Do Lerch zeta-functions satisfy the Lindelöf hypothesis?

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Abstract. We study the order of growth, and the zero distribution of Lerch zeta-functions. We show that the analogue of the density hypothesis fails to be true for a positive proportion of all Lerch zeta-functions. Further, we discuss the analogue of the Lindelöf hypothesis for Lerch zeta-functions and its connection with certain exponential sums and the distribution of nontrivial zeros.

Keywords: Lerch zeta-functions, Lindelöf hypothesis, zero distribution, exponential sums.

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1 Introduction

As usual, let \( s = \sigma + it \) be a complex variable and \( e(z) = \exp(2\pi i z) \). Denote by \( \mathbb{P} \) the set of prime numbers, and by \( \{ \lambda \} \) the fractional part of a real number \( \lambda \). We write \( f(x) = O(g(x)) \) and \( f(x) \ll g(x) \), resp., when \( \limsup_{x \to \infty} \frac{|f(x)|}{g(x)} \) is bounded, and \( f(x) = o(g(x)) \) if this limit equals 0. Further, \( f(x) \asymp g(x) \) denotes that the estimate \( g(x) \ll |f(x)| \ll g(x) \) holds.

For \( 0 < \alpha \leq 1, \lambda \in \mathbb{R} \), the Lerch zeta-function is given by

\[
L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e(\lambda n)}{(n + \alpha)^s}.
\]

This series converges absolutely for \( \sigma > 1 \). The analytic properties of \( L(\lambda, \alpha, s) \) are quite different depending on \( \lambda \in \mathbb{Z} \) or not. If \( \lambda \notin \mathbb{Z} \), then the series can be continued analytically

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to the whole complex plane, and \(L(\lambda, \alpha, s)\) turns out to be an entire function. For \(\lambda \in \mathbb{Z}\)
the Lerch zeta-function becomes the Hurwitz zeta-function \(\zeta(s, \alpha) := L(0, \alpha, s)\), which can be
continued analytically to the whole complex plane except for a simple pole at \(s = 1\) with residue
1. Setting
\[
\lambda^+ = 1 - \{\lambda\} \quad \text{and} \quad \lambda^- = \begin{cases} 1 \quad \text{if} \quad \lambda \in \mathbb{Z}, \\ \{\lambda\} \quad \text{if} \quad \lambda \notin \mathbb{Z}, \end{cases}
\]
for \(\lambda \in \mathbb{R}\), one can prove the functional equation
\begin{align}
L(\lambda, \alpha, 1 - s) &= \frac{\Gamma(s)}{(2\pi)^s} \left( e^{\left(\frac{s}{4} - \alpha \lambda^-\right)} L(-\alpha, \lambda^-, s) \\
&\quad + e^{\left(-\frac{s}{4} + \alpha \lambda^+\right)} L(\alpha, \lambda^+, s) \right).
\end{align}
\(\zeta(s, \alpha)\) was introduced by Hurwitz [9], \(L(\lambda, \alpha, s)\) by Lipschitz [14] and Lerch [13] (independently). Hurwitz and Lerch also gave the first proofs of the corresponding functional equations.

Many problems in number theory are connected with the location of the zeros of the Riemann
zeta-function, which is for \(\sigma > 1\) given by
\begin{equation}
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \left( 1 - \frac{1}{p^s} \right)^{-1}.
\end{equation}
Obviously, \(\zeta(s) = \zeta(s, 1) = L(1, 1, s)\). The famous, yet unproved \emph{Riemann hypothesis} states
that all nontrivial (complex) zeros of \(\zeta(s)\) lie on the critical line \(\sigma = \frac{1}{2}\). It seems that such
a zero distribution is connected with the Euler product representation (2). For a plenty of
number theoretical applications one does not need the truth of the Riemann hypothesis in its
full strength. As we will see in Theorem 9 below the Riemann hypothesis implies the so-called
\emph{Lindelöf hypothesis}, namely that
\begin{equation}
\zeta\left(\frac{1}{2} + it\right) \ll t^\varepsilon,
\end{equation}
as \(t \to \infty\). Moreover, using the Euler product (2), one can deduce from the Lindelöf hypothesis
the \emph{density hypothesis}, which states that, if \(N(\sigma, T)\) counts the zeros \(\rho = \beta + i\gamma\) of \(\zeta(s)\) with
\(\beta > \sigma\) and \(0 < \gamma \leq T\) (according multiplicities), the estimate
\begin{equation}
N(\sigma, T) \ll T^{2(1-\sigma) + \varepsilon} \quad \text{for} \quad \frac{1}{2} < \sigma \leq 1.
\end{equation}
holds. In the case of Lerch zeta-functions the analogue of the density hypothesis is not true in
general. For example,
\begin{align}
L\left(\frac{1}{2}, 1, s\right) = (1 - 2^{1-s})\zeta(s)
\end{align}
has infinitely many zeros off the critical line $\sigma = \frac{1}{2}$, at least the zeros $s = 1 - \frac{2\pi i k}{\log 2}, k \in \mathbb{Z}$, of the factor $1 - 2^{1-s}$, which violates the analogue of (3). Of course, this contradicts also the analogue of the Riemann hypothesis, but what is with the Lindelöf hypothesis? This question goes back at least to [1] where it was posed for Hurwitz zeta-functions. Obviously, $L(\frac{1}{2}, 1, s)$ satisfies the analogue of the Lindelöf hypothesis if $\zeta(s)$ satisfies the Lindelöf hypothesis. This indicates that, when $L(\lambda, \alpha, s)$ has no Euler product representation, the truth of the density hypothesis does not follow from the Lindelöf hypothesis.

The aim of this paper is to discuss the above quoted conjectures in the case of Lerch zeta-functions. We will show that for a positive proportion of Lerch zeta-functions the density hypothesis fails to be true. Further we investigate the order of growth of $L(\lambda, \alpha, s)$ in the critical strip and give some equivalent statement for the analogue of the Lindelöf hypothesis in terms of certain exponential sums, and of their zero distribution. By that one may conjecture that the Lindelöf hypothesis should hold for Lerch zeta-functions as well.

2 The density hypothesis

First, we recall some basic results on the zero distribution of Lerch zeta-functions. As in the case of the Riemann zeta-function one has to distinguish between so-called trivial and nontrivial zeros. The trivial zeros arise more or less directly from the functional equation and lie on the negative real axis if $\lambda = \frac{1}{2}$ or $\lambda = 1$, and close to the straight line, given by

$$\sigma = 1 + \frac{\pi t}{\log \frac{1-\lambda}{\lambda}}$$

in the left half of the complex plane, else (see [5]). In [5] and [15] it was proved that

(4) \quad $L(\lambda, \alpha, s) \neq 0$ \quad if \quad $\sigma \geq 1 + \alpha$,

and

(5) \quad $L(\lambda, \alpha, s) \neq 0$ \quad if \quad $\lambda = \frac{1}{2}, 1$, \quad $|t| \geq 1$ and $\sigma < -1$.

Also in [5] zero-free regions on the left side of the complex plane for others values $0 < \lambda < 1$ were obtained. In the next theorem we will indicate the exact regions for this case.

Let $l$ be a straight line on the complex plane $\mathbb{C}$, and denote by $\varrho(s, l)$ the distance of $s$ from $l$. Finally, define, for $\varepsilon > 0$,

$$L_\varepsilon(l) = \{s \in \mathbb{C} : \varrho(s, l) < \varepsilon\}.$$
Theorem 1 Let $0 < \lambda < 1$ and $\lambda \neq 1/2$. Then $L(\lambda, \alpha, s) \neq 0$ if $\sigma < -1$ and

$$s \not\in \frac{\log \lambda}{\pi} \left( \sigma = \frac{\pi t}{\log \frac{1-\lambda}{\lambda}} + 1 \right).$$

The following lemma will be useful for the proof of Theorem 1.

Lemma 2 Let $0 < \lambda < 1/2$ and $\sigma \leq -1$. Then

$$\sum_{m=1}^{\infty} \left( \left( \frac{\lambda + m}{\lambda} \right)^{-\sigma} + \left( \frac{1 - \lambda + m}{1 - \lambda} \right)^{-\sigma} \right) < \frac{3}{4}.$$

Proof of Lemma 2. Clearly, the sum in the lemma is less than

$$\left( \frac{1 + \lambda}{\lambda} \right)^{-\sigma} + \left( \frac{2 - \lambda}{1 - \lambda} \right)^{-\sigma} + \int_{1}^{\infty} \left( \frac{\lambda + x}{\lambda} \right)^{-\sigma} \, dx + \int_{1}^{\infty} \left( \frac{1 - \lambda + x}{1 - \lambda} \right)^{-\sigma} \, dx.$$

Since this function attains its maximum at $\sigma = -1$, $\lambda = 0$, we obtain Lemma 2.

Proof of Theorem 1. First we consider the case $0 < \lambda < 1/2$. Let the point $s_1 = \sigma_1 + t_1$ with $\sigma_1 < 0$ lie below the line

$$l : \quad \sigma = \frac{\pi t}{\log \frac{1-\lambda}{\lambda}} + 1,$$

i.e.

$$\sigma_1 \leq \frac{\pi t_1}{\log \frac{1-\lambda}{\lambda}} + 1.$$

We define the region

$$A(s_1) := \left\{ s \in C : \sigma \leq \sigma_1, \sigma - \sigma_1 \leq \frac{\pi (t - t_1)}{\log \frac{1-\lambda}{\lambda}} \right\},$$

lying below the line $l$ and left from the line $\sigma = \sigma_1$. From the functional equation (1) it follows (for details see [5]) that $L(\lambda, \alpha, s)$ does not vanish in $A(s_1)$ if the point $s_1$ satisfies also the condition

$$\exp \left\{ -\pi t_1 + (\sigma_1 - 1) \log \frac{1-\lambda}{\lambda} \right\} + \sum_{m=1}^{\infty} \left( \left( \frac{\lambda + m}{\lambda} \right)^{\sigma_1 - 1} + \left( \frac{1 - \lambda + m}{1 - \lambda} \right)^{\sigma_1 - 1} \right) < 1.$$

Thus, by Lemma 2, we can choose

$$s_1 = -1 - \frac{i}{\pi} \left( 2 \log \frac{1-\lambda}{\lambda} - \log 4 \right).$$
Now we consider the region lying below the line \( l \) and left from the imaginary axis. For \( s_2 = \sigma_2 + it_2 \) with
\[
\frac{\pi t_2}{\log \frac{1-\lambda}{\lambda}} + 1 < \sigma_2 < 0,
\]
we define the region
\[
B(s_2) := \left\{ s \in \mathbb{C} : \sigma \leq \sigma_2, \sigma - \sigma_2 \geq \frac{\pi (t - t_2)}{\log \frac{1-\lambda}{\lambda}} \right\}.
\]
Similarly as above, \( L(\lambda, \alpha, s) \) does not vanish in \( B(s_2) \) if \( s_2 \) satisfies the condition
\[
\exp \left\{ \pi t - (\sigma - 1) \log \frac{1-\lambda}{\lambda} \right\} + \sum_{m=1}^{\infty} \left( \left( \frac{\lambda + m}{\lambda} \right)^{\sigma-1} + \left( \frac{1-\lambda + m}{1-\lambda} \right)^{\sigma-1} \right) < 1.
\]
By Lemma 2 again, we choose
\[
s_2 = -1 - \frac{i}{\pi} \left( 2 \log \frac{1-\lambda}{\lambda} + \log 4 \right).
\]
Therefore, as the distance between regions \( A(s_1) \) and \( B(s_2) \) is less than \(|s_2 - s_1| = 2 \log 4 / \pi \), and \( \varrho(s_1, l) = \varrho(s_2, l) \), we can choose \( \varepsilon_0 = \log 4 / \pi \) and, clearly, \( \sigma_0 = -1 \). If \( \lambda > 1/2 \), then in view of the equality
\[
L(\lambda, \alpha, s) = L(1-\lambda, \alpha, s),
\]
the assertion of the theorem follows from the previous case. •

In view of (4), (5) and Theorem 1 we call a zero of the Lerch zeta-function nontrivial if its real part is between \(-1\) and \(1 + \alpha\). Let \( \varrho = \beta + i\gamma \) denote the nontrivial zeros of \( L(\lambda, \alpha, s) \). Applying ideas of Littlewood, Levinson and Montgomery [12], the authors proved in [8]
\[
\sum_{|\gamma| \leq T} (b + \beta) = \left( b + \frac{1}{2} \right) \frac{T}{\pi} \log \frac{T}{2\pi e \alpha \sqrt{\lambda + \lambda}} + \frac{T}{2\pi} \log \frac{\alpha}{\sqrt{\lambda + \lambda}} + O(\log T),
\]
as \( T \to \infty \), where \( b \) is a sufficiently large constant. As an immediate consequence one deduces for the number \( N(T; \lambda, \alpha) \) of nontrivial zeros \( \varrho \) of \( L(\lambda, \alpha, s) \) with \(|\gamma| \leq T \) (according multiplicities)
\[
N(T; \lambda, \alpha) = \frac{T}{\pi} \log \frac{T}{2\pi e \alpha \sqrt{\lambda + \lambda}} + O(\log T).
\]
This was first proved by Garunkštis and Laurinčikas [5] (using a slightly different method). Further, it follows that

\[
\sum_{|\gamma|\leq T} \left( \beta - \frac{1}{2} \right) = \frac{T}{2\pi} \log \frac{\alpha}{\sqrt{\lambda^+ \lambda^-}} + O(\log T).
\]

(8)

Therefore, \( L(\lambda, \alpha, s) \) has infinitely many zeros off the critical line, and they are asymmetrically distributed (with respect to the critical line) if \( \alpha^2 \neq \lambda^+ \lambda^- \). Thus almost all Lerch zeta-functions do not satisfy the analogue of the Riemann hypothesis. Upper and lower bounds for the zeros in the half plane \( \sigma > 1/2 \) were obtained in [3] and [4]. From these papers we quote

**Theorem 3** Denote by \( N(\sigma, T; \lambda, \alpha) \) the number of zeros \( \rho = \beta + i\gamma \) of \( L(\lambda, \alpha, s) \) with \( \beta > \sigma \) and \( |\gamma| \leq T \). If \( \alpha \) is an irrational number, then, for \( 1/2 < \sigma \leq 1 \),

\[
N(\sigma, T; \lambda, \alpha) \asymp T,
\]

as \( T \to \infty \), and the constants in \( \asymp \) depend only on \( \sigma \).

Thus almost all Lerch zeta-functions do not satisfy the analogue of the density hypothesis. The next theorem shows that the density hypothesis also is not valid for a positive proportion of the rational values of the parameter \( \alpha \).

**Theorem 4** If \( \alpha^2 > \lambda^+ \lambda^- \), then

\[
N \left( \frac{1}{2}, T; \lambda, \alpha \right) \geq \frac{T}{\pi(2\alpha + 1)} \log \frac{\alpha}{\sqrt{\lambda^+ \lambda^-}} + O(\log T).
\]

**Proof.** By (4) the number of zeros on the right of the critical line is bounded below by

\[
\sum_{|\gamma|\leq T} \frac{1}{1 + \alpha - \frac{1}{2}} \geq \frac{1}{\alpha + \frac{1}{2}} \sum_{|\gamma|\leq T} \left( \beta - \frac{1}{2} \right),
\]

and the assertion follows from (8). •

Note that a positive proportion of the parameters \( \alpha, \lambda \) satisfy the condition \( \alpha^2 > \lambda^+ \lambda^- \) of the theorem above, since

\[
\int_{0<\lambda, \alpha<1} \int_{\alpha^2>\lambda^+ \lambda^-} d\lambda d\alpha = 1 - \int_0^1 \sqrt{\lambda(1-\lambda)}d\lambda = 1 - \frac{\pi}{8}.
\]

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3 The order of growth and the Lindelöf hypothesis

Since in the case of the Riemann zeta-function the density hypothesis is deduced from the Lindelöf hypothesis by use of the Euler product (2), which does not exist for Lerch zeta-functions in general, it is not clear that the falsity of the density hypothesis in the general case of Lerch zeta-functions implies that these functions do not satisfy the analogue of the Lindelöf hypothesis, namely

\[ L \left( \lambda, \alpha, \frac{1}{2} + it \right) \ll |t|^\varepsilon \quad \text{for every } \varepsilon > 0, \text{ as } t \to \pm \infty. \]

Moreover, one may expect that the Lindelöf hypothesis for Lerch zeta-functions is as true as for the Riemann zeta-function: the best results in directions of the truth of Lindelöf’s hypothesis are deduced by exponential sum estimates from the approximate functional equation of the \( \zeta(s) \), and in [6] a more general approximate functional equation was established for Lerch zeta-functions.

Defining

\[ \mu(\sigma; \lambda, \alpha) = \limsup_{t \to \pm \infty} \frac{\log |L(\lambda, \alpha, \sigma + it)|}{\log |t|}, \]


\[ \mu(\sigma; \lambda, \alpha) \leq \begin{cases} 
1/2 - \sigma & \text{if } \sigma \leq 0, \\
(1 - \sigma)/2 & \text{if } 0 \leq \sigma \leq 1, \\
0 & \text{if } \sigma \geq 1. 
\end{cases} \]

This yields

\[ L \left( \lambda, \alpha, \frac{1}{2} + it \right) \ll |t|^\frac{1}{2} + \varepsilon. \]

The \( \varepsilon \) in the last estimate can be removed if we argue with the approximate functional equation [6], which states that if \( 0 < \lambda, \alpha \leq 1 \) and \( 0 \leq \sigma \leq 1 \) are fixed, then, for \( t \geq 1 \),

\[ L(\lambda, \alpha, s) = \sum_{0 \leq n \leq m(t)} \frac{e(\lambda n)}{(n + \alpha)^s} + \left( \frac{t}{2\pi} \right)^{\frac{1}{2} - \sigma - i\varepsilon} e^{it + \frac{t}{2} - 2\pi i(\lambda)\alpha} \sum_{0 \leq n \leq q(t)} \frac{e(-\alpha n)}{(n + \lambda)^{1-s}} \]

\[ + \left( \frac{t}{2\pi} \right)^{-\frac{t}{2}} e^{\pi i f(\lambda, \alpha, t)} \psi(g(\lambda, \alpha, t)) + O \left( t^{\frac{s}{2} - 1} \right), \]

where \( m(t) := \left\lfloor \sqrt{\frac{t}{2\pi}} - \alpha \right\rfloor \) and \( q(t) := \left\lfloor \sqrt{\frac{t}{2\pi}} \right\rfloor \).
4 Exponential sums

The Lindelöf hypothesis is closely connected with estimates of exponential sums.

**Theorem 5** Let \( \lambda, \alpha \in (0, 1] \) be fixed. Suppose that the estimates

\[
L(\lambda, \alpha, \frac{1}{2} \pm it) \ll t^\kappa
\]

holds for some \( \kappa \in (0, 1) \), and for arbitrary \( t \geq 2 \). Then the estimates

\[
\sum_{n \leq x} e(\lambda n) (n + \alpha)^{-s} it \ll x^{\frac{1}{2}} t^\kappa \log t \quad \text{and} \quad \sum_{n \leq x} e(\alpha n) (n + \lambda)^{-s} it \ll x^{\frac{1}{2}} t^\kappa \log t
\]

holds for \( 2 \leq x \leq t \). Conversely, if (13) holds for some \( \kappa \in (0, 1) \), and for arbitrary \( x, t \) with

\[
2 \leq x \leq t^{\frac{1}{2}}
\]

then the estimate

\[
L(\lambda, \alpha, \frac{1}{2} + it) \ll t^\kappa \log^2 t
\]

follows.

**Proof.** By Perron’s formula, one has

\[
\sum_{n \leq x} e(\lambda n) (n + \alpha)^{-s} it = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} L(\lambda, \alpha, s + it) \frac{x^s}{s} ds + O \left( \frac{x}{T} \log t \right),
\]

where \( b = 1 + \frac{1}{\log t}, T = \frac{1}{2} t \) and \( x = m + \alpha + 1/2, m \in \mathbb{N} \). If \( \Gamma \) is the rectangular contour with vertices \( b \pm iT, \frac{1}{2} \pm iT \), then Cauchy’s theorem yields

\[
\int_{\Gamma} L(\lambda, \alpha, s + it) \frac{x^s}{s} ds = 0.
\]

Hence we deduce

\[
\int_{b-iT}^{b+iT} L(\lambda, \alpha, s + it) \frac{x^s}{s} ds = \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} L(\lambda, \alpha, s + it) \frac{x^s}{s} ds
\]

\[+ O \left( \frac{1}{t} \int_{\frac{1}{2}}^{\frac{1}{2}+iT} |L(\lambda, \alpha, \sigma + iT)| \sigma^o d\sigma \right).\]

By (12) it turns out that

\[
\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} L(\lambda, \alpha, s + it) \frac{x^s}{s} ds \ll x^{\frac{1}{2}} t^\kappa \log t.
\]

We may approxiamte \( L(\lambda, \alpha, s) \) by a finite sum. If \( 0 \leq \sigma_0 \leq \sigma \leq 2, 0 < |t| \leq \pi \lambda N \), then

\[
L(\lambda, \alpha, s) = \sum_{0 \leq n \leq N} \frac{e(\lambda m)}{(n + \alpha)^s} + [\lambda] \frac{N^{1-s}}{s-1} + O(N^{-\sigma}).
\]
This yields the estimate
\[ L(\lambda, \alpha, s + it) \ll t^{1-\sigma} \log t \]
for \( \frac{1}{2} \leq \sigma \leq b \). Therefore we obtain
\[ \frac{1}{t} \int_{\frac{1}{2}}^{b} |L(\lambda, \alpha, \sigma + i(t \pm T))| x^\sigma \, d\sigma \ll \log t. \]
This proves the first estimate of (13). From this and from the approximate functional equation (11) it follows that also
\[ L(\alpha, \lambda, \frac{1}{2} \pm t) \ll t^\varepsilon. \]
Now in the same way as above we derive the second estimate of (13).

For the second part of the theorem we use the approximate functional equation to obtain
\[ L\left( \lambda, \alpha, \frac{1}{2} + it \right) \ll 1 + \left| \sum_{n \leq \sqrt{t/\pi}} \frac{e(\lambda n)}{(n + \alpha)^{1/2 + it}} \right| + \left| \sum_{n \leq \sqrt{t/\pi}} \frac{e(\alpha n)}{(n + \lambda)^{1/2 + it}} \right|. \]
By partial summation,
\[ \sum_{n \leq \sqrt{t/\pi}} \frac{e(\lambda n)}{(n + \alpha)^{1/2 + it}} = \frac{e(\lambda \sqrt{t/2\pi})}{(\sqrt{t/2\pi} + \alpha)^{1/2 + it}} \sum_{n \leq \sqrt{t/\pi}} \frac{e(\lambda n)}{(n + \alpha)^{1/2 + it}} \]
\[ + \frac{1}{2} \int_{1}^{\sqrt{t/\pi}} \sum_{n \leq u} \frac{e(\lambda n)}{(n + \alpha)^{1/2 + it}} \frac{du}{u^2}. \]
Assuming (13), the estimate (14) follows. 

We deduce from the theorem above

**Corollary 6** The truth of the Lindelöf hypothesis for \( L(\lambda, \alpha, s) \) is equivalent to the estimates
\[ \sum_{n \leq x} \frac{e(\lambda n)}{(n + \alpha)^{1/2 + it}} \ll x^{1/2} t^\varepsilon \quad \text{and} \quad \sum_{n \leq x} \frac{e(\alpha n)}{(n + \lambda)^{1/2 + it}} \ll x^{1/2} t^\varepsilon, \]
for \( 2 \leq x \leq t^{1/2} \).

**Corollary 7** For \( t \geq 2 \),
\[ L\left( \lambda, \alpha, \frac{1}{2} \pm it \right) \ll t^{1/2} \log t. \]
Proof. We use van der Corput’s method.

Lemma 8 Let \( f(x) \) be a real continues and three times continuously differentiable function. Let \( 0 < \beta_j \leq f^{(j)}(x) \leq h\beta_j, \ j = 1, 2, \) in \((a, b)\) with \( b - a \geq 1. \) Then

\[
\sum_{a < n \leq b} e(f(n)) \ll h(b - a)\beta_2^{1/2} + \beta_2^{-1/2},
\]

and

\[
\sum_{a < n \leq b} e(f(n)) \ll h^{1/2}(b - a)\beta_3^{1/2} + (b - a)^{1/2}\beta_3^{-1/2}.
\]

Proof of Lemma 8 can be found in §5.9 of [17].

In view of (17) we obtain for \( a \leq t^{2/3} \) that

\[
\sum_{a < n \leq b} e(\lambda n) \ll a \left( \frac{t}{a^2} \right)^{1/2} + a^{1/2} \left( \frac{t}{a^3} \right)^{-1/2} \ll a^{1/2} t^{1/6},
\]

and in view of (16) that

\[
\sum_{a < n \leq b} e(\lambda n) \ll t^{1/2} + at^{-1/2}.
\]

Thus the estimate (18) holds for \( t^{2/3} < a < t. \) Now Corollary 7 follows from Theorem 5. •

By partial summation we find

\[
\sum_{n \leq x} \frac{e(\lambda n)}{(n + \alpha)^it} = \sum_{n \leq x} n^{-it} e(\lambda n) \left( 1 + \frac{\alpha}{n} \right)^{-it}
\]

\[
= \sum_{n \leq x} n^{-it} e(\lambda x) \left( 1 + \frac{\alpha}{x} \right)^{-it} - i(2\pi i\lambda + \alpha t) \int_1^x \sum_{n \leq u} n^{-it} \exp \left( 2\pi i\lambda - it \log \left( 1 + \frac{\alpha}{u} \right) \right) \frac{du}{u(u + \alpha)}
\]

\[
\ll \left| \sum_{n \leq x} n^it \right| + \left| (2\pi \lambda + \alpha t) \sum_{n \leq x} n^{-it} \int_1^x \exp \left( 2\pi i\lambda - it \log \left( 1 + \frac{\alpha}{u} \right) \right) \frac{du}{u(u + \alpha)} \right|.
\]

It would be interesting to find (better) \( O \)- and \( \Omega \)-estimates of the second term in the last line. On the other side it seems impossible to deduce the truth of the Lindelöf hypothesis for Lerch zeta-functions from the truth of the Lindelöf hypothesis for the Riemann zeta-function.

However, by the mean square result [1]

\[
\int_0^1 \left| C \left( \frac{1}{2} + it, \alpha \right) - \alpha^{-1/2} - it \right|^2 \, d\alpha = \log t + O(1)
\]

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(which is a refinement of an older statement due to Lekkerkerker and Koksma), it follows that the mean value of all Hurwitz zeta-functions is small. A similar formula can be proved for Lerch zeta-functions.

5 Nontrivial zeros

Now we will extend Backlund’s equivalent statement [2] (or [17], §13.5) of the Lindelöf hypothesis for \( \zeta(s) \) in terms of the nontrivial zero distribution to Lerch zeta-functions.

**Theorem 9** The Lindelöf hypothesis (9) is true for \( L(\lambda, \alpha, s) \) if and only if

\[
N(\sigma, T + 1; \lambda, \alpha) - N(\sigma, T; \lambda, \alpha) = o(\log T)
\]

holds for every \( \sigma > \frac{1}{2} \) as \( T \to \infty \).

**Proof of Theorem 9.** First we assume the truth of Lindelöf’s hypothesis. Therefore, we make use of

**Lemma 10 (Jensen’s formula)** Let \( f(s) \) be analytic for \( |s| < R \). Suppose that \( f(0) \) is not zero, and let \( r_1, r_2, \ldots \) be the moduli of the zeros of \( f(s) \) in the circle \( |s| < R \), arranged as an non-decreasing sequence. Then, if \( r_n \leq r < r_{n+1} \),

\[
\log \frac{r^n |f(0)|}{r_1 \cdots r_n} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(r \exp(i\varphi))| \, d\varphi.
\]

For a proof see [16], §3.61.

Applying Jensen’s theorem to \( L(\lambda, \alpha, s) \) to the circle with centre \( 3 + it \) and radius \( \frac{5}{2} - \frac{\delta}{2} \), we obtain

\[
\sum_{|\varrho - 3 - it| < \frac{5}{2} - \frac{\delta}{2}} \log \frac{r_1}{|\varrho - 3 - it|} = \frac{1}{2\pi} \int_0^{2\pi} \log |L(\lambda, \alpha, 3 + it + \left( \frac{5}{2} - \frac{\delta}{4} \right) \exp(i\varphi))| \, d\varphi - \log |L(\lambda, \alpha, 3 + it)|.
\]

On the Lindelöf hypothesis (9) the right hand side is \( o(\log t) \). Further, if there are \( m \) zeros in the concentric circle of radius \( \frac{5}{2} - \frac{\delta}{2} \), the left hand side is bounded below by

\[
m \log \frac{\frac{5}{2} - \frac{\delta}{2}}{\frac{5}{2} - \frac{\delta}{4}} = m \log(1 + O(\delta)).
\]

Therefore, the number of zeros in the circle of radius \( \frac{5}{2} - \frac{\delta}{2} \) is \( o(\log t) \), and the result with \( \sigma > \frac{1}{2} \) follows by superposing a finite number (not depending on \( t \)) of such circles.

Now we have to prove the converse. Therefore we quote
Lemma 11 (Landau) If \( f(s) \) is regular, and

\[
\left| \frac{f(s)}{f(s_0)} \right| < e^M
\]

in \( \{ s : |s - s_0| \leq r \} \) with \( M > 1 \), then

\[
\left| \frac{f'(s)}{f(s)} - \sum \frac{1}{s - \rho} \right| < C \frac{M}{r}
\]

for \( |s - s_0| \leq \frac{T}{2} \), where \( C \) is some constant and \( \rho \) runs through the zeros of \( f(s) \) such that \( |\rho - s_0| \leq \frac{T}{2} \).

The proof depends mainly on the Hadamard product representation of \( f(s) \) and the Borel-Carathéodory theorem; see [17], §3.9 and [16], §5.5.

We apply this lemma with \( s_0 = 3 + iT \), where \( T \) is sufficiently large (see Theorem 1), and some bounded \( r \). Then we may choose \( M = c \log T \) (by the Phragmen-Lindelöf principle) and obtain

\[
\frac{L'}{L}(s, \lambda, \alpha) = \sum_{|\rho - (2iT)| \leq 1} \frac{1}{s - \rho} + O(\log T)
\]

for \( |s - s_0| \leq 3 \), and in particular for \( c \leq \sigma \leq 3, t = T \). We can replace \( T \) by \( t \) at the expense of an error \( O(\log t) \). By (7) we have \( N(t + c) - N(t - c) \ll \log t \). So that we can also change the condition of summation to obtain

\[
(19) \quad \frac{L'}{L}(s, \lambda, \alpha) = \sum_{|\gamma| \leq 1} \frac{1}{s - \gamma} + O(\log t)
\]

uniformly in \( \sigma, T \). Let \( C_1 \) be the circle with centre \( 3 + iT \) and radius \( \frac{5}{2} - \delta \) with some \( \delta > 0 \). Further, let \( C_2 \) be the concentric circle of radius \( \frac{5}{2} - 2\delta \). Then we have by (19)

\[
\Psi(s) := \frac{L'}{L}(s, \lambda, \alpha) - \sum_{\rho \in C_1} \frac{1}{s - \rho} = O \left( \frac{\log T}{\delta} \right),
\]

since for each of the \( O(\log T) \) terms which are in one of the sums

\[
\sum_{\rho \in C_1} \frac{1}{s - \rho}, \quad \sum_{|\gamma| \leq 1} \frac{1}{s - \gamma}
\]

we have \( |s - \rho| \geq \delta \). Now let \( C_3 \) be the concentric circle of radius \( \frac{5}{2} - 3\delta \), and \( C \) be the concentric circle of radius \( \frac{1}{2} \). Then \( \Psi(s) = o(\log T) \) by the hypothesis. Now we will use
Lemma 12 (Hadamard’s three circle theorem) Let \( f(s) \) be an analytic function, regular for \( r_1 \leq |s| \leq r_3 \). Let \( r_1 < r_2 < r_3 \), and let \( M_1, M_2, M_3 \) be the maxima of \(|f(s)|\) on the three circles \(|s| = r_1, r_2, r_3\) respectively. Then

\[
\log \frac{r_3}{r_1} \log M_2 \leq \log \frac{r_3}{r_2} \log M_1 + \log \frac{r_2}{r_1} \log M_3.
\]

For a proof see once more [16], §5.3.

Hadamard’s three circle theorem yields for \( s \in \mathbb{C}_3 \)

\[
\Psi(s) = (o(\log T))^\kappa \left( O \left( \frac{\log T}{\delta} \right) \right)^\iota,
\]

where \( \kappa + \iota = 1, 0 < \iota < 1 \) and \( \kappa, \iota \) depending on \( \delta \) only (and \( \lambda, \alpha \)). Hence we have \( \Psi(s) = o(\log t) \) for any given \( \delta \) in \( C_3 \). Since \( o(\log T) \) zeros lie inside \( C_1 \), we get

\[
\int_{1/2 + 3\delta}^{2} \Psi(s) d\sigma = \log L(\lambda, \alpha, 3 + it) - \log L \left( \lambda, \alpha, \frac{1}{2} + 3\delta \right) - \sum_{\varrho \in C_1} \left( \log(3 + it - \varrho) - \log \left( \frac{1}{2} + 3\delta + it - \varrho \right) \right)
\]

\[
= O(1) - \log L \left( \lambda, \alpha, \frac{1}{2} + 3\delta \right) + o(\log T) + \sum_{\varrho \in C_1} \log \left( \frac{1}{2} + 3\delta + it - \varrho \right).
\]

Now setting \( t = T \) the left-hand side is \( o(\log T) \). Taking the real parts we obtain

\[
\log \left| L \left( \lambda, \alpha, \frac{1}{2} + 3\delta \right) \right| = o(\log T) + \sum_{\varrho \in C_1} \log \left| \frac{1}{2} + 3\delta + it - \varrho \right|.
\]

Since \( |\frac{1}{2} + 3\delta + it - \varrho| < c \) in \( C_1 \), it follows that

\[
\log \left| L \left( \lambda, \alpha, \frac{1}{2} + 3\delta \right) \right| = o(\log T),
\]

which proves (9). •

When the zeros of \( L(\lambda, \alpha, s) \) on the right of the critical line are \textit{well} distributed, i.e. the spacing of the imaginary parts of consecutive exceptional zeros would satisfy the distribution law

\[
(20) \quad N(\sigma, T + H; \lambda, \alpha) - N(\sigma, T; \lambda, \alpha) \ll H \quad \text{for} \quad 1 \leq H \leq T,
\]

then the above theorem indicates that the Lindelöf hypothesis is true. In view to (20) we have a closer look on the upper bound in Theorem 3.
Theorem 13  As $T$ tends to infinity,

$$N(\sigma, T + H; \lambda, \alpha) - N(\sigma, T; \lambda, \alpha) \ll \begin{cases} H & \text{if } T^{1-\sigma} \leq H \leq T, \frac{1}{2} < \sigma \leq \frac{2}{3}, \\
H & \text{if } T^{\frac{2}{3}} \leq H \leq T, \frac{2}{3} < \sigma. \end{cases}$$

Proof. Let $H \leq T$ and $\frac{1}{2} < \Delta < \sigma$. Then

$$N(\sigma, T + H; \lambda, \alpha) - N(\sigma, T; \lambda, \alpha) \leq \frac{1}{\sigma - \Delta} \sum_{\beta > \Delta \atop T < \gamma \leq T + H} (\beta - \Delta).$$

By Littlewood’s lemma, we have

$$\sum_{\beta > \Delta \atop T < \gamma \leq T + H} (\beta - \Delta) = \frac{1}{2\pi} \left\{ \int_{-T}^{-T - (T + H)} + \int_{T}^{T + H} \right\} \log |L(\lambda, \alpha, \Delta + it)| dt + O(\log T).$$

With Jensen’s inequality we get

$$\int_{T}^{T + H} \log |L(\lambda, \alpha, \sigma + it)| dt \leq \frac{H}{2} \log \left( \frac{1}{H} \int_{T}^{T + H} |L(\lambda, \alpha, \sigma + it)|^2 dt \right).$$

In view to the mean square formula [7] we obtain

$$\frac{1}{H} \int_{T}^{T + H} |L(\lambda, \alpha, \sigma + it)|^2 dt = \zeta(2\sigma, \alpha) + (2 - 2\sigma) \frac{(2\pi)^{2\sigma - 1}}{2 - 2\sigma} \zeta(2 - 2\sigma, \lambda) T^{1 - 2\sigma} + O(HT^{-2\sigma} + H^{-1} T^{1 - \sigma} \log T + H^{-1} T^{\frac{2}{3}}).$$

The other integral in (22) can be treated similar. In view to (21) the assertion of the theorem follows. \bullet

References


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