

ON MAHLER MEASURES OF A SELF-INVERSIVE POLYNOMIAL AND ITS DERIVATIVE

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ABSTRACT. Let $M(f)$ be the Mahler measure of a polynomial $f \in \mathbb{C}[z]$. An old inequality (which is due to Mahler himself) asserts that $M(f') \leq dM(f)$ for each $f \in \mathbb{C}[z]$ of degree d . In contrast, we prove that if $f \in \mathbb{C}[z]$ is a self-inversive polynomial of degree $d \geq 2$ then $M(f') > \frac{d}{2}M(f)$. We also show that this inequality is best possible for d even, namely, that the quotient $M(f')/M(f)$ takes every value in the interval $(d/2, d]$ as f runs through reciprocal polynomials $f \in \mathbb{R}[z]$ of degree d . It seems likely that for d odd the constant $d/2$ is not optimal. For instance, for $d = 3$, the optimal value of the constant is $1.93867997\dots$ instead of $3/2$. For each odd $d \geq 5$, we prove that there exists a monic reciprocal polynomial $f \in \mathbb{Z}[z]$ of degree d such that $M(f') < \frac{d+1}{2}M(f)$. A corresponding problem for L_p norms of a self-inversive polynomial and its derivative is also considered.

1. INTRODUCTION

Recall that, for any polynomial

$$f(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_0 = a_d (z - \alpha_1) \dots (z - \alpha_d) \in \mathbb{C}[z]$$

of degree d , its *Mahler's measure* is defined as

$$M(f) = |a_d| \prod_{j=1}^d \max\{1, |\alpha_j|\}.$$

By Jensen's formula, the logarithm of the Mahler measure of $f \in \mathbb{C}[z]$ can be also expressed by the formulae

$$\log M(f) = \int_0^1 \log |f(e^{2\pi i t})| dt \tag{1}$$

(see [14]).

There is a classical inequality between the Mahler measure of a polynomial $f \in \mathbb{C}[z]$ of degree d and its derivative

$$M(f') \leq dM(f). \tag{2}$$

It was proved by Mahler himself [15]. See also Section D in [10] for another proof of (2) and two recent papers of Pereira [17] and Pritsker [18], where they showed that (2) can be derived from an earlier result of de Bruijn and Springer [9]. The example $f(z) = z^d + 1$ shows that the constant d in (2) is best possible.

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By the definition of Mahler's measure, we have

$$M(f') \geq d|a_d| \tag{3}$$

for every polynomial $f(z) = a_d z^d + \dots + a_0 \in \mathbb{C}[z]$ of degree d . The example $f(z) = a_d z^d + a_0$ with $a_d \neq 0$ and $a_0 \in \mathbb{C}$ shows that we can have $M(f') = d|a_d|$, so inequality (3) is sharp. Furthermore, by adding a 'large' positive integer n to any polynomial f we see that $\lim_{n \rightarrow \infty} M(f+n) = \infty$, whereas the derivative of $f+n$ is equal to f' . So one cannot replace the right-hand side of (3), $d|a_d|$, by, say, $\varepsilon dM(f)$ with some 'small' positive ε , unless there is some restriction on the constant term of a polynomial. The most natural restriction is $f(0) = 0$. Then a result of Storozhenko [27] asserts that

$$M(f') \geq s(d)M(f)$$

for each $f \in \mathbb{C}[z]$ of degree d satisfying $f(0) = 0$, where

$$s(d) = \frac{d}{\prod_{d/6 < k < 5d/6} (2 \sin(\pi k/d))} = \frac{d}{M((1+z)^d - 1)}.$$

The example $f(z) = (1+z)^d - 1$ shows that the bound of Storozhenko is best possible. However, $s(d)$ is approximately 1.4^{-d} for large d , so the constant $s(d)$ is very small. For which $f \in \mathbb{C}[z]$ there is a lower bound for $M(f')$ given in terms of $M(f)$ and d such that the dependence on d is similar to (2)? An important class of such polynomials is given in Theorem 1.

Let f^* denote the *reciprocal polynomial* of $f \in \mathbb{C}[z]$, defined by $f^*(z) = z^{\deg f} \overline{f}(1/z)$, where the bar denotes complex conjugation. More precisely, for $f(z) = a_d z^d + \dots + a_0$, where $a_d \neq 0$, we have $f^*(z) = \overline{a_0} z^d + \dots + \overline{a_d}$. Generally speaking, a polynomial $f \in \mathbb{C}[z]$ is called *self-inversive* if $f^*(z) = \theta f(z)$ for some $\theta \in \mathbb{C}$. Evidently, $\theta^2 = \overline{a_d a_0} / (a_d a_0)$, so such a number θ must be of modulus 1. In particular, a polynomial $f \in \mathbb{R}[z]$ is called *reciprocal* if its reciprocal polynomial f^* is $\pm f$, i.e., $f^*(z) = \pm f(z)$. In other words, f is reciprocal if it is a real self-inversive polynomial. The set of roots of a self-inversive polynomial is invariant under the map $z \mapsto 1/\overline{z}$, i.e., the multiset $\{\alpha_1, \dots, \alpha_d\}$ coincides with the multiset $\{1/\overline{\alpha_1}, \dots, 1/\overline{\alpha_d}\}$.

Reciprocal polynomials play a very important role in the theory of Mahler measure of algebraic numbers. Recall that the Mahler measure of an algebraic number is the Mahler measure of its minimal polynomial in $\mathbb{Z}[z]$. In 1933, Lehmer [13] asked whether for every $\varepsilon > 0$ there is a polynomial $f \in \mathbb{Z}[z]$ satisfying $1 < M(f) < 1 + \varepsilon$? In 1971, Smyth [25] showed a result which implies that such a polynomial f , if exists, must be reciprocal. It supports the opinion that the expected answer to Lehmer's question is negative. One should also mention the papers of Schinzel [21], Amoroso and Dvornicich [1], Borwein, Dobrowolski and Mossinghoff [6], where Lehmer's problem was resolved for some other classes of polynomials with integer coefficients. However, Smyth's result remains one of the most important results in this area, because it reduces Lehmer's problem to a 'small' class of reciprocal polynomials. Clearly, our next theorem, which gives a kind of reverse inequality to (2), is applicable to reciprocal polynomials in $\mathbb{Z}[z]$ and more generally to reciprocal polynomials in $\mathbb{R}[z]$:

Theorem 1. *Suppose that $f \in \mathbb{C}[z]$ is a self-inversive polynomial of degree $d \geq 2$. Then $M(f') > \frac{d}{2}M(f)$.*

We remark that two trivial cases $d = 0$ and $d = 1$ are excluded from Theorem 1. If $d = 1$ then $f(z) = a_1z + a_0 \in \mathbb{C}[z]$ is self-inversive if and only if $|a_0| = |a_1| \neq 0$. Then $M(f) = |a_1|$ and $M(f') = |a_1|$, giving $M(f') = M(f)$ for each linear self-inversive polynomial f . A key tool in the proof of Theorem 1 is Corollary 6. Although several versions of Corollary 6 (for example, Corollary 8) have already been used in several earlier papers (e. g., [3], [4]), it seems that neither Lemma 5 nor Corollary 6 were stated in this form and, moreover, were never applied in the context of Mahler measures of polynomials. (One can find many references on Mahler's measure in a recent survey of Smyth [26].)

Note that if $M(f) \leq 2|a_d|$ then the trivial inequality (3), which holds for every $f \in \mathbb{C}[z]$ of degree d , is stronger than the inequality of Theorem 1. In particular, Theorem 1 is of no use if $f \in \mathbb{Z}[z]$ is a monic reciprocal polynomial whose Mahler measure $M(f)$ satisfies $1 \leq M(f) \leq 2$. In this case, the trivial inequality $M(f') \geq d$ is stronger. Moreover, the inequality is strict $M(f') > d$ in case a self-inversive polynomial f of degree $d \geq 2$ has at least one root of modulus $\neq 1$. Indeed, by an old result of Marden (see Theorem (45,2) in [16]), a self-inversive polynomial f and its derivative f' have the same number of zeros in $|z| > 1$, so f' has at least one root in $|z| > 1$ provided that f has a root of modulus $\neq 1$. Combining Marden's inequality with Theorem 1 we thus obtain

$$M(f') > d \max\{M(f)/2, 1\}$$

for each self-inversive polynomial f of degree $d \geq 2$ having at least one root of modulus $\neq 1$. Self-inversive polynomials with all zeros on the unit circle have been studied by Bonsall and Marden [5], Lakatos and Losonczy [12], Schinzel [23], Sinclair and Vaaler [24]. (For those polynomials of degree d we obviously have $M(f') = dM(f)$.)

Combining Theorem 1 with Mahler's inequality (2), we have

$$\frac{d}{2} < \frac{M(f')}{M(f)} \leq d$$

for each self-inversive polynomial $f \in \mathbb{C}[z]$ of degree d . How close is the number $\inf \frac{M(f')}{M(f)}$, where the infimum is taken over every self-inversive polynomial $f \in \mathbb{C}[z]$ of degree d , to the value $d/2$? Our next theorem shows that for d even the lower bound $M(f')/M(f) > d/2$ is best possible even we are restricted to the class of monic integer reciprocal polynomials.

Theorem 2. *Let $d \geq 2$ be an even integer. Then there is a sequence of monic integer reciprocal polynomials f_n , $n = 1, 2, 3, \dots$, of degree d such that $M(f'_n)/M(f_n) \rightarrow d/2$ as $n \rightarrow \infty$. Furthermore, for every $w \in (d/2, d]$ there is a monic reciprocal polynomial $f \in \mathbb{R}[z]$ of degree d such that $M(f')/M(f) = w$.*

For $d \geq 3$ odd we prove the following:

Theorem 3. *Let $d \geq 3$ be an odd integer. Then there is a monic integer reciprocal polynomial f of degree d such that $M(f') < \frac{d+1}{2}M(f)$.*

For $d = 3$ we shall find the exact value of $\inf \frac{M(f')}{M(f)}$, where the infimum is taken over every cubic self-inversive polynomial $f \in \mathbb{C}[z]$:

Theorem 4. *Let $f \in \mathbb{C}[z]$ be a cubic self-inversive polynomial. Then*

$$M(f')/M(f) \geq \frac{3(827 + 384\sqrt{2})^{1/3}}{32} + \frac{219}{32(827 + 384\sqrt{2})^{1/3}} + \frac{9}{32} = 1.93867997\dots \quad (4)$$

The equality in (4) is attained for the polynomial $f(z) = z^3 - B_0z^2 - B_0z + 1 \in \mathbb{R}[z]$, where

$$B_0 = \frac{(827 + 384\sqrt{2})^{1/3}}{8} + \frac{73}{8(827 + 384\sqrt{2})^{1/3}} + \frac{11}{8} = 3.58490663\dots \quad (5)$$

One can also express B_0 as $B_0 = t + 1/t - 1$ with

$$t = (3 + 2\sqrt{2})^{1/3} + (3 - 2\sqrt{2})^{1/3} + 2 = 4.355301398\dots$$

Then

$$f(z) = z^3 - B_0z^2 - B_0z + 1 = (z + 1)(z - t)(z - 1/t),$$

where t satisfies

$$t^3 - 6t^2 + 9t - 8 = 0.$$

The minimal polynomial of B_0 is

$$8z^3 - 33z^2 + 18z - 9. \quad (6)$$

By (5), the right hand side of (4) is equal to $3(B_0 - 1)/4$. The minimal polynomial of the cubic number $3(B_0 - 1)/4$ is

$$32z^3 - 27z^2 - 54z - 27. \quad (7)$$

Let \mathcal{S}_d be the set of complex self-inversive polynomials of degree d , and let $\mathcal{R}_d \subset \mathcal{S}_d$ be the subset of (real) reciprocal polynomials. Theorems 1, 2 and 3 imply that $\inf_{f \in \mathcal{S}_d} \frac{M(f')}{M(f)} = (d + \tau_d)/2$, where $\tau_d = 0$ for d even and $0 \leq \tau_d < 1$ for $d \geq 3$ odd. Theorem 4 asserts that $\min_{f \in \mathcal{S}_3} \frac{M(f')}{M(f)} = 1.93867997\dots$. Their proofs will be given in Sections 3 and 4. In Section 5 we shall study the quotients of L_p norms $\|f'\|_p/\|f\|_p$ for $f \in \mathcal{S}_d$ and $f \in \mathcal{R}_d$. We also give some numerical examples of polynomials of odd degree and low value of the quotient $M(f')/M(f)$ in Section 6.

2. SOME LEMMAS

Lemma 5. *Let $f \in \mathbb{C}[z]$ be a self-inversive polynomial of degree $d \geq 1$, and let z_0 be a complex number of modulus 1. Then*

$$|f'(z_0)|^2 = \left(\frac{d}{2}\right)^2 |f(z_0)|^2 + |z_0 f'(z_0) - \frac{d}{2} f(z_0)|^2. \quad (8)$$

If, in addition, z_0 is not a root of f then $\Re(z_0 f'(z_0)/f(z_0)) = d/2$.

Proof: Since $f \in \mathbb{C}[z]$ is self-inversive, $f^*(z) = z^d \bar{f}(1/z) = \theta f(z)$ for some $\theta \in \mathbb{C}$ of modulus 1. Calculating the derivative of θf , we obtain

$$\theta f'(z) = dz^{d-1} \bar{f}'(1/z) - z^{d-2} \bar{f}'(1/z) = \theta dz^{-1} f(z) - z^{d-2} \bar{f}'(1/z)$$

for every non-zero complex number z . Dividing both sides of the above equality by $\theta z^{-1} f(z)$, we deduce that

$$d = \frac{zf'(z)}{f(z)} + \frac{z^{d-1} \bar{f}'(1/z)}{\theta f(z)} = \frac{zf'(z)}{f(z)} + \frac{\bar{f}'(1/z)}{z \bar{f}(1/z)} \quad (9)$$

if z is non-zero and not a root of f .

Fix any z_0 of modulus 1 which is not a root of f . Then $1/z_0 = \bar{z}_0$, because $|z_0| = 1$. Setting $z = z_0$ into the right hand side of (9), we see that the second term is equal to the complex conjugate $\overline{z_0 f'(z_0)/f(z_0)}$ of the first term $z_0 f'(z_0)/f(z_0)$. It follows that $d = 2\Re(z_0 f'(z_0)/f(z_0))$, giving $\Re(z_0 f'(z_0)/f(z_0)) = d/2$. This proves the second part of the lemma.

It is clear that equality (8) holds for each z_0 of modulus 1 which is a root of f . Suppose that $|z_0| = 1$, where z_0 is not a root of f . Then $\Re(z_0 f'(z_0)/f(z_0)) = d/2$ implies that

$$z_0 f'(z_0)/f(z_0) = d/2 + i\Im(z_0 f'(z_0)/f(z_0)).$$

Hence $|z_0 f'(z_0)/f(z_0) - d/2| = |\Im(z_0 f'(z_0)/f(z_0))|$. Since $|z_0| = 1$, we have

$$\begin{aligned} |f'(z_0)/f(z_0)|^2 &= |z_0 f'(z_0)/f(z_0)|^2 = \Re(z_0 f'(z_0)/f(z_0))^2 + \Im(z_0 f'(z_0)/f(z_0))^2 \\ &= (d/2)^2 + |z_0 f'(z_0)/f(z_0) - d/2|^2. \end{aligned}$$

Multiplying by $|f(z_0)|^2 \neq 0$ yields (8). \square

We remark that identity (8) of Lemma 5 implies that if $f \in \mathbb{C}[z]$ is a self-inversive polynomial then its derivative f' has no zeros on the unit circle $|z| = 1$ except for the multiple zeros of f (if any). This is an old result of Bonsall and Marden [5] (see Lemma (45,2) in [16]) which was proved by an entirely different argument. Below, we shall use the following corollary:

Corollary 6. *Let $f \in \mathbb{C}[z]$ be a self-inversive polynomial of degree $d \geq 1$, and let z_0 be a complex number of modulus 1. Then $|f'(z_0)| \geq \frac{d}{2}|f(z_0)|$, where equality holds if and only if z_0 is one of at most d points of the circle $|z| = 1$ satisfying $z_0 f'(z_0) = \frac{d}{2} f(z_0)$.*

Proof: From (8) we see that $|f'(z_0)| \geq \frac{d}{2}|f(z_0)|$, where equality holds if and only if $z_0 f'(z_0) = \frac{d}{2} f(z_0)$. Note that $zf'(z) - \frac{d}{2}f(z)$ is a polynomial of degree d , so it has at most d complex roots. In particular, the equality $z_0 f'(z_0) = \frac{d}{2} f(z_0)$ holds for at most d complex numbers z_0 lying on the unit circle $|z| = 1$. \square

Lemma 7. *Let D be a fixed positive integer, and let $g_1, g_2, g_3, \dots \in \mathbb{C}[z]$ be a sequence of polynomials satisfying $\deg g_n \leq D$ and $\lim_{n \rightarrow \infty} L(g_n) = 0$. Then $\lim_{n \rightarrow \infty} M(g + g_n) = M(g)$ for each $g \in \mathbb{C}[z]$.*

Here and below, for any polynomial $f(z) = \sum_{j=0}^d a_j z^j \in \mathbb{C}[z]$, its *length* $L(f)$ is defined by the formulae $L(f) = \sum_{j=0}^d |a_j|$. The inequality for the difference of two Mahler measures

$$|M(f)^{1/d} - M(g)^{1/d}| \leq 2L(f-g)^{1/d}, \quad (10)$$

where $f, g \in \mathbb{C}[z]$ are any polynomials of degree $\leq d$, was obtained by Chern and Vaaler [8]. Setting $d = \max(D, \deg g)$ and $f = g + g_n$ into (10) yields Lemma 7. The fact that $\lim_{n \rightarrow \infty} M(f_n) = M(f)$ if $\lim_{n \rightarrow \infty} L(f - f_n) = 0$ and the polynomials f, f_1, f_2, \dots have the same degree was earlier proved by Boyd [7]. See also Corollary 14 on p. 251 in [22].

3. PROOFS OF THEOREMS 1, 2 AND 3

Proof of Theorem 1: Suppose that $f \in \mathbb{C}[z]$ is a self-inversive polynomial of degree $d \geq 1$. By Corollary 6, there exists a finite set Λ of at most $2d + 1$ points of the interval $I = [0, 1)$ which includes 0 and all the points $t \in I$ for which either $|f'(e^{2\pi it})| = \frac{d}{2}|f(e^{2\pi it})|$ or $f(e^{2\pi it}) = 0$. Note that, by Lemma 5, every $t \in I$ satisfying $f'(e^{2\pi it}) = 0$ also belongs to Λ .

Fix a small positive number ε . Let $I_\varepsilon = \cup_{\lambda \in \Lambda} I_{\lambda, \varepsilon}$ be a finite union of intervals $I_{\lambda, \varepsilon} = (\lambda - \varepsilon/2, \lambda + \varepsilon/2)$, where $I_{0, \varepsilon}$ is defined as $I_{0, \varepsilon} = [0, \varepsilon) \cup (1 - \varepsilon, 1)$. All those intervals $I_{\lambda, \varepsilon}$ are disjoint for sufficiently small ε . So I_ε is the union of at most $2d + 2$ intervals of length ε , and $I \setminus I_\varepsilon$ is the union of at most $2d + 1$ closed intervals.

By (1), we have

$$\log M(f) = \int_0^1 \log |f(e^{2\pi it})| dt = \int_{I \setminus I_\varepsilon} \log |f(e^{2\pi it})| dt + \int_{I_\varepsilon} \log |f(e^{2\pi it})| dt.$$

Similarly,

$$\log M(f') = \int_0^1 \log |f'(e^{2\pi it})| dt = \int_{I \setminus I_\varepsilon} \log |f'(e^{2\pi it})| dt + \int_{I_\varepsilon} \log |f'(e^{2\pi it})| dt.$$

Note that, by Corollary 6, for every $t \in I \setminus I_\varepsilon$, we have $|f'(e^{2\pi it})| > \frac{d}{2}|f(e^{2\pi it})| > 0$. Thus

$$\int_{I \setminus I_\varepsilon} \log |f'(e^{2\pi it})| dt - \int_{I \setminus I_\varepsilon} \log |f(e^{2\pi it})| dt > (1 - |I_\varepsilon|) \log(d/2). \quad (11)$$

Also, $|f'(e^{2\pi it})| \geq \frac{d}{2}|f(e^{2\pi it})|$ for $t \in I_\varepsilon$. Hence

$$\int_{I_\varepsilon} \log |f'(e^{2\pi it})| dt - \int_{I_\varepsilon} \log |f(e^{2\pi it})| dt \geq |I_\varepsilon| \log(d/2). \quad (12)$$

(In fact, the singularities of the integrands in (12) are logarithmic, because

$$\log |e^{2\pi it} - e^{2\pi i\lambda}| = \log |2 \sin(\pi(t - \lambda))| \sim \log(2\pi|t - \lambda|) \quad \text{as } t \rightarrow \lambda.$$

So, using the fact that I_ε is a finite union of intervals of length ε , as on p. 8 of [10], one can show that there is a constant $C(f)$ independent of ε such that $|\int_{I_\varepsilon} \log |f'(e^{2\pi it})| dt| < C(f)\varepsilon \log(1/\varepsilon)$ and $|\int_{I_\varepsilon} \log |f(e^{2\pi it})| dt| < C(f)\varepsilon \log(1/\varepsilon)$.)

Adding (11) with (12) and using the formulas for $\log M(f')$ and $\log M(f)$, we deduce that

$$\log M(f') - \log M(f) > (1 - |I_\varepsilon|) \log(d/2) + |I_\varepsilon| \log(d/2) > \log(d/2).$$

This proves the inequality $M(f')/M(f) > d/2$. \square

The referee observed that by using Lemma 5 one can prove a slightly stronger inequality, namely,

$$M(f')^2 \geq \left(\frac{d}{2}\right)^2 M(f)^2 + M\left(zf'(z) - \frac{d}{2}f(z)\right)^2 \quad (13)$$

for every self-inversive polynomial $f \in \mathbb{C}[z]$.

We give a sketch of the proof. Suppose that the polynomials $g, h \in \mathbb{C}[z]$ have no zeros on the unit circle. The Mahler's inequality

$$\prod_{k=1}^n (x_k + y_k)^{1/N} \geq \prod_{k=1}^n x_k^{1/N} + \prod_{k=1}^n y_k^{1/N},$$

where $x_k, y_k > 0, k = 1, \dots, N$, for the corresponding Riemann integral sums implies that

$$\begin{aligned} & \exp\left(\int_0^1 \log(|g(e^{2\pi i\theta})| + |h(e^{2\pi i\theta})|)d\theta\right) \geq \\ & \exp\left(\int_0^1 \log|g(e^{2\pi i\theta})|d\theta\right) + \exp\left(\int_0^1 \log|h(e^{2\pi i\theta})|d\theta\right) = M(g) + M(h). \end{aligned}$$

Selecting here $g(z) = (df(z)/2)^2$ and $h(z) = (zf'(z) - df(z)/2)^2$ for $z = e^{2\pi i\theta}$ and using (8) we obtain

$$\begin{aligned} M(f')^2 &= \exp\left(\int_0^1 \log|f'(e^{2\pi i\theta})|^2 d\theta\right) = \exp\left(\int_0^1 \log(|g(e^{2\pi i\theta})| + |h(e^{2\pi i\theta})|)d\theta\right) \geq \\ & M(g) + M(h) = \left(\frac{d}{2}\right)^2 M(f)^2 + M\left(zf'(z) - \frac{d}{2}f(z)\right)^2, \end{aligned}$$

which proves (13). The case when the polynomial gh has some zeros on the unit circle can be treated by introducing I_ε as in the proof of Theorem 1.

Proof of Theorem 2: Put

$$f_n(z) = z^d - nz^{d/2} + 1.$$

The Mahler measure of f_n is equal to the Mahler measure of the polynomial $z^2 - nz + 1$, so $M(f_n) = (n + \sqrt{n^2 - 4})/2$ if $n \geq 2$. From $f'_n(z) = dz^{d/2-1}(z^{d/2} - n/2)$ it follows that $M(f'_n) = dn/2$ for $n \geq 2$. Thus

$$\frac{M(f'_n)}{M(f_n)} = \frac{d}{1 + \sqrt{1 - 4/n^2}} \rightarrow \frac{d}{2} \quad \text{as } n \rightarrow \infty,$$

which proves the first part of the theorem.

For the second part, fix $w \in (d/2, d]$ and select a real number $u \geq 2$ such that $d/w - 1 = \sqrt{1 - 4/u^2}$. By the above, for the polynomial $f(z) = z^d - uz^{d/2} + 1 \in \mathbb{R}[z]$, we have $M(f')/M(f) = du/(u + \sqrt{u^2 - 4}) = w$. \square

Proof of Theorem 3: For $d = 3$ we can take $f(z) = z^3 - 4(z^2 + z) + 1$ and find that $M(f')/M(f) = 1.939249\dots$ is smaller than 2.

Suppose that $d \geq 5$. Set

$$f_n(z) = z^d - ng(z) + 1 \quad \text{with} \quad g(z) = (z+1)(z^2 - 4z + 1)z^{(d-3)/2}.$$

Using $L(f_n/n + g) = 2/n \rightarrow 0$ as $n \rightarrow \infty$, by Lemma 7, we obtain

$$\lim_{n \rightarrow \infty} \frac{M(f_n)}{n} = M(g) = M(z^2 - 4z + 1) = 2 + \sqrt{3}.$$

Note that

$$g'(z) = ((d+3)z^3 - 3(d+1)z^2 - 3(d-1)z + d-3)z^{(d-5)/2}/2.$$

Hence, using $L(f'_n/n + g') = d/n \rightarrow \infty$ as $n \rightarrow \infty$, we find that

$$\lim_{n \rightarrow \infty} \frac{M(f'_n)}{n} = M(g') = \frac{1}{2}M((d+3)z^3 - 3(d+1)z^2 - 3(d-1)z + d-3) = \frac{1}{2}M(h),$$

where $h(z) = (d+3)z^3 - 3(d+1)z^2 - 3(d-1)z + d-3$.

In case the limit $\lim_{n \rightarrow \infty} M(f'_n)/M(f_n) = (2 - \sqrt{3})M(h)/2$ is smaller than $(d+1)/2$, we can take $f = f_n$ with sufficiently large n . Indeed, if $(2 - \sqrt{3})M(h) = d+1 - \kappa(d)$ with some positive number $\kappa(d)$ then $M(f'_n)/M(f_n) < (d+1 - \kappa(d)/2)/2$ for each sufficiently large n , say, $n \geq n_0$. Taking $f = f_n$, where $n \geq n_0$, we would obtain

$$M(f')/M(f) = M(f'_n)/M(f_n) < (d+1 - \kappa(d)/2)/2 < (d+1)/2,$$

as claimed.

So it remains to prove the inequality $M(h) < (2 + \sqrt{3})(d+1)$. Note that $h(-1) = -12$, $h(0) = d-3 > 0$ and $h(1) = -4d < 0$. So h has a root in $(-1, 0)$, a root in $(0, 1)$, and a root $\beta > 1$. Hence $M(h) = (d+3)\beta$. The required inequality is equivalent to the inequality $\beta < (2 + \sqrt{3})(d+1)/(d+3)$. For this, it suffices to show that $h((2 + \sqrt{3})(d+1)/(d+3)) > 0$. Indeed, by a simple computation, we have

$$h((2 + \sqrt{3})(d+1)/(d+3)) = 4(3(\sqrt{3} + 1)d + 3\sqrt{3} - 1)/(d+3)^2 > 0,$$

which completes the proof. \square

4. THE CUBIC CASE: PROOF OF THEOREM 4

If all three roots of f lie on the unit circle then, by the Gauss-Lucas theorem, all roots of f' lie in the unit circle $|z| \leq 1$, hence $M(f')/M(f) = 3$. This is more than we claim in (4).

Next, since $M(uf) = |u|M(f)$ and $M(uf') = |u|M(f')$, $u \in \mathbb{C}$, we can assume that f is monic. The Mahler measure of $f(z)$ and that of $f(ze^{i\phi})$, $\phi \in \mathbb{R}$, are equal. The same holds for their derivatives. So we may assume that f has a real root t greater than 1. Since the polynomial f is self-inversive, its two other roots are $1/t$ and $e^{i\phi}$, where $\phi \in [0, 2\pi)$. However, as $M(f) = M(\bar{f})$ and $M(f') = M(\bar{f}')$, without loss of generality, we can assume that $\phi \in [0, \pi]$.

It remains to minimize the quotient $M(f')/M(f)$, where f runs through the polynomials $f(z) = (z - e^{i\phi})(z - t)(z - 1/t)$ with $t > 1$ and $\phi \in [0, \pi]$. Evidently, $M(f) = t$. Set

$$w = e^{i\phi} \quad \text{and} \quad \lambda = t + 1/t.$$

Since

$$f(z) = (z - w)(z^2 - \lambda z + 1) = z^3 - (w + \lambda)z^2 + (w\lambda + 1)z - w,$$

we find that

$$f'(z) = 3z^2 - 2(w + \lambda)z + w\lambda + 1.$$

By the above mentioned theorem of Marden, f' has a unique root outside the unit circle, say, $\beta = \beta(\phi, t)$. Hence

$$M(f')/M(f) = 3|\beta|/t. \quad (14)$$

Clearly,

$$\beta = \beta(\phi, t) = \frac{\lambda + e^{i\phi} + \sqrt{\lambda^2 - \lambda e^{i\phi} + e^{2i\phi} - 3}}{3}, \quad (15)$$

where

$$3\beta(\phi, t)^2 - 2(t + 1/t + e^{i\phi})\beta(\phi, t) + (t + 1/t)e^{i\phi} + 1 = 0. \quad (16)$$

The sign of the square root in (15) is chosen so that its real part is positive. We claim that then $|\beta(\phi, t)| > 1$. Indeed, this is the case if $\lambda = t + 1/t$ is large enough. Suppose for some $t > 1$ and $\phi \in [0, \pi]$ we have $|\beta(\phi, t)| < 1$. Then, by continuity, there exist $t_0 > 1$ and $\phi_0 \in [0, \pi]$ such that $|\beta(\phi_0, t_0)| = 1$. However, this is impossible, because f has no multiple roots on $|z| = 1$ (see a remark after Lemma 5).

Set

$$\gamma = \gamma(\phi, t) = \frac{1}{\beta(\phi, t)} = \frac{\lambda + e^{-i\phi} + \sqrt{\lambda^2 - \lambda e^{-i\phi} + e^{-2i\phi} - 3}}{3}.$$

Then

$$3\gamma(\phi, t)^2 - 2(t + 1/t + e^{-i\phi})\gamma(\phi, t) + (t + 1/t)e^{-i\phi} + 1 = 0. \quad (17)$$

Note that if $t \leq 3/2$ then $M(f')/M(f) \geq 3/t \geq 2$. Also, $|\beta(\phi, t)|/t \rightarrow 2/3$ as $t \rightarrow \infty$, hence $M(f')/M(f)$ tends to 2 as $t \rightarrow \infty$. The extremal polynomial $f(z) = z^3 - B_0 z^2 - B_0 z + 1$ given in the statement of the theorem shows that $M(f')/M(f)$ attains smaller value. We can thus restrict t to the interval $3/2 \leq t \leq t_0$ with some absolute constant t_0 . By (14), we have $M(f')^2/9M(f)^2 = |\beta|^2/t^2 = \beta\gamma/t^2$, so we need to find the minimum of the function $h(\phi, t) = \beta(\phi, t)\gamma(\phi, t)/t^2$ in the rectangle $\phi \in [0, \pi]$, $t \in [3/2, t_0]$.

Suppose that the minimum is attained at the point (ϕ, t) , where $0 < \phi < \pi$ and $3/2 \leq t \leq t_0$. (It will be clear from the context when the same letters ϕ, t are used to denote the variables of the functions $\beta(\phi, t)$, $\gamma(\phi, t)$ and $h(\phi, t)$ and when the pair (ϕ, t) is used to denote the point, where the minimum of h is attained.) Then this point is a critical point of the function $h(\phi, t)$, so

$$\frac{\partial h}{\partial \phi} = \frac{\partial \beta}{\partial \phi} \gamma + \frac{\partial \gamma}{\partial \phi} \beta = 0 \quad \text{and} \quad \frac{\partial h}{\partial t} = \frac{1}{t^2} \left(\frac{\partial \beta}{\partial t} \gamma + \frac{\partial \gamma}{\partial t} \beta \right) - \frac{2\beta\gamma}{t^3} = 0. \quad (18)$$

Our goal is to prove that this is not the case. Suppose for a moment that this is already established. Then, as we know, the minimum is not attained at $t = 3/2$ and at $t = t_0$, so

it must be attained at some point (ϕ, t) , where $\phi \in \{0, \pi\}$ and $3/2 < t < t_0$. Moreover, it is easy to check that $h(0, t) > h(\pi, t)$. Indeed,

$$t\sqrt{h(0, t)} = \beta(0, t) = \frac{\lambda + 1 + \sqrt{\lambda^2 - \lambda - 2}}{3} > \frac{\lambda - 1 + \sqrt{\lambda^2 + \lambda - 2}}{3} = \beta(\pi, t) = t\sqrt{h(\pi, t)},$$

because by squaring the inequality $2 + \sqrt{\lambda^2 - \lambda - 2} > \sqrt{\lambda^2 + \lambda - 2}$ we have $2\sqrt{\lambda^2 - \lambda - 2} > \lambda - 2$. By squaring again, we see that this inequality holds, because $\lambda > 2$. Thus the minimum of h must be attained at some point (π, t) with $3/2 < t < t_0$. Then

$$\frac{M(f')}{M(f)} = 3\sqrt{h(\pi, t)} = \frac{3\beta(\pi, t)}{t} = \frac{2(\lambda - 1 + \sqrt{\lambda^2 + \lambda - 2})}{\lambda + \sqrt{\lambda^2 - 4}}.$$

Setting $x = \lambda - 1$, we thus need to find the minimum of the function

$$\Psi(x) = \frac{2(x + \sqrt{x^2 + 3x})}{x + 1 + \sqrt{(x + 1)^2 - 4}}$$

for $x > \lambda - 1 = t + 1/t - 1 > 7/6$. A simple computation with Maple shows that the function $\Psi(x)$ has only one critical point in $[1, \infty)$. From

$$\Psi'(x) = \frac{2 + (2x + 3)/\sqrt{x^2 + 3x}}{x + 1 + \sqrt{x^2 + 2x - 3}} - \frac{2(x + \sqrt{x^2 + 3x})(1 + (x + 1)/\sqrt{x^2 + 2x - 3})}{(x + 1 + \sqrt{x^2 + 2x - 3})^2} = 0$$

we find that the minimum of $\Psi(x)$ in $[1, \infty)$ is attained at the point

$$x = B_0 = \frac{(827 + 384\sqrt{2})^{1/3}}{8} + \frac{73}{8(827 + 384\sqrt{2})^{1/3}} + \frac{11}{8} = 3.58490663\dots,$$

where $\Psi'(B_0) = 0$. It is equal to $\Psi(B_0) = 1.93867997\dots$. One of the extremal polynomials at which the equality $M(f')/M(f) = \Psi(B_0)$ is attained is the polynomial $f(z) = z^3 - B_0z^2 - B_0z + 1$. The minimal polynomials for B_0 and $\Psi(B_0)$, namely, (6) and (7) have been found with Maple. This proves the theorem.

In the remainder of this section we will show that for no point (ϕ, t) , where $0 < \phi < \pi$ and $3/2 < t < t_0$ all four equalities given in (16), (17), (18) can hold.

Differentiating (16) with respect to the variable ϕ , we obtain

$$6\beta \frac{\partial \beta}{\partial \phi} - 2(\lambda + e^{i\phi}) \frac{\partial \beta}{\partial \phi} - 2i\beta e^{i\phi} + ie^{i\phi}\lambda = 0.$$

Hence

$$(6\beta - 2\lambda - 2e^{i\phi}) \frac{\partial \beta}{\partial \phi} = ie^{i\phi}(2\beta - \lambda). \quad (19)$$

Analogously, (17) yields

$$6\gamma \frac{\partial \gamma}{\partial \phi} - 2(\lambda + e^{-i\phi}) \frac{\partial \gamma}{\partial \phi} + 2i\gamma e^{-i\phi} - ie^{-i\phi}\lambda = 0,$$

thus

$$(6\gamma - 2\lambda - 2e^{-i\phi}) \frac{\partial \gamma}{\partial \phi} = ie^{-i\phi}(-2\gamma + \lambda). \quad (20)$$

Since, by (18), $\frac{\partial \beta}{\partial \phi} \gamma + \frac{\partial \gamma}{\partial \phi} \beta = 0$, multiplying (19) by $(3\gamma - \lambda - e^{-i\phi})\gamma$ and (20) by $(3\beta - \lambda - e^{i\phi})\beta$ and then adding, we obtain

$$0 = (3\gamma - \lambda - e^{-i\phi})\gamma i e^{i\phi} (2\beta - \lambda) + (3\beta - \lambda - e^{i\phi})\beta i e^{-i\phi} (-2\gamma + \lambda).$$

Hence

$$(2\beta - \lambda)(3\gamma^2 - (\lambda + e^{-i\phi})\gamma) e^{i\phi} = (2\gamma - \lambda)(3\beta^2 - (\lambda + e^{i\phi})\beta) e^{-i\phi}.$$

By (16) and (17), we know that $3\beta^2 - (\lambda + e^{i\phi})\beta = (\lambda + e^{i\phi})\beta - \lambda e^{i\phi} - 1$ and $3\gamma^2 - (\lambda + e^{-i\phi})\gamma = (\lambda + e^{-i\phi})\gamma - \lambda e^{-i\phi} - 1$. So

$$(2\beta - \lambda)((\lambda + e^{-i\phi})\gamma - \lambda e^{-i\phi} - 1) e^{i\phi} = (2\gamma - \lambda)((\lambda + e^{i\phi})\beta - \lambda e^{i\phi} - 1) e^{-i\phi}.$$

Here, the left hand side is equal to

$$2\beta\gamma(1 + \lambda e^{i\phi}) - \lambda(1 + \lambda e^{i\phi})\gamma - 2\beta(\lambda + e^{i\phi}) + \lambda(\lambda + e^{i\phi}),$$

and the right hand side is equal to

$$2\beta\gamma(1 + \lambda e^{-i\phi}) - \lambda(1 + \lambda e^{-i\phi})\beta - 2\gamma(\lambda + e^{-i\phi}) + \lambda(\lambda + e^{-i\phi}).$$

Subtracting one side from another and multiplying by $w = e^{i\phi}$ yields

$$2\lambda\beta\gamma(w^2 - 1) + \beta(-\lambda w - 2w^2 + \lambda^2) + \gamma(\lambda w + 2 - \lambda^2 w^2) + \lambda(w^2 - 1) = 0. \quad (21)$$

Differentiating (16) with respect to the variable t and using $\frac{\partial \lambda}{\partial t} = 1 - 1/t^2$, we obtain

$$6\beta \frac{\partial \beta}{\partial t} - 2(\lambda + w) \frac{\partial \beta}{\partial t} - 2(1 - 1/t^2)\beta + w(1 - 1/t^2) = 0.$$

Since, by (16), $\beta(3\beta - \lambda - w) = (\lambda + w)\beta - (\lambda w + 1)$, this yields

$$\frac{1}{\beta} \frac{\partial \beta}{\partial t} = \frac{(2\beta - w)(1 - 1/t^2)}{2((\lambda + w)\beta - \lambda w - 1)}.$$

Taking the complex conjugates of both sides, we obtain

$$\frac{1}{\gamma} \frac{\partial \gamma}{\partial t} = \frac{(2\gamma - \bar{w})(1 - 1/t^2)}{2((\lambda + \bar{w})\gamma - \lambda \bar{w} - 1)}.$$

By (18), their sum $\frac{1}{\beta} \frac{\partial \beta}{\partial t} + \frac{1}{\gamma} \frac{\partial \gamma}{\partial t}$ must be equal to $2/t$, hence

$$\frac{(2\beta - w)(1 - 1/t^2)}{2((\lambda + w)\beta - \lambda w - 1)} + \frac{(2\gamma - \bar{w})(1 - 1/t^2)}{2((\lambda + \bar{w})\gamma - \lambda \bar{w} - 1)} = \frac{2}{t} \quad (22)$$

Expressing $\bar{w} = 1/w$, $\lambda = t + 1/t$ and multiplying by common denominators, we can rewrite (22) in the form

$$p_3(w, t)\beta\gamma + p_2(w, t)\beta + p_1(w, t)\gamma + p_0(w, t) = 0, \quad (23)$$

where $p_j(w, t) \in \mathbb{Z}[w, t]$ for $j = 0, 1, 2, 3$.

Clearly, (16) can be written as

$$3\beta^2 - 2(\lambda + w)\beta + \lambda w + 1 = 0 \quad (24)$$

and (17) can be written as

$$3w\gamma^2 - 2(\lambda w + 1)\gamma + \lambda + w = 0. \quad (25)$$

Observe that (21), (24), (25) are three equations in four variables $\beta, \gamma, \lambda, w$. Here, β and γ are the values of the corresponding functions at the critical point (ϕ, t) of h . By computing the resultant of (21) and (24) with respect to β we shall obtain a polynomial F_0 in γ, λ, w . Then, by computing the resultant of this polynomial F_0 and the polynomial on the right hand side of (25) with respect to γ , we shall obtain the polynomial

$$F_1(w, \lambda) = -3w^2(w-1)^2(w+1)^2(\lambda-2)^2(\lambda+2)^2(w^2 - \lambda w + 1) \\ ((\lambda^4 - 2\lambda^2 - 3)w^2 - (\lambda^5 - 6\lambda^3 + 13\lambda)w + \lambda^4 - 2\lambda^2 - 3).$$

Similarly, we can substitute $\lambda = t + 1/t$ into (24) and (25), multiply them by t and then obtain a polynomial $G_1(w, t)$ as a result of first computing the resultant G_0 of (23) (obtained from (22)) and (24) with respect to β and then the resultant of this polynomial G_0 in γ, w, t and the polynomial (25) with respect to γ . The polynomial

$$G_1(w, t) = 8(w^7 + w^5)t^{23} + (21w^8 + 54w^6 + 21w^4)t^{22} + \dots - 888(w^7 + w^5)t + 1152w^6$$

is to ‘large’ to be reproduced here. It has 120 nonzero terms, its total degree is 30, its degree in w is 12, and its degree in t is 23. (Of course, the polynomial $G_1(w, t)$ can be quickly computed with Maple exactly as it is described above.)

Since $w \neq \pm 1$ and $\lambda > 2$, only factor of F_1 which may vanish at points w with modulus 1 is

$$F_2(w, \lambda) = (\lambda^4 - 2\lambda^2 - 3)w^2 - (\lambda^5 - 6\lambda^3 + 13\lambda)w + \lambda^4 - 2\lambda^2 - 3.$$

We substitute $\lambda = t + 1/t$ into F_2 and eliminate t by computing the resultant of $t^5 F_2(w, t + 1/t)$ and $G_1(w, t)$ with respect to t . This resultant $F_3(w)$ factors in $\mathbb{Z}[w]$ into irreducible factors as follows:

$$F_3(w) = -4398046511104(w-1)^{32}(w+1)^{32}F_4(w)F_5(w)F_6(w)F_6^*(w),$$

where

$$F_4(w) = 121w^{16} - 1184w^{14} + 3086w^{12} - 4544w^{10} \\ + 19379w^8 - 4544w^6 + 3086w^4 - 1184w^2 + 121, \\ F_5(w) = 7623w^{24} - 51426w^{22} + 150903w^{20} - 294044w^{18} \\ + 469180w^{16} - 620138w^{14} + 694272w^{12} - 620138w^{10} \\ + 469180w^8 - 294044w^6 + 150903w^4 - 51426w^2 + 7623, \\ F_6(w) = 5887w^{20} + 140216w^{18} + 48780w^{16} + 358498w^{14} \\ - 1562910w^{12} + 1867560w^{10} - 914417w^8 \\ + 127414w^6 - 15237w^4 + 21096w^2 + 4356.$$

The polynomials F_6, F_6^* are irreducible in $\mathbb{Z}[w]$ and non-reciprocal, so they have no zeros of modulus 1. A simple numerical computation with Maple with extended precision shows that neither F_4 nor F_5 have such zeros. It follows that the function $h(\phi, t)$ has no critical points if $0 < \phi < \pi$. This proves our claim and completes the proof of Theorem 4. \square

Another way to compute the extremal value t is to use (22). Since it is already established that the minimum is attained at $w = -1$ (and so $\beta = \gamma$), by (22), we have

$$\frac{2\beta + 1}{(\lambda - 1)(\beta + 1)} = \frac{2t}{t^2 - 1}.$$

Thus $\beta = (t^2 - 2t + 3)/2(t - 2)$. Substituting this expression into $3\beta^2 - (t + 1/t - 1)(2\beta + 1) = 0$ (which is obtained from (24) with $w = -1$), we find that $3t(t^2 - 2t + 3)^2 = 4(t^2 - t + 1)^2(t - 2)$. Hence $(t^3 - 6t^2 + 9t - 8)(t + 1)^2 = 0$, giving $t^3 - 6t^2 + 9t - 8 = 0$. Thus

$$t = (3 + 2\sqrt{2})^{1/3} + (3 - 2\sqrt{2})^{1/3} + 2 = 4.355301398 \dots$$

is the critical point of $h(\pi, t)$ and $f(z) = (z + 1)(z - t)(z - 1/t)$ is a corresponding extremal polynomial.

5. L_p NORMS OF A POLYNOMIAL AND ITS DERIVATIVE

Suppose that $0 < p \leq \infty$. It is well-known that the Mahler measure of a polynomial $f \in \mathbb{C}[z]$ is the limit of its L_p norm defined by

$$\|f\|_p = \left(\int_0^1 |f(e^{2\pi it})|^p dt \right)^{1/p},$$

namely,

$$\lim_{p \rightarrow 0^+} \|f\|_p = M(f)$$

(see, e.g., [11]). Now, a direct analogue of (2) is the upper bound

$$\|f'\|_p \leq d\|f\|_p, \tag{26}$$

which holds for every $p > 0$ and every $f \in \mathbb{C}[z]$ of degree d . For $p = \infty$ inequality (26) is a classical inequality of Bernstein. Later, Zygmund [28] proved (26) for $1 \leq p < \infty$ and Arestov [2] for $0 < p < 1$. Fix any $p > 0$. Which values the quotient $\|f'\|_p/\|f\|_p$ takes as f runs through self-inversive polynomials of degree d ?

For $p = \infty$, we have $\|f\|_\infty = \max_{|z|=1} |f(z)|$. Theorem 14.3.1 of [20] asserts that if $f \in \mathbb{C}[z]$ is a self-inversive polynomial of degree $d \geq 1$ then

$$\|f'\|_\infty = \frac{d}{2}\|f\|_\infty.$$

So the quotient $\|f'\|_\infty/\|f\|_\infty$, where $f \in \mathcal{S}_d$, takes only one value $d/2$.

The upper bound for $\|f'\|_p/\|f\|_p$, where $f \in \mathbb{C}[z]$ is a self-inversive polynomial of degree d , was established by Rahman and Schmeisser [19]. Combining Theorem 14.6.5 and Remark 14.6.6 of [20], we have

$$\frac{\|f'\|_p}{\|f\|_p} \leq \frac{d}{\|z^d + 1\|_p} = \frac{d}{\|z + 1\|_p} = \frac{d}{2} \left(\frac{\sqrt{\pi}\Gamma(p/2 + 1)}{\Gamma(p/2 + 1/2)} \right)^{1/p} \tag{27}$$

for every $f \in \mathcal{S}_d$. Equality in (27) is attained for the polynomial $f(z) = z^d + 1 \in \mathcal{R}_d$.

Since the integrands $|f(e^{2\pi it})|^p$ and $|f'(e^{2\pi it})|^p$, where f is a polynomial, are defined for every point $t \in [0, 1]$, Corollary 6 immediately implies the following lower bound for L_p -norms of a self-inversive polynomial and its derivative:

Corollary 8. *Suppose that $0 < p < \infty$. If $f \in \mathbb{C}[z]$ is a self-inversive polynomial of degree $d \geq 2$ then $\|f'\|_p > \frac{d}{2}\|f\|_p$.*

The bound of Corollary 8 is best possible for d even. Indeed, one can take the same example $f_n(z) = z^d - nz^{d/2} + 1$ with n large as in the proof of Theorem 2. Then, for every fixed $p > 0$, we have $\|f_n\|_p \sim n$ and $\|f'_n\|_p \sim dn/2$ as $n \rightarrow \infty$. Hence $\|f'_n\|_p/\|f_n\|_p \rightarrow d/2$ as $n \rightarrow \infty$. Combining with (27), by continuity, we deduce that for every $0 < p < \infty$ the quotient $\|f'\|_p/\|f\|_p$ takes every value in the interval $(d/2, (\sqrt{\pi}\Gamma(p/2 + 1)/\Gamma(p/2 + 1/2))^{1/p}d/2]$ as f runs through \mathcal{S}_d (or \mathcal{R}_d), where $d \geq 2$ is even. For example, one can take $f(z) = z^d - uz^{d/2} + 1 \in \mathcal{R}_d$ with $u \in [0, +\infty)$. Corollary 8 for $p \geq 1$ was already established in [4]: see Theorem 2 and Remark 2 in [4], and also an earlier paper [3].

The question of finding $\inf_{f \in \mathcal{S}_d} \|f'\|_p/\|f\|_p$ and $\inf_{f \in \mathcal{R}_d} \|f'\|_p/\|f\|_p$ remains open for $d \geq 3$ odd. One case when both infimums can be established is $p = 2$. For $f(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_0$ we have $\|f\|_2 = \sqrt{|a_d|^2 + |a_{d-1}|^2 + \dots + |a_0|^2}$. Thus if $f \in \mathbb{C}[z]$ is a self-inversive polynomial of degree $d = 1$ then $f(z) = a(z + \theta)$ with $a \neq 0$ and $|\theta| = 1$. Hence $\|f'\|_2 = \frac{1}{\sqrt{2}}\|f\|_2$.

For $d \geq 3$ we prove the following:

Theorem 9. *If $f \in \mathbb{C}[z]$ is a self-inversive polynomial of odd degree $d \geq 3$ then $\|f'\|_2 > \frac{\sqrt{d^2+1}}{2}\|f\|_2$. This inequality is best possible.*

Proof: Without loss of generality we may assume that f is monic. Then

$$f(z) = z^d + a_1 z^{d-1} + \dots + a_{(d-1)/2} z^{(d+1)/2} + \theta \overline{a_{(d-1)/2}} z^{(d-1)/2} + \dots + \theta \overline{a_1} z + \theta$$

with some $\theta \in \mathbb{C}$ of modulus 1. Hence

$$\|f\|_2^2 = 2(1 + |a_1|^2 + |a_2|^2 + \dots + |a_{(d-1)/2}|^2)$$

and

$$\|f'\|_2^2 = d^2 + ((d-1)^2 + 1^2)|a_1|^2 + ((d-2)^2 + 2^2)|a_2|^2 + \dots + \frac{d^2 + 1}{2}|a_{(d-1)/2}|^2.$$

The coefficients $(d-j)^2 + j^2$ are greater than $(d^2 + 1)/2$ for each j in the range $0 \leq j < (d-1)/2$. Therefore,

$$\|f'\|_2^2 > \frac{d^2 + 1}{2}(1 + |a_1|^2 + |a_2|^2 + \dots + |a_{(d-1)/2}|^2) = \frac{d^2 + 1}{4}\|f\|_2^2.$$

This yields the required inequality $\|f'\|_2 > \frac{\sqrt{d^2+1}}{2}\|f\|_2$ for $d \geq 3$ odd.

In order to show that this inequality is tight let us take the reciprocal polynomial $f_n(z) = z^d + n(z^{(d+1)/2} + z^{(d-1)/2}) + 1 \in \mathbb{Z}[z]$. We have $\|f_n\|_2^2 = 2(1+n^2)$ and $\|f'_n\|_2^2 = d^2 + (d^2+1)n^2/2$. So $\lim_{n \rightarrow \infty} \|f'_n\|_2/\|f_n\|_2 = \sqrt{d^2+1}/2$. \square

Theorem 9 implies that

$$\inf_{f \in \mathcal{S}_d} \|f'\|_2 / \|f\|_2 = \inf_{f \in \mathcal{R}_d} \|f'\|_2 / \|f\|_2 = \sqrt{d^2 + 1} / 2$$

for $d \geq 3$ odd. Note that the right hand side of (27) for $p = 2$ is equal to $d/\sqrt{2}$. By continuity, i.e., taking

$$f(z) = z^d + u(z^{(d+1)/2} + z^{(d-1)/2}) + 1 \in \mathcal{R}_d,$$

where $u \in [0, +\infty)$, we deduce that if f runs through every reciprocal polynomial f of odd degree $d \geq 3$ then the quotient $\|f'\|_2 / \|f\|_2$ takes every value in the interval $(\sqrt{d^2 + 1}/2, d/\sqrt{2}]$.

6. NUMERICAL EXAMPLES

By the above results, it seems that the constant $d/2$ in the L_p norm inequality $\|f'\|_p > \frac{d}{2}\|f\|_p$, where $0 \leq p < \infty$, is not optimal for self-inversive polynomials $f(z)$ of odd degree d . In general, the computation of the infimum $\|f'\|_p / \|f\|_p$ over self-inversive polynomials f of odd degree d , is non-trivial except in two cases $p = 2$ (Theorem 9) and $p = \infty$.

We give several examples of integer monic self-reciprocal polynomials with low value of $M(f')/M(f)$ (corresponding to the case $p = 0$), computed with Maple.

TABLE 1. Monic reciprocal polynomials $f \in \mathbb{Z}[z]$ of small odd degree d and low value $M(f')/M(f)$.

d	$f(z)$	$M(f)$	$M(f')$	$M(f')/M(f)$
3	$z^3 - 4z^2 - 4z + 1$	4.79128	9.29150	1.93924
5	$z^5 - 2z^4 + 7z^3 + 7z^2 - 2z + 1$	9.08609	26.58617	2.92602
7	$z^7 - z^6 + 2z^5 - 8z^4 - 8z^3 + 2z^2 - z + 1$	10.60148	41.56526	3.92070
9	$z^9 - z^8 + 2z^7 - 3z^6 + 11z^5 + 11z^4 - 3z^3 + 2z^2 - z + 1$	15.15276	74.51235	4.91740

Theorem 4 gives a precise answer for cubic self-reciprocal polynomials $f(z) \in \mathbb{C}[z]$. The cubic polynomial in Table 1 is the monic polynomial with integer coefficients closest to the optimal cubic polynomial in Theorem 4. The value $1.93924\dots$ is the minimal value of $M(f')/M(f)$ among all monic cubic reciprocal polynomials with integer coefficients. For $d = 5$, we searched for optimal monic reciprocal polynomial with integer coefficients in the interval $[-100, 100]$. For $d = 7$ and $d = 9$, the search interval was reduced to $[-20, 20]$ and to $[-15, 15]$, respectively. We must note that the values of $M(f')/M(f)$ in Table 1 are not far from $(d+1)/2$ (see also the Theorem 3), but the distance from those values to $(d+1)/2$ is increasing with d .

In the case $d = 5$, we have also computed the polynomial

$$g(z) = z^5 - 1.732z^4 + 6.165z^3 + 6.165z^2 - 1.732z + 1$$

with $M(g')/M(g) = 2.92557564\dots$. It seems that this value is close to the infimum of $M(f')/M(f)$, where $f \in \mathcal{R}_5$.

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