Mathematical Competition for Students (MIFMO)
of the Department of Mathematics and Informatics
of Vilnius University

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Problem 1. Find all real y for which the equation $x^2 + x\sin(\pi y) + 2\cos(\pi y) = 0$ has two roots of the form $x_1 = \sin z$ and $x_2 = \cos z$, where $z = z(y) \in [0, 1]$.

Problem 2. Suppose $a_0 > a_1 > a_2 > a_3 > \dots$ is a decreasing sequence of positive numbers satisfying $\sum_{k=0}^{\infty} a_k = 1$. Is there a constant C for which the inequality

$$(n+1)^2 \sum_{k=n}^{\infty} a_k^3 \leqslant C$$

holds for each integer $n \geqslant 0$? If so, find the smallest such constant.

Problem 3. Let $a \geqslant 2$ and b be two integers. Prove that the sequence $a^{n^{2014}} + b$, $n = 1, 2, 3, \dots$, contains infinitely many composite numbers. (An integer $n \geqslant 2$ is called *composite* if it is not a prime number.)

Problem 4. Let S be a nonempty set, and let $*$ be an operation which to any $a, b \in S$ assigns some element $a*b \in S$ and satisfies the associativity property $(a*b)*c = a*(b*c)$ for all $a, b, c \in S$. Assume that for each $a \in S$ there is a unique $b = b(a) \in S$ satisfying $a * b * a = a$.

a) Prove that S contains an idempotent. (An element $e \in S$ is called *idempotent* if $e * e = e$.)

b) Prove that S contains a unique idempotent.

Each problem is worth 10 points.