \) is the Euler gamma-function.Moreover, the function \( \zeta(s) \) has infinitely many the so-called non-trivial zeros which are complex and are located in the critical strip \( \{ s \in \mathbb{C} : 0 < \sigma < 1 \} \).

The Riemann hypothesis (RH) asserts that all non-trivial zeros of \( \zeta(s) \) lie in the critical line \( \sigma = \frac{1}{2} \).

At the moment, it is known [4] that at least 41 percent of all non-trivial zeros in the sense of density
are on the critical line.

The function \( \zeta(s) \) is one of remarkable analytic objects and has a series of interesting properties.
One of them is universality discovered by S.M. Voronin in [17]. He proved that a wide class of
analytic functions can be approximated by shifts \( \zeta(s + i\tau) \), \( \tau \in \mathbb{R} \). More precisely, Voronin obtained
that if the function \( f(s) \) is continuous and non-vanishing in the disc \( |s| \leq r \), \( 0 < r < \frac{1}{4} \), and analytic
in the interior of this disc, then, for every \( \varepsilon > 0 \), there exists a real number \( \tau = \tau(\varepsilon) \) such that
\[
\max_{|s| \leq r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon.
\]

The Voronin theorem is high estimated by number theorists, it is improved and extended for other
zeta and \( L \)-functions. Let \( D = \{ s \in \mathbb{C} : \frac{1}{4} < \sigma < 1 \} \). Denote by \( \mathcal{K} \) the class of compact subsets
of the strip \( D \) with connected complements, and by \( H_0(K) \) with \( K \in \mathcal{K} \) the class of continuous
non-vanishing functions on \( K \) which are analytic in the interior of \( K \). Let \( \text{meas} A \) stand for the
Lebesgue measure of a measurable set \( A \subset \mathbb{R} \). Then the modern version of the Voronin theorem
has the following form.

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Theorem 1.1. Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$, then, for every $\varepsilon > 0$,
\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s+i\tau) - f(s)| < \varepsilon \right\} > 0.
\]

The theorem shows that there are infinitely many shifts $\zeta(s+i\tau)$ approximating a given function: the set of these shifts has a positive lower density.

Different proofs of Theorem 1.1 were proposed by S.M. Gonek [8] and B. Bagchi [1] (for slightly different sets). The proof can be also found in [10] and [14].

Theorem 1.1 is of continuous type, the shifts $\tau$ in $\zeta(s+i\tau)$ can take arbitrary real values. Also, a discrete universality theorem for $\zeta(s)$ is known. In this case, $\tau$ takes values from a certain discrete set, for example, from arithmetical progression $\{kh : k = 0, 1, 2, \ldots\}$, where $h > 0$ is a fixed number. Let #A denote the cardinality of the set A. Then the discrete version of Theorem 1.1 has the following form.

Theorem 1.2. Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$, then, for every $\varepsilon > 0$,
\[
\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 1 \leq k \leq N : \sup_{s \in K} |\zeta(s+ikh) - f(s)| < \varepsilon \right\} > 0.
\]

Discrete universality for zeta-functions was proposed by A. Reich. In [12], he obtained a discrete universality theorem for Dedekind zeta-functions. Theorem 1.2 in a slightly different form was proved in [1], and, by different way, in [13].

Discrete universality theorems for zeta-functions in some sense are more convenient for practical applications. For example, a discrete version of the Voronin theorem was applied [3] for estimation of complicated integrals over analytic curves used in quantum mechanics.

A problem arises to replace the set $\{kh\}$ by more interesting sets. In [6], this was done for the set $\{k^\alpha h\}$ with a fixed $0 < \alpha < 1$. Let $\gamma_1 \leq \gamma_2 \leq \cdots$ be the ordinates of non-trivial zeros of $\zeta(s)$.

The aim of this paper is to prove that the shifts $\zeta(s+i\gamma_k)$, $k = 1, 2, \ldots$, approximate the functions from the class $H_0(K)$.

Theorem 1.3. Suppose that RH is true. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,
\[
\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 1 \leq k \leq N : \sup_{s \in K} |\zeta(s+i\gamma_k) - f(s)| < \varepsilon \right\} > 0.
\]

Proof of Theorem 1.3 is probabilistic, it is based on limit theorems for weakly convergent probability measures in the space of analytic functions. For this, discrete moments of $\zeta(s)$ are needed.

2. Discrete moments of $\zeta(s)$

Lemma 2.1. Suppose that RH is true. Then, for fixed $\frac{1}{2} < \sigma < 1$ and all $t \in \mathbb{R}$,
\[
\sum_{k=1}^{N} |\zeta(\sigma+i\gamma_k+it)| \ll N(1+|t|).
\]

Proof. It is known [16] that
\[
\gamma_n \sim \frac{2\pi n}{\log n}, \quad n \to \infty. \tag{2.1}
\]

By [7] we have that, under RH,
\[
\sum_{\gamma \leq T} |\zeta(\sigma+i\gamma+it)|^2 \ll T \log T(1+|t|).
\]

Therefore, the lemma follows from (2.1). \qed

We note that RH is used only in Lemma 2.1.
3. A LIMIT THEOREM

Let $\hat{\gamma}$ be the unit circle $\{s \in \mathbb{C} : |s| = 1\}$, and
\[ \Omega = \prod_p \hat{\gamma}_p, \]
where $\hat{\gamma}_p = \hat{\gamma}$ for all primes $p$. With the product topology and pointwise multiplication, the infinite-dimensional torus $\Omega$ is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ ($\mathcal{B}(X)$ stands for the Borel $\sigma$-field of the space $X$), the probability Haar measure $m_H$ exists, and we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ be the projection of an element $\omega \in \Omega$ to the circle $\hat{\gamma}_p$. Denote by $H(D)$ the space of analytic functions on $D$ endowed with the topology of uniform convergence on compacta, and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s, \omega)$ by the formula
\[ \zeta(s, \omega) = \prod_p \left( 1 - \frac{\omega(p)}{p^s} \right)^{-1}. \]
We observe that the above product, for almost all $\omega \in \Omega$, converges uniformly on compact subsets of the strip $D$, and thus define the $H(D)$-valued random element $\zeta(s, \omega)$ by the formula

We start the proof of Theorem 3.1 with some lemmas. First we remind that the sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is called uniformly distributed modulo 1 if, for every interval $I = [a, b) \subset [0, 1)$ of length $|I|$,}
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_I(x_k) = |I|, \]
where $\chi_I$ is the indicator function of the interval $I$, and $\{u\}$ denotes the fractional part of a real $u$. For sequences uniformly distributed modulo 1, the Weyl criterion is true, see, for example, [9].

Lemma 3.1. A sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is uniformly distributed modulo 1 if and only if, for all $m \in \mathbb{Z} \setminus \{0\}$,
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i k x_k m} = 0. \]

Lemma 3.2. For every real $a \neq 0$, the sequence $\{a x_k : k \in \mathbb{N}\}$ is uniformly distributed modulo 1.

Proof. The lemma follows from Theorem 1.2 of [15] and Lemma 3.1. □

Denote by $\mathbb{P}$ the set of all prime numbers.
Lemma 3.3. Let, for \( A \in B(\Omega) \),
\[
Q_N(A) = \frac{1}{N} \#\{1 \leq k \leq N : (p^{-i\gamma_k} : p \in \mathbb{P}) \in A\}.
\]
Then \( Q_N \) converges weakly to the Haar measure \( m_H \) as \( N \to \infty \).

Proof. Consider the Fourier transform \( g_N(\mathbf{k}) \), \( \mathbf{k} = (k_2, k_3, \ldots) \), of \( Q_N \) which is defined by
\[
g_N(\mathbf{k}) = \int_{\Omega} \prod_p p^{k_p} dQ_N,
\]
where only a finite number of integers \( k_p \) are distinct from zero. By the definition of \( Q_N \), we have
\[
Q_N = \frac{1}{N} \sum_{k=1}^N \prod_p p^{-i\gamma_k} \exp \left\{-i\gamma_k \sum_p k_p \log p \right\}.
\]

(3.1)

Obviously, \( g_N(0) = 1 \).

(3.2)

Since the logarithms \( \log p \) are linearly independent over the field of rational numbers,
\[
\sum_p k_p \log p \neq 0
\]
for \( \mathbf{k} \neq 0 \). Hence, in view of Lemmas 3.1 and 3.2, and (3.1),
\[
\lim_{N \to \infty} g_N(\mathbf{k}) = 0
\]
for \( \mathbf{k} \neq 0 \). This together with (3.2) shows that
\[
\lim_{N \to \infty} g_N(\mathbf{k}) = \begin{cases} 1 & \text{if } \mathbf{k} = 0, \\ 0 & \text{if } \mathbf{k} \neq 0. \end{cases}
\]

Since the right-hand side of the later equality is the Fourier transform of the measure \( m_H \), the lemma follows from a continuity theorem for probability measures on compact topological groups.

\[\square\]

Let \( \theta > \frac{1}{2} \) be a fixed number, \( v_n(m) = \exp \left\{-\left(\frac{m}{n}\right)^{\theta} \right\} \) with \( m, n \in \mathbb{N} \), and
\[
\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s},
\]
\[
\zeta_n(s, \omega) = \sum_{m=1}^{\infty} \omega(m)v_n(m) \frac{m^s}{m^s},
\]
with \( \omega \in \Omega \). Then the above series are absolutely convergent for \( \sigma > \frac{1}{2} \) [10].

Lemma 3.4. Let, for \( A \in B(H(D)) \),
\[
P_{N,n}(A) = \frac{1}{N} \#\{1 \leq k \leq N : \zeta_n(s + i\gamma_k) \in A\}.
\]
Then \( P_{N,n} \), as \( N \to \infty \), converges weakly to the measure \( \hat{P}_n = m_H u_n^{-1} \), where \( u_n : \Omega \to H(D) \) is given by \( u_n(\omega) = \zeta_n(s, \omega) \), and \( m_H u_n^{-1}(A) = m_H(u_n^{-1}A) \), \( A \in B(H(D)) \).

Proof. Since the series for \( \zeta_n(s, \omega) \) is absolutely convergent, the function \( u_n \) is continuous one. Therefore, \( u_n \) is \((B(\Omega), B(H(D)))\)-measurable, and \( \hat{P}_n \) is a probability measure on \( H((D), B(H(D))) \). Thus, the assertion of the lemma follows from Lemma 3.3 and Theorem 5.1 of [2].

\[\square\]
Let \( \{K_l : l \in \mathbb{N}\} \) be a sequence of compact subsets of the strip \( D \) such that
\[
D = \bigcup_{l=1}^{\infty} K_l,
\]
\( K_l \subset K_{l+1} \) for all \( l \in \mathbb{N} \), and if \( K \subset D \) is a compact set, then \( K \subset K_l \) for some \( l \in \mathbb{N} \). The existence of such a sequence is proved, for example, in [5]. For \( g_1, g_2 \in H(D) \), define
\[
\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \sup_{s \in K_l} |g_1(s) - g_2(s)| / \left(1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|\right).
\]
Then \( \rho \) is a metric in \( H(D) \) inducing its topology of uniform convergence on compacta.

The next lemma is devoted to the approximation of the function \( \zeta(s) \) by \( \zeta_n(s) \).

**Lemma 3.5.** Suppose that RH is true. Then
\[
\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \rho(\zeta(s+i\gamma_k), \zeta_n(s+i\gamma_k)) = 0.
\]

**Proof.** Let
\[
l_n(s) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) n^s,
\]
where the number \( \theta \) is from the definition of \( v_n(m) \). Then it is known [10] that
\[
\zeta_n(s) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z) l_n(z) \frac{dz}{z}.
\]
Then, for \( \theta < \sigma < 1 \), the residue theorem gives
\[
\zeta_n(s) - \zeta(s) = \frac{1}{2\pi i} \int_{\theta-\sigma-i\infty}^{\theta+\sigma+i\infty} \zeta(s+z) l_n(z) \frac{dz}{z} + R(s),
\]
where
\[
R(s) = \operatorname{Res}_{z=1-s} \left(\zeta(s+z) \frac{l_n(z)}{z}\right).
\]
Now let \( K \) be an arbitrary compact subset of \( D \). Then, using (3.3), we find that, for sufficiently large \( N \),
\[
\frac{1}{N} \sum_{k=1}^{N} \sup_{s \in K} |\zeta(s+i\gamma_k) - \zeta_n(s+i\gamma_k)|
\]
\[
\ll \int_{-\infty}^{\infty} |l_n(\sigma_1 - \sigma + i\tau)| \frac{1}{N} \sum_{k=1}^{2N} |\zeta(\sigma + i\gamma_k + i\tau) + i\tau)| d\tau
\]
\[
+ \frac{1}{N} \sum_{k=1}^{N} |R(\sigma + i\gamma_k + it)| ,
\]
where \( \frac{1}{2} < \sigma_1 < \sigma < 1 \), and \( t \) is bounded by a constant depending on \( K \). Clearly
\[
R(s) = \frac{l_n(1-s)}{1-s} .
\]
The estimate \( \Gamma(\sigma + it) \ll e^{-c|t|} \) with \( c > 0 \) and the convergence of the series
\[
\sum_{k=1}^{\infty} \frac{1}{l_k}
\]
show that the second term in the right-hand side of (3.4) is estimated as \( o(1) \) for \( N \to \infty \).

Therefore, in view of Lemma 2.1 and (3.4), we have that

\[
\limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \sup_{s \in K} |\zeta(s + i\gamma_k) - \zeta_n(s + i\gamma_k)| \ll \int_{-\infty}^{\infty} |\ln(\sigma_2 - \sigma + i\tau)| (1 + |\tau|) d\tau.
\]

Hence, by the definition of \( l_n(s) \),

\[
\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \sup_{s \in K} |\zeta(s + i\gamma_k) - \zeta_n(s + i\gamma_k)| = 0.
\]

This together with the definition of the metric \( \rho \) proves the lemma. \( \square \)

Proof of Theorem 3.1. We apply quite standard arguments, therefore, we omit the details. First, Lemma 3.4 and the absolute convergence of the series for \( \zeta_n(s) \) in the half plane \( \sigma > \frac{1}{2} \) imply that the family of probability measures \( \{ \hat{P}_n : n \in \mathbb{N} \} \) is tight, i.e., for every \( \varepsilon > 0 \) there exists a compact set \( K = K(\varepsilon) \subset H(D) \) such that, for all \( n \in \mathbb{N} \),

\[
\hat{P}_n(K) > 1 - \varepsilon.
\]

Hence, in view of the Prokhorov theorem, Theorem 6.1 of [2], we have that the family \( \{ \hat{P}_n \} \) is relatively compact. Thus, every sequence of \( \{ \hat{P}_n \} \) contains a subsequence \( \hat{P}_{n_r} \) which weakly converges to a certain probability measure \( P \) on \( (H(D), \mathcal{B}(H(D))) \) as \( r \to \infty \).

Let \( \theta_N \) be a random variable defined on a certain probability space \( (\hat{\Omega}, \mathcal{A}, \mu) \) and having the distribution \( \mu(\theta_N = \gamma_k) = \frac{1}{N}, \ k = 1, \ldots, N \).

On the above probability space, define the \( H(D) \)-valued random element \( X_{N,n} = X_{N,n}(s) \) by

\[
X_{N,n}(s) = \zeta_n(s + i\theta_N).
\]

Moreover, let \( \hat{X}_n = \hat{X}_n(s) \) be the \( H(D) \)-valued random element having the distribution \( \hat{P}_n \). Then, denoting by \( \overset{D}{\rightarrow} \) the convergence in distribution, by Lemma 3.4 and the above remark, we obtain the relations

\[
X_{N,n} \overset{D}{\rightarrow} \hat{X}_n \quad (3.5)
\]

and

\[
\hat{X}_{n_r} \overset{D}{\rightarrow} P \quad (3.6)
\]

Define one more random element \( X_N = X_N(s) \) by

\[
X_N(s) = \zeta(s + i\theta_N).
\]

Then the relations (3.5) and (3.6), Lemma 3.5 and Theorem 4.2 of [2] show that

\[
X_N \overset{D}{\rightarrow} P \quad (3.7)
\]

This means that \( P_N \) converges weakly to \( P \) as \( N \to \infty \). The relation (3.7) also shows that the measure \( P \) is independent of the sequence \( \hat{P}_{n_r} \). Therefore, the relation

\[
\hat{X}_n \overset{D}{\rightarrow} P
\]

is true. Consequently, we have that \( P_N \), as \( N \to \infty \), converges weakly to the measure \( P \), where \( P \) is a limit measure of \( \hat{P}_n \) as \( n \to \infty \). However, it is known [10] that

\[
\frac{1}{T} \text{meas} \{ \tau \in [0, T] : \zeta(s + i\tau) \in A \}, \ A \in \mathcal{B}(H(D)),
\]
also, as $T \to \infty$, converges weakly to the limit measure $P$ of $\hat{P}_n$ as $n \to \infty$, and $P = P_\zeta$. Therefore, $P_N$ also converges weakly to $P_\zeta$. Moreover, in [10], it is obtained that the support of $P_\zeta$ is the set $S$. The theorem is proved. 

4. PROOF OF UNIVERSALITY

Theorem 1.3 is a consequence of Theorem 3.1 and the Mergelyan theorem on approximation of analytic functions by polynomials [11].

Thus, by the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$
\sup_{s \in K} \left| f(s) - e^{p(s)} \right| < \frac{\varepsilon}{2}.
$$

Let

$$
G = \left\{ g \in H(D) : \sup_{s \in K} \left| g(s) - e^{p(s)} \right| < \frac{\varepsilon}{2} \right\}.
$$

Then $G$ is an open set in $H(D)$, and using the equivalent of weak convergence in terms of open sets, and Theorem 3.1, we find that

$$
\liminf_{N \to \infty} P_N(G) \geq P_\zeta(G). \tag{4.2}
$$

Since, $e^{p(s)} \in S$, again by Theorem 3.1, we have that

$$
P_\zeta(G) > 0. \tag{4.3}
$$

Therefore, the definition of $G$ and $P_N$, and (4.2) and (4.3) give the inequality

$$
\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq k \leq N : \sup_{s \in K} \left| \zeta(s + i\gamma_k) - e^{p(s)} \right| < \frac{\varepsilon}{2} \right\} > 0.
$$

Combining this with inequality (4.1) gives the assertion of Theorem 1.3.

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