Pisot–Salem numbers, interlacing and \(\{0, 1\}\)-words

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Pisot and Salem numbers

- **Pisot numbers**
  A real algebraic integer $\alpha > 1$ whose algebraic conjugates over $\mathbb{Q}$ $\alpha' \neq \alpha$ satisfy $|\alpha'| < 1$.

- **Salem number:** a real algebraic integer $\alpha > 1$ whose conjugates $\alpha' \neq \alpha$ satisfy $|\alpha'| \leq 1$ with at least one conjugate being of modulus $|\alpha'| = 1$. 
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Examples

- **Siegel’s number:** Smallest Pisot number (proved such by Siegel in 1938):
  \[
  \theta = \frac{1}{6} \left( \sqrt[3]{108 + 12\sqrt{69}} + \sqrt[3]{108 - 12\sqrt{69}} \right) = 1.3247 \ldots ,
  \]
  with minimal polynomial:
  \[
  P(z) = z^3 - z - 1.
  \]

- **Lehmer’s number:** the positive root \( \mu = 1.17628 \ldots \) of
  \[
  L(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1,
  \]
  discovered by Lehmer in 1933.
Figure: $z^3 - z - 1$  
Figure: $L(z)$
Theorem 1 (Salem 1945)

Let $P(z)$ be a minimal polynomial of a Pisot number. Then, for every sufficiently large $n > n_0$, the roots of polynomials

$$Q(z) = z^n P(z) + P^*(z), \quad R(z) = z^n P(z) - P^*(z),$$

where $P^*(z) = z^{\deg P} P(1/z)$, are Salem numbers and (possibly) some roots of unity.

Example: take $P(z) = z^3 - z - 1$, $n = 8$. Then

$$R(z) = z^8(z^3 - z - 1) - (z^3 - z - 1)^* =$$

$$= z^{11} - z^9 - z^8 + z^3 + z^2 - 1 = (z - 1)L(z)$$
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Salem’s construction reversed

- **Theorem 2 (Boyd 1977)**
  
  _Salem’s construction produces all possible Salem numbers._
  
  - **Boyd**: can we get all Pisot numbers back by Salem’s method?
  - **Start** with Salem minimal polynomials $S(z)$ and $T(z)$.
  - **Multiply** by $z \pm 1$, $z^2 - 1$, $z^k$ to get $Q(z)$ and $R(z)$.
  - **Solve back**:
    
    $$P(z) := \frac{Q(z) + R(z)}{2z^n}, \quad P^*(z) = \frac{Q(z) - R(z)}{2}.$$ 
  - **Question**: Is $P(z)$ Pisot?

- **Theorem 3 (Boyd 1977, Bertin, Boyd 1995)**
  
  _YES if the unimodular roots of $Q(z)$ and $R(z)$ interlace. Moreover, for each Salem $S(z)$ there exists corresponding pair $Q(z)$ and $R(z)$ with interlaced roots._
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Theorem 4 (McKee, Smyth, 2012)

Assume that unimodular roots of $Q(z)$, $R(z)$ are interlaced according to one of the 3 types:

- Cyclotomic-Cyclotomic (CC);
- Cyclotomic-Salem (CS);
- Salem-Salem (SS).

Then $P(z)$ obtained from $Q(z)$ and $R(z)$ by reversing the Salem’s construction is a Pisot polynomial. Moreover, all Pisot numbers can be obtained by using this constructions.

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Example: construction using CS pattern

- Let \( Q(z) = z^6 - z^4 - 2z^3 - z^2 + 1 \)
- Let \( R(z) = (z^2 - 1)(z^4 + 1) \).

Figure: CS-type pattern: black - roots of \( Q(z) \), grey - roots of \( R(z) \)
Reversing Salem’s construction:
\[ P(z) := \frac{Q(z) + R(z)}{2z^3} = z^3 - z - 1. \]

**Figure:** \( P(z) \) produced by bad interlacing
Example: failed interlacing

- Let $Q(z) = z^{12} + 1$
- Let $R(z) = z(z^{12} - 1)/(z^2 + 1) = z(z^6 - 1)(z^4 - z^2 + 1)$.

Figure: Bad pattern: black - roots of $Q(z)$, grey - roots of $R(z)$
Bad example continued

- **Reversing Salem’s construction**

\[ 2P(z) = Q(z) + R(z) = z^{12} + z^{11} - z^9 + z^7 - z^5 + z^3 - z + 1 \]

- **Failed interlacing**: roots of \( Q(z) \) and \( R(z) \) do not interlace according to \( CC \), \( CS \) or \( SS \) pattern – not interlaced near \( z = i \) and \( z = -i \).

- **Naive expectation**: \( Z(P) = \deg(P) - 2 \) (say, a complex Pisot number).
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- **Naive expectation**: \( Z(P) = \deg(P) - 2 \) (say, a complex Pisot number).
**Reality:** $Z(P) = \deg P/2 = 6$): *skew-symmetric:* $P^*(-z) = P(z)$.

**Question:** Why this is so?

**Root number:** let $Z(P) = \#$ zeros of $P(z)$ with $|z| < 1$, counted with multiplicities.
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Binary words (basic topology)

- Binary words: \( w = w = w_1w_2 \ldots w_n \), where \( w_j \in \{0, 1\} \).
- Length: \( |w| = n \).
- Empty word \( w = \emptyset \) with \( |\emptyset| = 0 \)
- Concatenation: \( w = uv \), powers: \( w = v^m = \underbrace{v \ldots v}_m \), \( v^0 := \emptyset \)
- Inversion: \( w^{-1} = w_n \ldots w_1 \)
- Conjugation: \( \bar{w} = \bar{w}_1 \ldots \bar{w}_n \), where \( \bar{w}_j = 1 - w_j \).
- Properties: \( (uv)w = u(vw) \), \( \bar{v} \bar{w} = \bar{v} \bar{w} \), \( (vw)^{-1} = w^{-1}v^{-1} \), \( \bar{w}^{-1} = w^{-1} \).
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Irreducible words

- Word $w$ is irreducible $\iff$ does not contain 00 or 11. 
  *Examples*: $w = 0101$ or $w = \emptyset$.
- Reductions: $00 \mapsto \emptyset$, $11 \mapsto \emptyset$.
- Sequence of reductions:
  
  $$w = 011010001 \mapsto 0010001 \mapsto 10001 \mapsto 101 = w'.$$

- Irreducible word $w'$ obtained from $w$ is called reduced form of $w$.
- One can think about irreducible words as quotient group of a free group on $\{0, 1\}$ under concatenation modulo the reduction map.
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  $$w = 011010001 \mapsto 0010001 \mapsto 10001 \mapsto 101 = w'.$$

- Irreducible word $w'$ obtained from $w$ is called *reduced form* of $w$.

- One can think about irreducible words as quotient group of a free group on $\{0, 1\}$ under concatenation modulo the reduction map.
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  *Examples:* $w = 0101$ or $w = \emptyset$.

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Theorem 5

Let \( w' \) be the reduced form of a binary word \( w \). Then \( w' \) is unique and the function \( l(w) \) measures the length of the reduced form: one has \( l(w) = l(w') \) and \( |l(w)| = |w'| \).

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\]

and \( l(v) = l(w) \).

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Elementary properties of $l(w)$

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g) \[ l(ww^{-1}) = \begin{cases} (-1)^{|w|}l(v), & \text{if } |v| \text{ is even,} \\ 2l(w) + (-1)^{|w|}l(v), & \text{if } |v| \text{ is odd.} \end{cases} \]

h) Suppose that \( w \) is a cyclic shift (the rotation) of \( v \) by \( k \) positions. If \( |v| \) is even, then \( l(w) = (-1)^k l(v) \).
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Main Results: Counting zeros and poles

**Theorem 6**
Let $D \subset \mathbb{C}$ be a simply connected domain whose boundary $\partial D$ is a Jordan curve. Assume that a meromorphic function $F : \overline{D} \to \mathbb{C}$ is holomorphic and $\neq 0$ on $\partial D$ with $\Re F$ and $\Im F$ vanishing at most at finitely many points of $\partial D$. Then

$$\# \text{Zeros}_D(F) - \# \text{Poles}_D(F) = \frac{1}{4} \cdot \varepsilon \cdot I(w),$$

where $w$ is a binary word on $\{0, 1\}$, with 1's representing the sign change points of $\Re F$ and 0's representing the sign change points of $\Im F$ on $\partial D$ in a positive direction. The $\varepsilon = \pm 1$ denotes the sign of $\Re F \cdot \Im F$ on the boundary $\partial D$ right before the first sign change recorded in $w$ occurs.
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Theorem 7
Let $P(z) \in \mathbb{C}[z]$ be of degree $d$. Suppose that $P(z) \neq \pm P^*(z)$ and that $P(z) \neq 0$ for any $z$ of modulus $|z| = 1$. Set

$$Q(z) = P(z) + P^*(z), \quad R(z) = P(z) - P^*(z).$$

Then

$$\#\text{Zeros}_\mathbb{D}(P) = \frac{d}{2} + \frac{\varepsilon \cdot l(w)}{4}, \quad Z_\mathbb{D}(P^*) = \frac{d}{2} - \frac{\varepsilon \cdot l(w)}{4},$$

and

$$\#\text{Zeros}_\mathbb{D}(P) - \#\text{Zeros}_\mathbb{D}(P^*) = \frac{\varepsilon \cdot l(w)}{2},$$

where $w$ represents the interlacing pattern of the unimodular zeros of polynomials $Q(z)$ and $R(z)$ odd multiplicities in positive direction.
Polynomials on the unit circle, II

Theorem 8

The number $\varepsilon$ in Theorem 7 is defined by

$$
\varepsilon := \text{sgn} \frac{Q(z)R(z)}{iz^d} = \text{sgn} \frac{P^2(z) - P^*2(z)}{iz^d},
$$

on the unit circle $|z| = 1$ just before the first zero in $w$. If $R(z)$ vanishes at $z = 1$ with multiplicity $m = 2k + 1$, then the initial sign in Theorem 7 just above the point $z = 1$ is

$$
\varepsilon = (-1)^k \text{sgn} P(1) \frac{\partial^m}{\partial z} R(1).
$$

If $m = 1$, then

$$
\varepsilon = \text{sgn} \left( \frac{P'(1)}{P(1)} - \frac{d}{2} \right).
$$
**Theorem 9**

Let $P(z) \in \mathbb{R}[z]$ be of degree $d \geq 1$. $P(z)$ has $Z_D(P) = d - 1$ roots of modulus $|z| < 1$ and $Z_D(P^*) = 1$ root of modulus $|z| > 1$, iff the odd-multiplicity roots of $Q(z) = P(z) + P^*(z)$ and $R(z) = P(z) - P^*(z)$ interlace according to one of the patterns:

<table>
<thead>
<tr>
<th>$d$</th>
<th>$d$</th>
<th>Sign $\varepsilon$</th>
<th>Pattern</th>
<th># of 1’s</th>
<th># of 0’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>odd</td>
<td>1</td>
<td>10</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>even</td>
<td>$-1$ or 1</td>
<td>00</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$\geq 3$</td>
<td>either</td>
<td>1</td>
<td>$(10)^{d-3}10$</td>
<td>$d - 2$</td>
<td>$d - 2$</td>
</tr>
<tr>
<td></td>
<td>either</td>
<td>$-1$</td>
<td>$(01)^{d-2}00$</td>
<td>$d - 2$</td>
<td>$d$</td>
</tr>
<tr>
<td></td>
<td>even</td>
<td>1</td>
<td>$(10)^{d/2-1}0(01)^{d/2-1}0$</td>
<td>$d - 2$</td>
<td>$d$</td>
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<tr>
<td></td>
<td>odd</td>
<td>1</td>
<td>$(10)^{(d-3)/2}1^3(01)^{(d-3)/2}0$</td>
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Here 1’s represent roots of $Q(z)$, 0’s – roots of $R(z)$, written counterclockwise starting after $z = 1$. 
Further Research

- Classification of interlacing patterns that produce complex Pisot numbers.
- Complex Pisot numbers from pairs of cyclotomic polynomials.
- Look for extensions of Beukers-Heckman result on finite reflection groups.
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