Pisot–Salem numbers, interlacing and $\{0, 1\}$-words

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Papers

- J. Jankauskas,
  Binary words, winding numbers and polynomials with interlaced roots (submitted).

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Pisot and Salem numbers

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  A real algebraic integer $\alpha > 1$ whose algeraic conjugates over $\mathbb{Q}$ $\alpha' \neq \alpha$ satisfy $|\alpha'| < 1$.
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- **Salem number:** a real algebraic integer $\alpha > 1$ whose conjugates $\alpha' \neq \alpha$ satisfy $|\alpha'| \leqslant 1$ with at least one conjugate being of modulus $|\alpha'| = 1$.

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Examples

- **Siegel's number:** Smallest Pisot number (proved such by Siegel in 1938):

$$\theta = \frac{1}{6}\left(\sqrt[3]{108+12\sqrt{69}}+\sqrt[3]{108-12\sqrt{69}}\right) = 1.3247\ldots,$$

  with minimal polynomial:

$$P(z) = z^3 - z - 1.$$

- **Lehmer's number:** the positive root $\mu = 1.17628\ldots$ of

$$L(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1,$$

  discovered by Lehmer in 1933.

Complex roots of Pisot and Salem polynomials

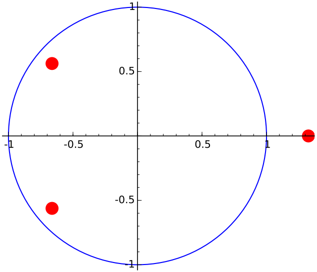

Figure: $z^3 - z - 1$

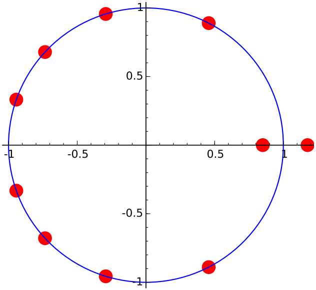

Figure: $L(z)$

Arithmetical neighbours: Salem's construction

- Theorem 1 (Salem 1945)

  Let $P(z)$ be a minimal polynomial of a Pisot number. Then, for every sufficiently large $n > n_0$, the roots of polynomials

$$Q(z) = z^n P(z) + P^*(z), \qquad R(z) = z^n P(z) - P^*(z),$$

  where $P^*(z) = z^{\deg P} P(1/z)$, are Salem numbers and (possibly) some roots of unity.

  - **Example**: take $P(z) = z^3 - z - 1$, $n = 8$. Then

$$R(z) = z^8(z^3 - z - 1) - (z^3 - z - 1)^* =$$
$$= z^{11} - z^9 - z^8 + z^3 + z^2 - 1 = (z-1)L(z)$$

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Salem's construction reversed

- Theorem 2 (Boyd 1977)

  Salem's construction produces all possible Salem numbers.

  - **Boyd**: can we get all Pisot numbers back by Salem's method?
  - **Start** with Salem minimal polynomials $S(z)$ and $T(z)$.
  - **Multiply** by $z \pm 1$, $z^2 - 1$, z^k to get $Q(z)$ and $R(z)$.
  - **Solve back**:

    $$P(z) := (Q(z)+R(z))/(2z^n), \quad P^*(z) = (Q(z)-R(z))/2.$$

  - **Question**: Is $P(z)$ Pisot?

- Theorem 3 (Boyd 1977, Bertin, Boyd 1995)

  YES if the unimodular roots of $Q(z)$ and $R(z)$ interlace. Moreover, for each Salem $S(z)$ there exists corresponding pair $Q(z)$ and $R(z)$ with interlaced roots.

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- Theorem 4 (McKee, Smyth, 2012)

  Assume that unimodular roots of $Q(z)$, $R(z)$ are interlaced according to one of the 3 types:

  - *Cyclotomic-Cyclotomic (CC);*
  - *Cyclotomic-Salem (CS);*
  - *Salem-Salem (SS).*

  Then $P(z)$ obtained from $Q(z)$ and $R(z)$ by reversing the Salem's construction is a Pisot polynomial. Moreover, all Pisot numbers can be obtained by using this constructions.

  - The proof is quite complicated (uses limit functions from quotients of graphs, Beukers-Heckman classification of finite reflection groups, earlier results of Boyd).

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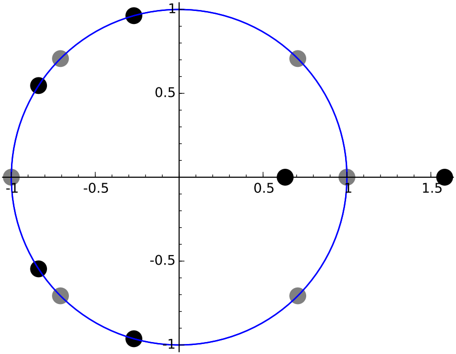
Example: construction using CS pattern

- Let $Q(z) = z^6 - z^4 - 2z^3 - z^2 + 1$
- Let $R(z) = (z^2 - 1)(z^4 + 1)$.

Figure: *CS*-type pattern: black - roots of Q(z), grey - roots of R(z)



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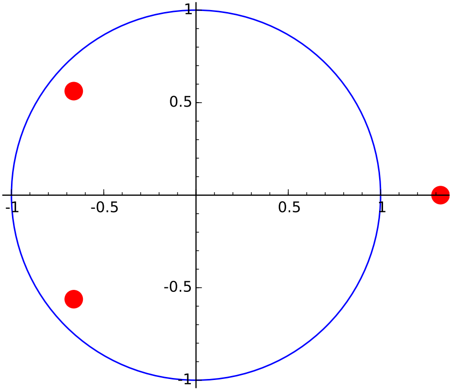
CS interlacing example (continued)

- Reversing Salem's construction:
  $P(z) := (Q(z) + R(z))/(2z^3) = z^3 - z - 1.$

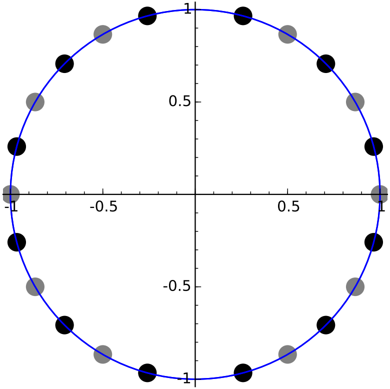
Figure: $P(z)$ produced by bad interlacing



Example: failed interlacing

- Let $Q(z) = z^{12} + 1$
- Let $R(z) = z(z^{12} - 1)/(z^2 + 1) = z(z^6 - 1)(z^4 - z^2 + 1)$.

Figure: **Bad pattern**: black - roots of Q(z), grey - roots of R(z)



Bad example continued

- Reversing Salem's construction

$$2P(z) = Q(z) + R(z) = z^{12} + z^{11} - z^9 + z^7 - z^5 + z^3 - z + 1$$

- **Failed interlacing**: roots of $Q(z)$ and $R(z)$ do not interlace according to CC, CS or SS pattern – not interlaced near $z = i$ and $z = -i$.
- **Naive expectation**: $Z(P) = \deg(P) - 2$ (say, a complex Pisot number).

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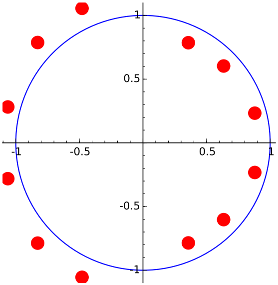

- **Question**: Why this is so?
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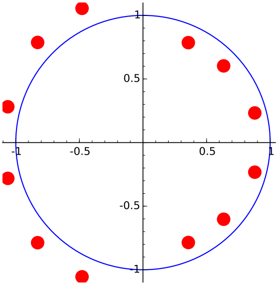

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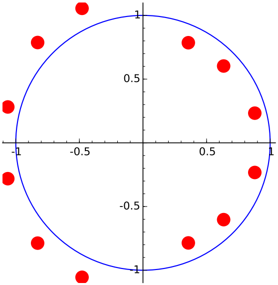

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Binary words (basic topology)

- Binary words: $w = w = w_1 w_2 \dots w_n$, where $w_j \in \{0, 1\}$.
- Length: $|w| = n$.
- Empty word $w = \varnothing$ with $|\varnothing| = 0$
- Concatenation: $w = uv$, powers: $w = v^m = \underbrace{v \dots v}_{m \text{ times}}$,

  $v^0 := \varnothing$
- Inversion: $w^{-1} = w_n \dots w_1$
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- Word w is *irreducible* $\iff$ does not contain 00 or 11. *Examples*: $w = 0101$ or $w = \varnothing$.
- Reductions: $00 \mapsto \varnothing$, $11 \mapsto \varnothing$.
- Sequence of reductions:

$$w = 0\mathbf{11}010001 \mapsto \mathbf{00}10001 \mapsto 10\mathbf{00}1 \mapsto 101 = w'.$$

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Reduced length formula

- *Signed reduced length*: for $w = w_1 w_2 \ldots w_n$ of length $|w| = n$, define

$$l(w) = \sum_{j=1}^{|w|} (-1)^{j+w_j}.$$

- **Theorem 5**
  Let w' be the reduced form of a binary word w. Then w' is unique and the function $l(w)$ measures the length of the reduced form: one has $l(w) = l(w')$ and $|l(w)| = |w'|$.
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Elementary properties of $I(w)$

Let v, w be finite binary words on 0, 1. Then:

a) $|w| \geqslant I(w)$.

b) if w can be obtained from v by reductions, then

$$|v| \equiv |w| \equiv I(v) \equiv I(w) \pmod 2$$

and $I(v) = I(w)$.

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g) $I(wvw^{-1}) = \begin{cases} (-1)^{|w|} I(v), & \text{if } |v| \text{ is even,} \\ 2I(w) + (-1)^{|w|} I(v), & \text{if } |v| \text{ is odd.} \end{cases}$

h) Suppose that w is a cyclic shift (the rotation) of v by k positions. If $|v|$ is even, then $I(w) = (-1)^k I(v)$.

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Main Results: Counting zeros and poles

Theorem 6

Let $\mathcal{D} \subset \mathbb{C}$ be a simply connected domain whose boundary ∂D is a Jordan curve. Assume that a meromorphic function $F : \overline{\mathcal{D}} \to \mathbb{C}$ is holomorphic and $\neq 0$ on ∂D with $\Re F$ and $\Im F$ vanishing at most at finitely many points of ∂D. Then

$$\#Zeros_{\mathcal{D}}(F) - \#Poles_{\mathcal{D}}(F) = \frac{1}{4} \cdot \varepsilon \cdot I(w),$$

where w is a binary word on $\{0, 1\}$, with 1's representing the sign change points of $\Re F$ and 0's representing the sign change points of $\Im F$ on ∂D in a positive direction. The $\varepsilon = \pm 1$ denotes the sign of $\Re F \cdot \Im F$ on the boundary ∂D right before the first sign change recorded in w occurs.

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Polynomials on the unit circle, I

Theorem 7

Let $P(z) \in \mathbb{C}[z]$ be of degree d. Suppose that $P(z) \neq \pm P^*(z)$ and that $P(z) \neq 0$ for any z of modulus $|z| = 1$. Set

$$Q(z) = P(z) + P^*(z), \qquad R(z) = P(z) - P^*(z).$$

Then

$$\#Zeros_{\mathbb{D}}(P) = \frac{d}{2} + \frac{\varepsilon \cdot I(w)}{4}, \qquad Z_{\mathbb{D}}(P^*) = \frac{d}{2} - \frac{\varepsilon \cdot I(w)}{4},$$

and

$$\#Zeros_{\mathbb{D}}(P) - \#Zeros_{\mathbb{D}}(P^*) = \frac{\varepsilon \cdot I(w)}{2},$$

where w represents the interlacing pattern of the unimodular zeros of polynomials $Q(z)$ and $R(z)$ odd multiplicities in positive direction.

Polynomials on the unit circle, II

Theorem 8

The number ε in Theorem 7 is defined by

$$\varepsilon := sgn\frac{Q(z)R(z)}{iz^d} = sgn\frac{P^2(z) - P^{*2}(z)}{iz^d},$$

on the unit circle $|z| = 1$ just before the first zero in w.
If $R(z)$ vanishes at $z = 1$ with multiplicity $m = 2k + 1$, then the initial sign in Theorem 7 just above the point $z = 1$ is

$$\varepsilon = (-1)^k sgn P(1)\frac{\partial^m}{\partial z}R(1).$$

If $m = 1$, then

$$\varepsilon = sgn\left(\frac{P'(1)}{P(1)} - \frac{d}{2}\right).$$

McKee-Smyth revisited

Theorem 9

Let $P(z) \in \mathbb{R}[z]$ be of degree $d \geqslant 1$. $P(z)$ has $Z_{\mathbb{D}}(P) = d-1$ roots of modulus $|z| < 1$ and $Z_{\mathbb{D}}(P^*) = 1$ root of modulus $|z| > 1$, iff the odd-multiplicity roots of $Q(z) = P(z) + P^*(z)$ and $R(z) = P(z) - P^*(z)$ interlace according to one of the patterns:

d	d	*Sign* ε	*Pattern*	# *of* 1*'s*	# *of* 0*'s*
1	*odd*	1	10	1	1
2	*even*	-1 *or* 1	00	0	2
$\geqslant 3$	*either*	1	$(10)^{d-3}10$	$d-2$	$d-2$
	either	-1	$(01)^{d-2}00$	$d-2$	d
	even	1	$(10)^{d/2-1}0(01)^{d/2-1}0$	$d-2$	d
	odd	1	$(10)^{(d-3)/2}1^3(01)^{(d-3)/2}0$	d	$d-2$

Here 1's represent roots of $Q(z)$, 0's – roots of $R(z)$, written counterclockwise starting after $z = 1$.

Further Research

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- Look for extensions of Beukers-Heckman result on finite reflection groups.

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