On some inequalities concerning $\pi(x)$

R. Garunkštis*

January 2001

Abstract. Here we investigate inequalities $\pi(M+N) \leq a\pi(M/a)+\pi(N)$ and $\pi(M+N) \leq a(\pi(M/a)+\pi(N/a))$ with $a \geq 1$.

Keywords: distribution of prime numbers, prime counting function, Hardy-Littlewood’s conjecture.

AMS subject classification: 11N05, 11A41.

1 Introduction and statement of results

Let $\pi(x)$ as usual denote the number of primes not exceeding $x$. Further by $M, N, K$ and $x, y$ we mean, respectively, positive integers and positive real numbers.

The conjecture that

$$\pi(M+N) \leq \pi(M) + \pi(N)$$

for $M, N \geq 2$ takes its origin from Hardy and Littlewood [1923]. There are many results concerning this conjecture. We will mention a few of them. Schinzel and Sierpinski [1958] (see also [Schinzel 1961]) proved the inequality (1) for $2 \leq \min(M, N) \leq 146$ and from [Gordon and Rodemich 1998] it follows that inequality (1) is valid in a wider region,

$$2 \leq \min(M, N) \leq 1731$$

Dusart [1998, Theorem 2.6] obtained that if $x \leq y \leq \frac{17}{5}x \log x \log \log x$, then

$$\pi(x+y) \leq \pi(x) + \pi(y).$$

*Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, 2600 Vilnius, Lithuania, e-mail: ramunas.garunkstis@maf.vu.lt

Partially supported by Grant from Lithuanian Foundation of Studies and Science.
However, in general it is believed that (1) is not valid, as Hensley and Richards [1974] have shown that this inequality is incompatible with another Hardy-Littlewood conjecture, the so called

**Prime k-tuples conjecture:** Let $b_1 < b_2 < \ldots < b_k$ be a set of integers, such that for each prime $p$, there is some congruence class (mod $p$) which contains none of the integers $b_i$. Then there exist infinitely many integers $n > 0$ for which all of the numbers $n + b_1, \ldots, n + b_k$ are prime.

More precisely, Hensley and Richards [1974], under prime $k$-tuples conjecture, proved that

$$\limsup_{y \to \infty} (\pi(y + x) - \pi(y)) - \pi(x) \geq (\log 2 - \varepsilon) \frac{x}{\log^2 x} \quad \text{for} \quad x \geq x_0.$$  

From this it follows easy, that the inequality

$$\pi(M + N) \leq a\pi\left(\frac{M}{a}\right) + \pi(N)$$

is not valid for $1 \leq a < 2$. Under the same assumption Clark and Jarvis [2001] showed that it is not valid for $a = 2$ also.

The inequality

$$\pi(M + N) \leq 2\pi(M) + \pi(N) \quad \text{for} \quad M \geq 1, N \geq 2,$$

proved by Montgomery and Vaughan [1973], suggests some $a$ for which (3) is satisfied.

**Theorem 1** Let $M$ and $N$ are integers. If $a \geq \sqrt{M}$, then

$$\pi(M + N) \leq a\pi\left(\frac{M}{a}\right) + \pi(N)$$

for $\frac{M}{a} \geq 3$ and $N \geq 1$.

If $a \geq 2\sqrt{M}$, then this inequality is true for $\frac{M}{a} \geq 2$ and $N \geq 1$.

For $M \geq N$, a much smaller coefficient $a$ can be chosen in the inequality (3). Panaitopol [2000] proved that

$$\pi(M + N) \leq 2\pi\left(\frac{M}{2}\right) + \pi(N) \quad \text{for} \quad M \geq N \geq 2 \text{ and } M \geq 6.$$

We prove

**Theorem 2** If $M \geq N \geq 7$ are integers, then

$$\pi(M + N) \leq 1.11\pi\left(\frac{M}{1.11}\right) + \pi(N).$$
The proof of Theorem 2 requires some computer calculations, and we also make use of Dusart’s [1998; 1999] evaluations for the prime counting function:

\[ \pi(x) \geq \frac{x}{\log x - 1}, \quad \text{for } x \geq 5393, \quad (4) \]

\[ \pi(x) \leq \frac{x}{\log x - 1.1}, \quad \text{for } x \geq 60184, \quad (5) \]

\[ \pi(x) \geq \frac{x}{\log x} \left( 1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x} \right), \quad \text{for } x \geq 32299, \quad (6) \]

\[ \pi(x) \leq \frac{x}{\log x} \left( 1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x} \right), \quad \text{for } x \geq 355991. \quad (7) \]

It is easy to obtain the symmetric version of Theorem 2.

**Corollary 3** If \( M, N \geq 13 \) are integers, then

\[ \pi(M + N) \leq 1.11\pi \left( \frac{M}{1.11} \right) + 1.11\pi \left( \frac{N}{1.11} \right). \]

Udrescu [1975] has proved that (1) is ‘\( \varepsilon \)-exact’, i.e. that for any \( \varepsilon > 0 \) and any \( x, y \geq 17 \) with \( x + y \geq 1 + e^{4(1+1/\varepsilon)} \),

\[ \pi(x + y) \leq (1 + \varepsilon)(\pi(x) + \pi(y)). \]

Using estimates (6), (7) we obtain

**Theorem 4** For any \( 0 < \varepsilon < 1 \) and any \( x, y \geq 32299 \) with \( x + y \geq e^{3(1-\varepsilon/2)} + 13 \),

\[ \pi(x + y) \leq (1 + \varepsilon) \left( \pi \left( \frac{x}{1 + \varepsilon} \right) + \pi \left( \frac{y}{1 + \varepsilon} \right) \right). \]

**Acknowledgement.** The author wishes to express his thanks to Professor A. Schinzel for useful comments on the author’s talk on the subject, given at the conference ”Analytic and Probabilistic Methods in Number Theory”, arranged in honour of Professor J. Kubilius in 2001.

## 2 Proofs of theorems

To prove Theorem 1 we first obtain several auxiliary inequalities.

**Lemma 5** Let \( x \) be a real number and \( c > b \geq 1 \). Then

\[ b\pi \left( \frac{x}{b} \right) < c\pi \left( \frac{x}{c} \right) \]

for \( x > e^{\frac{4}{c} + \varepsilon} \).
Proof. The lemma follows immediately from the following result of Panaitopol [2000]: If \( a > 1 \) and \( x > e^{4(\log a)^{-2}} \) then \( \pi(ax) < a\pi(x) \).

**Lemma 6** Let \( M \) be an integer. If \( 1 \leq a \leq 12, \frac{\sqrt{M}}{a} \geq 3 \) and \( M \leq 1731 \), then

\[
\pi(M) \leq a\sqrt{M}\pi\left(\frac{\sqrt{M}}{a}\right). \tag{8}
\]

The latter inequality is also true for \( 2 \leq a \leq 12, \frac{\sqrt{M}}{a} \geq 2 \) and \( M \leq 1731 \).

**Proof.** Let \( b \geq 1, \ c \geq 0 \) and \([x]\) denotes the greatest integer not exceeding \( x \). If

\[
\pi(M) \leq b\sqrt{M}\pi\left(\frac{\sqrt{M}}{b + c}\right),
\]

then the inequality (8) is valid for \( a \in [b, b + c] \). Hence in order to prove the lemma we check with a computer the following inequalities,

\[
\pi(i) \leq (1 + 0.21j)\sqrt{i}\pi\left(\max\left(\frac{\sqrt{i}}{1 + 0.21j + 0.21}, 3\right)\right)
\]

for \( j = 0, 1, \ldots, 5; \ i = [3^2(1 + 0.21j)^2] + 1, \ldots, 1731 \) and

\[
\pi(i) \leq (2 + 0.091j)\sqrt{i}\pi\left(\max\left(\frac{\sqrt{i}}{2 + 0.091j + 0.091}, 2\right)\right)
\]

for \( j = 0, 1, \ldots, 121; \ i = [2^2(1 + 0.091j)^2] + 1, \ldots, 1731 \). By this the lemma is proved.

**Lemma 7** Let \( M \) be an integer. If \( 2.44 \leq a \leq 4 \) and \( \frac{\sqrt{1720}}{a} \leq \frac{\sqrt{M}}{a} \leq \min\left(17, e^{\frac{4}{\log 2a}}\right) \), then

\[
\frac{2M}{\log M} \leq a\sqrt{M}\pi\left(\frac{\sqrt{M}}{a}\right). \tag{9}
\]

The proof is analogous to the proof of Lemma 8. Here we check the inequalities

\[
\frac{2i}{\log i} \leq (2.44 + j)\sqrt{i}\pi\left(\frac{\sqrt{i}}{2.44 + j + 1}\right)
\]

where \( j = 0, 1; \ i = 1720, \ldots, \min\left(2.44 + j)^2172, \left[(2.44 + j)^2e^{\frac{8}{\log (2.44+j)}}\right]\right) \).

**Proof of Theorem 1.** Montgomery and Vaughan [1973] have shown that

\[
\pi(M + N) = \pi(M) + \pi(N) \leq \frac{2M}{\log M} \quad \text{for} \quad M \geq 2, N \geq 1. \tag{10}
\]
Then, in view of the inequality ([Rosser and Shoenfeld 1962])

$$\pi(x) > \frac{x}{\log x} \quad \text{for} \quad x \geq 17,$$

we have that if \( d \geq 1 \), then

$$\pi(M + N) - \pi(N) \leq \frac{M}{\log \sqrt{M}} < d\sqrt{M} \pi \left( \frac{\sqrt{M}}{d} \right) \quad \text{for} \quad M \geq 17^2d^2, \ N \geq 1. \ (11)$$

By (2) and Lemma 6 we have

$$\pi(M + N) - \pi(N) \leq \pi(M) < d\sqrt{M} \pi \left( \frac{\sqrt{M}}{d} \right) \quad \text{for} \quad M \geq 17^2d^2, \ N \geq 1. \ (12)$$

From (12) and (13), since \( 17^2d^2 \) is less than 1731 if \( 1 \leq d \leq 2.44 \), we prove the theorem for \( \sqrt{M} \leq a \leq 2.44\sqrt{M} \).

By Lemma 5 we obtain

$$\sqrt{M}\pi(\sqrt{M}) < d\sqrt{M} \pi \left( \frac{\sqrt{M}}{d} \right) \quad \text{for} \quad \sqrt{M} \geq e^{4\log d},$$

and we have already proven, that

$$\pi(M + N) - \pi(N) \leq \sqrt{M} \pi \left( \sqrt{M} \right) \quad \text{for} \quad \sqrt{M} \geq 3, \ N \geq 1.$$

By this, (12), (13), (10) and Lemma 7, in view of \( e^{4\log d} \leq \frac{1731}{d} \) if \( d \geq 4 \), we obtain the theorem for the remaining case \( a > 2.44\sqrt{M} \).

The next two lemmas will be useful in the proof of Theorem 2.

**Lemma 8** If \( x \geq y \geq 5393 \) and \( x + y \geq 60184 \), then

$$\pi(x + y) < 1.11\pi \left( \frac{x}{1.11} \right) + \pi(y)$$

**Proof.** From (4) and (5) we have

$$\pi(x + y) \geq \frac{1 + a}{\log \left( \frac{x}{1 + a} \right)} + \pi(y) - \pi(x + y)$$

$$\geq x \frac{\log (1 + \frac{x}{a}) + \log(1 + a) - 0.1}{\log \left( \frac{x}{1 + a} - 1 \right) (\log(x + y) - 1.1)} + y \frac{\log \left( 1 + \frac{x}{y} \right) - 0.1}{(\log y - 1)(\log(x + y) - 1.1)} > 0$$

when \( a \geq 0.106 \).
Lemma 9 If $M \geq 619,901$, then
\[ 1.11\pi \left( \frac{M}{1.11} \right) > \pi (M + 5393) \]

Proof. Most of the calculations below were made using a computer.

For $619,901 \leq M < 1,040,000$ we check the lemma directly. For the remaining range we will use P. Dusart’s inequalities for the prime counting function. Let us define
\[ f(x) := \frac{x}{\log_{1.11} x} \left( 1 + \frac{1}{\log_{1.11} x} + \frac{1.8}{\log^2_{1.11} x} \right) \]
and
\[ g(x) := \frac{x + 5393}{\log(x + 5393)} \left( 1 + \frac{1}{\log(x + 5393)} + \frac{2.51}{\log^2(x + 5393)} \right). \]

Then by (6) and (7), the lemma for $M \geq 1,040,000$ will follow from the inequality
\[ f(x) > g(x) \text{ if } x \geq 1,040,000. \] (14)

As $f(1,040,000) > g(1,040,000)$, it is enough to prove that, for $x \geq 1,040,000$,
\[ (f(x) - g(x))' > 0. \] (15)

After removing the denominator we see that, for $x > 5393$, inequality (15) becomes equivalent to the inequality
\[ \Delta(x) := 100 \log^4(5393 + x) \log^2 \frac{x}{1.11} - 100 \log^4 \frac{x}{1.11} \log^3(5393 + x) - 20 \log^4(5393 + x) \log \frac{x}{1.11} - 51 \log^4 \frac{x}{1.11} \log(5393 + x) - 540 \log^4(5393 + x) + 753 \log^4 \frac{x}{1.11} > 0. \] (16)

Now using
\[ \log \frac{x}{1.11} = \log x - \log 1.11, \]
\[ \log(5393 + x) =: \log x + \frac{5393 a}{x}, \text{ where } a = a(x), \text{ and } |a| \leq 1, \]
we rewrite $\Delta(x)$ as
\[ \Delta(x) = M(\log x) + R \left( \log x, \frac{a}{x} \right), \] (17)
where
\[ M(y) = 753 \log 1.11 - (3012 \log 1.11 + 51 \log 1.11) y + \\
+ (4518 \log 1.11 + 204 \log 1.11) y^2 - \\
- (3012 \log 1.11 + 306 \log 1.11 + 100 \log 1.11) y^3 + \\
+ (213 + 224 \log 1.11 + 300 \log 1.11) y^4 - \\
- (71 + 300 \log 1.11) y^5 + 100 \log(1.11) y^6, \]

and \( R(\log x, \frac{a}{x}) \) is the remaining, 'small', part of \( \Delta(x) \). If \( x \geq 1040000 \) then it is easy to compute that
\[
|R(\log x, \frac{a}{x})| = \left| \sum_{0 \leq i \leq 4} b_{ijk} \log^i 1.11 \log^j x \left( \frac{a}{x} \right)^k \right| \\
\leq \sum |b_{ijk}| \log^i 1.11 \log^j x \left( \frac{1}{x} \right)^k < 4 \times 10^6, \tag{18}
\]

where \( b_{ijk} \) are appropriate coefficients.

Considering the main part, we have that \( M'(y) > 0 \) for \( y > 2 \) and \( M(\log 1040000) > 4 \times 10^7 \). Then

\[ M(\log x) > 4 \times 10^7 \text{ for } x \geq 1040000. \]

By this and (14)–(18) we obtain the lemma for \( x \geq 1040000 \). This finishes the proof.

**Proof of Theorem 2.** From Lemma 8 it follows that the inequality of the theorem holds if \( M \geq N \geq 5393 \) and \( M + N \geq 60184 \). By Lemma 9 it also holds if \( M \geq 619901 \) and \( 7 \leq N \leq 5393 \). Computer check for the remaining cases completes the proof of the theorem.

**Proof of Corollary 3.** For \( 13 \leq M \leq N \leq 1644 \) we check the inequality of the corollary with a computer.

By (6) and (7) we have that \( 1.11\pi(N/1.11) \geq \pi(N) \) for \( N \geq 355991 \) and computer check gives that this inequality is true for \( N \geq 1644 \). Now Corollary 3 follows from Theorem 2.

We will use the following lemma in the proof of Theorem 4.

**Lemma 10** Let \( f''(x) \leq 0 \) for \( x \geq x_0 \geq 0 \) and let \( f'(x_0)x_0 \leq f(x_0) \). Then, if \( x_1, x_2 \geq x_0 \),

\[ f(x_1 + x_2) \leq f(x_1) + f(x_2). \]
Proof. Let the line \( l : y = kx + c \) cut the curve \( y = f(x) \) at points \((x_1, f(x_1))\) and \((x_2, f(x_2))\). Then the point \((x_1 + x_2, f(x_1) + f(x_2) - c)\) lies on \( l \) and, because of the concavity down of \( f(x) \), this point is above the curve \( y = f(x) \). Thus

\[
f(x_1) + f(x_2) - c \geq f(x_1 + x_2).
\]

Now we will prove that \( c \geq 0 \). Let \( x_1 \leq x_2 \) (the case \( x_1 \geq x_2 \) is analogous). By Lagrange’s theorem we obtain that there exists \( x_1 \leq \xi \leq x_2 \), such that \( k = f’(\xi) \). Then

\[
c = f(x_1) - f’(\xi)x_1.
\]

Let the line \( y = k_0x + c_0 \) be a tangent to the curve \( y = f(x) \) at \((x_0, y_0)\). By the decreasing of \( f’(x) \)

\[
c_0 = f(x_0) - f’(x_0)x_0 \leq f(x_0) - f’(\xi)x_0.
\]

Once again, by Lagrange’s theorem, we have that there exist \( x_0 \leq \xi_0 \leq x_1 \) and \( \xi_0 \leq \xi_1 \leq \xi \), such that

\[
c - c_0 \geq (f’(\xi_0) - f’(\xi))(x_1 - x_0) = f''(\xi_1)(\xi_0 - \xi)(x_1 - x_0).
\]

Thus \( c - c_0 \geq 0 \). Since \( c_0 \geq 0 \), we obtain the lemma.

**Proof of Theorem 4.** Let’s define

\[
f(x) := \frac{x}{\log \frac{x}{1+\varepsilon}} \left( 1 + \frac{1}{\log \frac{x}{1+\varepsilon}} + \frac{1.8}{\log^2 \frac{x}{1+\varepsilon}} \right)
\]

and

\[
g(x) := \frac{x}{\log x} \left( 1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x} \right).
\]

Then, if \( x \geq 32299 \),

\[
\frac{(f(x) - g(x))100}{x} \log^3 x \log^3 \frac{x}{1+\varepsilon} \geq 100 \log(1 + \varepsilon) \log^4 x - 71 \log^3 x.
\]

Thus, we have that \( f(x + y) \geq g(x + y) \), if the conditions of the theorem is satisfied. As \( f''(x) \leq 0 \) and

\[
f(x) - f’(x)x = \frac{27x}{5 \log^4 \frac{x}{1+\varepsilon}} + \frac{2x}{\log^3 \frac{x}{1+\varepsilon}} + \frac{x}{\log^2 \frac{x}{1+\varepsilon}} \geq 0,
\]

hence by Lemma 10 we see that \( f(x) + f(y) \geq f(x + y) \geq g(x + y) \). From this and (6), (7) the theorem follows.
References


[Dusart 1999] P. DUSART, Inegalites explicites pour $\psi(X)$, $\theta(X)$, $\pi(X)$ et les nombres premiers, Comptes Rendus Mathématiques de l’Académie des Sciences. La Société Royale du Canada 2 (1999), 53-59.


[Udrescu 1975] V. UDRESCU, *Some remarks concerning the conjecture* $\pi(x+y) \leq \pi(x) + \pi(y)$, *Revue Roumaine de Mathématiques Pures et Appliquées* 20 (1975), 1201-1208.