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THE MAXIMAL VALUE OF POLYNOMIALS WITH RESTRICTED COEFFICIENTS

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Abstract. Let ζ be a fixed complex number. In this paper, we study the quantity $S(\zeta, n) := \max_{f\in\Lambda_n} |f(\zeta)|$, where Λ_n is the set of all real polynomials of degree at most $n-1$ with coefficients in the interval $[0,1]$. We first show how, in principle, for any given $\zeta \in \mathbb{C}$ and $n \in \mathbb{N}$, the quantity $S(\zeta, n)$ can be calculated. Then we compute the limit $\lim_{n\to\infty} S(\zeta, n)/n$ for every $\zeta \in \mathbb{C}$ of modulus 1. It is equal to $1/\pi$ if ζ is not a root of unity. If $\zeta = \exp(2\pi ik/d)$, where $d \in \mathbb{N}$ and $k \in [1, d-1]$ is an integer satisfying $\gcd(k, d) = 1$, then the answer depends on the parity of d. More precisely, the limit is 1, $1/(d \sin(\pi/d))$ and $1/(2d \sin(\pi/2d))$ for $d = 1$, d even and $d > 1$ odd, respectively.

1. Introduction

A nonzero polynomial with $0, 1$ coefficients is called a *Newman polynomial* after [6]. There is a variety of different problems in number theory and analysis related to Newman polynomials. See, for instance, [2], [3], [4], [7], [8].

This paper is motivated by the work of Akiyama, Brunotte, Pethö, and Steiner [1] which, at the first glance, has nothing to do with Newman polynomials. They investigate the sequence of integers satisfying $a_{n+1} = -[\lambda a_n] - a_{n-1}$, $n = 1, 2, \ldots$. It is conjectured in [1] that, for any $a_0, a_1 \in \mathbb{Z}$ and $\lambda \in [-2, 2]$, the sequence a_n, $n = 0, 1, 2, \ldots$ is periodic. The nontrivial case is when $\lambda \in (-2, 2)\backslash\{-1, 0, 1\}$. This problem seems to be very difficult, especially, when the number ζ, defined by the equality $\zeta + \zeta^{-1} = -\lambda$ (so that $|\zeta| = 1$), is not a root of unity. In fact, the only case when the periodicity of the sequence a_n, $n = 0, 1, 2, \ldots$, is proved and published [1] is when $\lambda = (1+\sqrt{5})/2 = 2\cos(\pi/5)$, so that ζ corresponding to λ is a root of unity. It seems that similar methods can be applied to some other λ of the form $2\cos(\pi r)$ with $r \in \mathbb{Q}$. However, for $\lambda \neq 2\cos(\pi r)$, i.e., when ζ is not a root of unity, the periodicity problem seems to be completely out of reach.

We now explain how this periodicity problem is related to polynomials with coefficients in $[0, 1]$ and, in particular, with Newman polynomials. Rewrite the

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recurrence equation as $a_{j+1} + \lambda a_j + a_{j-1} = \{\lambda a_j\}$. Multiplying each equality by ζ^j and adding all obtained equalities for $j = 1, \dots, n$, using $\zeta + \zeta^{-1} = -\lambda$, we get

$$(a_{n+1} - \zeta a_n)\zeta^n = \sum_{j=1}^{n} \{\lambda a_j\}\zeta^j + (a_1 - \zeta a_0).$$

Put $r_n := |a_{n+1} - \zeta a_n|$. Then

$$|r_n| \leqslant |\sum_{j=1}^{n} \{\lambda a_j\}\zeta^j| + |r_0| = |\sum_{j=1}^{n} \{\lambda a_j\}\zeta^{j-1}| + |r_0|.$$

One can show easily (see Proposition 2.4 in [1]) that the periodicity of the sequence a_n, $n = 0, 1, 2, \dots$, would follow from the inequality

$$\limsup_{n\to\infty} \frac{r_n^2}{n} < \frac{\sqrt{4-\lambda^2}}{\pi}.$$

The sum $\sum_{j=1}^{n} \{\lambda a_j\}\zeta^{j-1}$ is equal to the value at ζ of a certain polynomial of degree $\leqslant n-1$ whose coefficients are all in the interval $[0, 1)$. This suggests the problem of finding the maximum $S(\zeta, n)$ over all degree $\leqslant n-1$ polynomials with coefficients in the interval $[0, 1]$ at a fixed point of the unit circle ζ. We shall prove below that $\lim_{n\to\infty} S(\zeta, n)/n = 1/\pi$ for every ζ of modulus 1 which is not a root of unity, so that $\limsup_{n\to\infty} |r_n|/n \leqslant 1/\pi$ which is too weak to solve the above problem of periodicity.

Finally, let us consider the case $\lambda = 1/2$. Then $\zeta = (-1+i\sqrt{15})/4$ satisfying $\zeta + \zeta^{-1} = -1/2$ is not a root of unity. We claim that the sequence a_n, $n = 0, 1, 2, \dots$, defined by $a_{n+1} = -[a_n/2] - a_{n-1}$, $n = 1, 2, \dots$, contains at least four equal elements. Indeed, without loss of generality suppose that the sequence $|a_n|$, $n = 0, 1, 2, \dots$, is unbounded. Then, for any $N \in \mathbb{N}$, there is an index $n > N$ such that $|a_n| \geqslant |a_j|$ for $j = 0, 1, \dots, n-1$. The corresponding polynomial $f(z) := \sum_{j=1}^{n} \{a_j/2\} z^{j-1}$ is a Newman polynomial multiplied by $1/2$. The inequality

$$|r_n| = |a_{n+1} - \zeta a_n| \leqslant |f(\zeta)|/2 + |a_1 - \zeta a_0|$$

combined with the inequality $|a_{n+1} - \zeta a_n| \geqslant |\Im(\zeta a_n)| = |a_n|\sqrt{15}/4$ implies that

$$|a_n| \leqslant 2|f(\zeta)|/\sqrt{15} + 4|a_1 - \zeta a_0|/\sqrt{15}.$$

Hence, by Theorem 4 below, for any $\varepsilon > 0$ and any sufficiently large $n > n(\varepsilon)$, we have $|a_n| < (2/(\pi\sqrt{15}) + \varepsilon)n < 0.165n$. The interval $[-0.165n, 0.165n]$ contains at most $0.33n + 1 < 0.333n < n/3$ distinct integers. Since $|a_n| \geqslant |a_j|$, $j = 0, 1, \dots, n-1$, it includes all integers $a_0, a_1, \dots, a_n$. If none of them is repeated more than three times then the set $\{a_0, a_1, \dots, a_n\}$ is of cardinality $\geqslant (n+1)/3 > n/3$, a contradiction.

2. Main results

Let Λ_n be the set of real polynomials of degree $\leqslant n-1$ whose coefficients all lie in the interval $[0,1]$. Set

$$S(\zeta, n) := \max_{f \in \Lambda_n} |f(\zeta)|$$

for any $\zeta \in \mathbb{C}$. It is clear that

$$S(\zeta, n) = 1 + \zeta + \cdots + \zeta^{n-1}$$

for each nonnegative real number ζ.

We remark first that, for any fixed $\zeta \in \mathbb{C}$, the maximum $S(\zeta, n)$ is attained for some polynomial $f(z) = c_0 + c_1 z + \cdots + c_{n-1} z^{n-1} \in \Lambda_n$. Indeed, treating $f(\zeta)$ as a complex continuous function in n real variables $c_0, \dots, c_{n-1} \in [0,1]$, by a standard argument of compactness, we see that its modulus $|f(\zeta)|$ attains its maximum for some fixed values of the coefficients $c_0, \dots, c_{n-1} \in [0,1]$. It follows that, for any $\zeta \in \mathbb{C}$, there exist a (not necessarily unique) polynomial $f \in \Lambda_n$ such that $S(\zeta, n) = |f(\zeta)|$.

Below, we sometimes use the vector representation of complex numbers. Let us denote the value $f(\zeta)$ whose modulus $|f(\zeta)|$ is the largest among all $f \in \Lambda_n$ by the vector $\mathbf{s}$. As we already said above, the vector $\mathbf{s}$ satisfying $|\mathbf{s}| = S(\zeta, n)$ is not necessarily unique. We begin with the following simple, but important observation:

Theorem 1. *Let $\zeta \neq 0$, and let $\mathbf{s} = f(\zeta) = \sum_{j=0}^{n-1} c_j \zeta^j$ be one of the vectors of maximal length, where $f \in \Lambda_n$. Then f is a Newman polynomial. Moreover, for each $j = 0, 1, \dots, n-1$, we have $c_j = 1$ if the projection of the vector ζ^j to the vector $\mathbf{s}$ is positive, and $c_j = 0$ otherwise.*

In particular, if $\mathbf{s}$ is one of the extremal vectors, then the line passing through the origin and orthogonal to $\mathbf{s}$ contains none of the points $1, \zeta, \dots, \zeta^{n-1}$. Therefore, Theorem 1 suggests the following practical method for the computation of $S(\zeta, n)$. Suppose that $\zeta \neq 0$. Let ℓ be any line passing through the origin but through none of the n points $D_n := \{1, \zeta, \dots, \zeta^{n-1}\}$. Let us rotate the line ℓ, say, counterclockwise until it reaches at least one of the points of D_n. Then rotate ℓ again by an angle so small that no point of D_n lies on ℓ and stop. At this, first, stop we calculate the sums r_1 and l_1 of the numbers from D_n that lie on both sides, say, 'right hand side' and 'left hand side' of ℓ. (Note that $r_1 + s_1 = 1 + \zeta + \cdots + \zeta^{n-1}$.) Then rotate ℓ until it reaches at least one point of D_n again, slightly pass this point, stop for the second time, and calculate r_2, l_2, where $r_2 + l_2 = 1 + \zeta + \cdots + \zeta^{n-1}$, and so on. The last, say, kth stop will be when ℓ is rotated by the angle π, so that it reaches its original position (but changes its direction). It is easy to see that $k \leqslant n$, where the value n for k is attained when no two points of D_n lie on a line passing through the origin. Theorem 1 implies that

$$S(\zeta, n) = \max\big(|r_1|, |l_1|, |r_2|, |l_2|, \dots, |r_k|, |l_k|\big).$$

In particular, if ζ is a negative real number, then all of its powers are positive and negative real numbers. Let us start with a line, say, orthogonal to the real axis and begin the process described above. Then there is only one stop, giving $r_1 = 1 + \zeta^2 + \cdots + \zeta^u$, where $u \leqslant n-1$ is the largest even integer, and $l_1 = -\zeta - \zeta^3 - \cdots - \zeta^v$, where $v \leqslant n-1$ is the largest odd integer. The formula $S(\zeta, n) = \max\big(|r_1|, |l_1|\big)$ yields the following corollary:

Corollary 2. *Let u and v be the largest even and odd numbers, respectively, satisfying $u, v \leqslant n-1$. If ζ is a negative real number then*

$$S(\zeta, n) = \max\big(1 + \zeta^2 + \cdots + \zeta^u, -\zeta(1 + \zeta^2 + \cdots + \zeta^{v-1})\big).$$

Suppose that ζ is a complex number of modulus 1. In the evaluation of $S(\zeta, n)$ there are two different cases depending on whether ζ is or is not a root of unity. Let throughout $\zeta_d := \exp(2\pi i/d)$ be a primitive dth root of unity. Let also U_d be the set of its conjugates over $\mathbb{Q}$, so that $|U_d| = \varphi(d)$, where $\varphi(d)$ stands for the Euler totient function. In the next theorem, we calculate the value $S(\zeta, md)$ for every $\zeta \in U_d$ and $m \in \mathbb{N}$.

Theorem 3. *Suppose that $m \in \mathbb{N}$ and $\zeta \in U_d$, where $d \geqslant 2$. Then $S(\zeta, md) = m/\sin(\pi/d)$ if d is even and $S(\zeta, md) = m/(2\sin(\pi/2d))$ if d is odd.*

The main theorem of this paper can be stated as follows:

Theorem 4. *Let $\zeta \in \mathbb{C}$ be a complex number of modulus 1. If $\zeta \in U_d$, where $d \in \mathbb{N}$, then*

$$\lim_{n\to\infty} S(\zeta, n)/n = \begin{cases} 1, & \text{if } d = 1, \\ 1/(d\sin(\pi/d)) & \text{if } d \text{ is even}, \\ 1/(2d\sin(\pi/2d)) & \text{if } d > 1 \text{ is odd}. \end{cases}$$

If ζ is not a root of unity, then $\lim_{n\to\infty} S(\zeta, n)/n = 1/\pi$.

In the next section, we shall prove Theorems 1, 3 and 4. Some numerical examples will be given in Section 4.

3. Proofs

Proof of Theorem 1. The vector $\mathbf{s}$ is the sum of the vectors ζ^j, where $j = 0, \ldots, n-1$, scaled by $c_j \in [0,1]$. Clearly, $|\mathbf{s}| > 0$. Put $\mathbf{s}_j := \zeta^j$. If there is an index $j \in \{0, \ldots, n-1\}$ such that the projection of $\mathbf{s}_j = \zeta^j$ to $\mathbf{s}$ is positive (i.e., the scalar product $(\mathbf{s}_j, \mathbf{s})$ is positive) and $c_j < 1$ then, by replacing c_j by 1, we obtain that the vector $\mathbf{s} - c_j\mathbf{s}_j + \mathbf{s}_j = \mathbf{s} + (1 - c_j)\mathbf{s}_j$ has greater length than $|\mathbf{s}|$, a contradiction. Similarly, suppose that there is an index $j \in \{0, \ldots, n-1\}$ such that the projection of $\mathbf{s}_j = \zeta^j$ to $\mathbf{s}$ is negative or zero (i.e., $(\mathbf{s}_j, \mathbf{s}) \leqslant 0$) and $c_j > 0$. Then, by replacing c_j by 0, we obtain that the vector $\mathbf{s} - c_j\mathbf{s}_j$ has greater length than $|\mathbf{s}|$, because $|\mathbf{s} - c_j\mathbf{s}_j|^2 - |\mathbf{s}|^2 = c_j^2|\mathbf{s}_j|^2 - 2c_j(\mathbf{s}_j, \mathbf{s}) \geqslant c_j^2|\mathbf{s}_j|^2 > 0$, a contradiction again. □

The following simple lemma will be used in the proof of Theorem 3 and in numerical examples of Section 4:

Lemma 5. *Let* Γ_d *be the set of complex roots of* $z^d - 1 = 0$, *where* $d \geqslant 2$, *and let* ℓ *be a line passing through the origin but through none of the points of* Γ_d. *Then the sum of all numbers from* Γ_d *that lie on one side of* ℓ *belongs to some axis of symmetry of a regular* d*-gon with vertices in* Γ_d, *and the modulus of this sum is equal to* $1/\sin(\pi/d)$ *for* d *even, and to* $1/(2\sin(\pi/2d))$ *for* d *odd.*

Proof. Consider a half plane in that side of ℓ, where exactly $k = [d/2]$ points of Γ_d are lying. Take $\zeta_d = \exp(2\pi i/d)$. Let r be the smallest positive integer such that ζ_d^r is the first vertex of Γ_d in that half plane counterclockwise. Then the points of Γ_d in this half plane are the powers ζ_d^j, where $j = r, \dots, r+k-1$. Note that all sums $\zeta_d^{r+j} + \zeta_d^{r+k-1-j}$, where $j = 0, \dots, [(k-1)/2]$, lie on the same axis of symmetry of a regular d-gon, hence so does their sum $\sum_{j=r}^{r+k-1} \zeta_d^j = \frac{1}{2}\sum_{j=0}^{k-1}(\zeta_d^{r+j} + \zeta_d^{r+k-1-j})$ on the same side of ℓ.

Next, recall that $1 + \zeta_d + \cdots + \zeta_d^{d-1} = 0$. Hence on both sides of ℓ we get the sums lying on the same axis of symmetry whose moduli are

$$|1 + \zeta_d + \cdots + \zeta_d^{[d/2]-1}| = |(\zeta_d^{[d/2]} - 1)/(\zeta_d - 1)| = \frac{\sin(\pi[d/2]/d)}{\sin(\pi/d)}.$$

This is equal to $\frac{1}{\sin(\pi/d)}$ for d even, and to $\frac{\cos(\pi/2d)}{\sin(\pi/d)} = \frac{1}{2\sin(\pi/2d)}$ for d odd. □

Proof of Theorem 3. Suppose that $\zeta \in U_d$, where $d \geqslant 2$ is an integer. Since $\zeta^d = 1$, we can write the value $f(\zeta)$ of the polynomial $f \in \Lambda_{md}$ at $z = \zeta$ as

$$f(\zeta) = f_1(\zeta) + \cdots + f_m(\zeta),$$

where $f_1, \dots, f_m \in \Lambda_d$. Hence $S(\zeta, md) \leqslant mS(\zeta, d)$. Moreover, if $f_0 \in \Lambda_d$ is a polynomial for which $S(\zeta, d) = |f_0(\zeta)|$ then, by setting $f(z) := f_0(z)(1 + z^d + \cdots + z^{(m-1)d}) \in \Lambda_{md}$, we find that $f(\zeta) = mf_0(\zeta)$. Hence $S(\zeta, md) = mS(\zeta, d)$. It remains to show that $S(\zeta, d) = 1/\sin(\pi/d)$ if d is even and $S(\zeta, d) = 1/(2\sin(\pi/2d))$ if $d > 1$ is odd.

Let f be a Newman polynomial of degree $\leqslant d-1$ for which we have $S(\zeta, d) = |f(\zeta)|$. Put $\mathbf{s} = f(\zeta)$. By Theorem 1, $\mathbf{s}$ is the sum of all numbers ζ^j, where $j \in \{0, \dots, d-1\}$, that lie on one side of a line ℓ orthogonal to $\mathbf{s}$ but not on ℓ itself. Moreover, none of the points ζ^j lies on ℓ. Since $\zeta \in U_d$, the set $\{\zeta^j : j = 0, \dots, d-1\}$ is precisely the set of roots of $z^d - 1$, i.e., Γ_d. By Lemma 5, $|\mathbf{s}| = 1/\sin(\pi/d)$ for d even and $|\mathbf{s}| = 1/(2\sin(\pi/2d))$ for $d > 1$ odd. This completes the proof of the theorem. □

Proof of Theorem 4. The case $\zeta = 1$ is obvious. The maximal sum is $1 + \zeta + \cdots + \zeta^{n-1}$, so $S(1, n) = n$ for every positive integer n. Suppose that $\zeta \in U_d$ with $d \geqslant 2$. Choose an integer m such that $md \leqslant n < (m+1)d$. Since $S(\zeta, n)$ is a nondecreasing function in n, we have $S(\zeta, md) \leqslant S(\zeta, n) \leqslant S(\zeta, (m+1)d)$.

Thus, by Theorem 3, for even $d \geqslant 2$, we have

$$\frac{1-d/n}{d\sin(\pi/d)} = \frac{n/d-1}{n\sin(\pi/d)} < \frac{m}{n\sin(\pi/d)} = \frac{S(\zeta,md)}{n} \leqslant \frac{S(\zeta,n)}{n} \leqslant \frac{S(\zeta,(m+1)d)}{n} = \frac{m+1}{n\sin(\pi/d)} \leqslant \frac{n/d+1}{n\sin(\pi/d)} = \frac{1+d/n}{d\sin(\pi/d)}.$$

It follows that $\lim_{n\to\infty} S(\zeta,n)/n = 1/(d\sin(\pi/d))$ for each even $d \geqslant 2$. The proof of the case when $d > 1$ is odd is similar: one just uses the 'odd' part of Theorem 3 instead of its 'even' part.

Finally, suppose that $\zeta = e^{i\phi}$, where $0 < \phi < 2\pi$, is a complex number of modulus 1 which is not a root of unity. Then $\phi/\pi \notin \mathbb{Q}$. Suppose that $\mathbf{s} = f(\zeta) = \sum_{j=0}^{n-1} c_j\zeta^j$ is one of the vectors of maximal length. Then, by Theorem 1, $c_j \in \{0,1\}$ with $c_j = 1$ if and only if the projection of ζ^j to $\mathbf{s}$ is positive. Let ℓ be the line passing through the origin and orthogonal to $\mathbf{s} = |\mathbf{s}|e^{i\tau}$. The line ℓ divides the complex plane into two half planes. Let us divide the open half plane with the point $e^{i\tau}$ into $2M$ equal sectors, where for each $k \in \{-M,\dots,-1,1,\dots,M\}$ the kth sector consists of complex numbers whose arguments belong to the interval $[\tau+\pi(k-1)/2M, \tau+\pi k/2M)$ for $k > 0$ and to the interval $[\tau+\pi k/2M, \tau+\pi(k+1)/2M)$ for $k < 0$. (Since this half plane needs to be open, one exception is that the interval corresponding to $k = -M$ is open $(\tau-\pi/2, \tau-\pi(M-1)/2M)$.)

For any $j \in \{0,1,\dots,n-1\}$ the vector ζ^j is belongs to the sum $\mathbf{s}$ if and only if it lies in one of the above $2M$ sectors. The sum of the vectors $\zeta^j = \cos(j\phi)+i\sin(j\phi)$ is $f(\zeta) = \mathbf{s} = |\mathbf{s}|e^{i\tau}$, hence $f(\zeta)e^{-i\tau}$ is a real number. Using the fact that the number

$$f(\zeta)e^{-i\tau} = \sum_{j=0}^{n-1} c_j\zeta^j e^{-i\tau} = \sum_{j=0}^{n-1} c_j(\cos(j\phi-\tau)+i\sin(j\phi-\tau))$$

is real, we obtain that $\sum_{j=0}^{n-1} c_j\sin(j\phi-\tau) = 0$, so

$$|f(\zeta)| = f(\zeta)e^{-i\tau} = \sum_{j=0}^{n-1} c_j\cos(j\phi-\tau).$$

Suppose that the sector corresponding to the index k contains n_k vectors of the set $\{1,\dots,\zeta^{n-1}\}$, say, ζ^j with $j \in N_k$, where N_k is a subset of $\{0,1,\dots,n-1\}$ of cardinality n_k. Then $\sum_{j\in N_k}\cos(j\phi-\tau)$ is at least $n_k\cos(|k|\pi/2M)$ and at most $n_k\cos((|k|-1)\pi/2M)$. It follows that

$$\sum_{k=1}^{M}(n_k+n_{-k})\cos(k\pi/2M) \leqslant |f(\zeta)| \leqslant \sum_{k=1}^{M}(n_k+n_{-k})\cos((k-1)\pi/2M).$$

By an old result of Weyl [9] (see, e.g., Example 2.1 in [5]), the sequence of fractional parts $\{m\phi/2\pi\}$, $m = 0,1,2,\dots$, is uniformly distributed in the interval $[0,1)$, because $\phi/2\pi \notin \mathbb{Q}$. Fix $\varepsilon > 0$. Then fix any $M = M(\varepsilon) \in \mathbb{N}$ satisfying

$$\frac{1}{4M}\left(1+\frac{1}{\tan(\pi/4M)}\right) < \frac{1+\varepsilon}{\pi} \quad \text{and} \quad \frac{1}{4M}\left(-1+\frac{1}{\tan(\pi/4M)}\right) > \frac{1-\varepsilon}{\pi}.$$

Such an M exists, because $\lim_{x\to\infty} x\tan(\pi/x) = \pi$. Given $k \in \{1,\dots,M\}$, ζ^j belongs to the kth sector if an only if there is an $l \in \mathbb{Z}$ such that

$$\tau + \pi(k-1)/2M \leqslant j\phi - 2\pi l < \tau + \pi k/2M,$$

i.e., $(k-1)/4M \leqslant \{j\phi/2\pi - \tau/2\pi\} < k/4M$. Using uniform distribution of $\{j\phi/2\pi - \tau/2\pi\}$, $j = 0,1,\dots$, in $[0,1)$, we deduce that $(1-\varepsilon)n/4M < n_k < (1+\varepsilon)n/4M$ for each sufficiently large $n \in \mathbb{N}$. The same bounds hold for $k \in \{-M,\dots,-1\}$. Hence

$$(1-\varepsilon)\frac{n}{2M}\sum_{k=1}^{M}\cos(k\pi/2M) \leqslant |f(\zeta)| \leqslant (1+\varepsilon)\frac{n}{2M}\sum_{k=1}^{M}\cos((k-1)\pi/2M).$$

Setting $x = \pi/2M$ into the identity

$$1/2 + \cos(x) + \dots + \cos((M-1)x) = \frac{\sin((M-1/2)x)}{2\sin(x/2)},$$

we derive that

$$\sum_{k=1}^{M}\cos((k-1)\pi/2M) = \frac{1}{2}\left(1+\frac{1}{\tan(\pi/4M)}\right)$$

and

$$\sum_{k=1}^{M}\cos(k\pi/2M) = \frac{1}{2}\left(-1+\frac{1}{\tan(\pi/4M)}\right).$$

Hence

$$(1-\varepsilon)\frac{n}{4M}\left(-1+\frac{1}{\tan(\pi/4M)}\right) \leqslant |f(\zeta)| \leqslant (1+\varepsilon)\frac{n}{4M}\left(1+\frac{1}{\tan(\pi/4M)}\right).$$

By the choice of M, this implies that $(1-\varepsilon)^2 n/\pi \leqslant |f(\zeta)| \leqslant (1+\varepsilon)^2 n/\pi$. Thus

$$(1-\varepsilon)^2/\pi \leqslant S(\zeta,n)/n = |f(\zeta)/n| \leqslant (1+\varepsilon)^2/\pi$$

for each $n \geqslant n(\varepsilon)$. However, ε can be arbitrarily small, so $\lim_{n\to\infty} S(\zeta,n)/n = 1/\pi$, as claimed. □

4. Practical computations

Take $\zeta = \exp(2\pi i/5)$ and $n = 5$. By Lemma 5, we can take any ℓ which goes through none of the roots of $z^5 - 1 = 0$. Take ℓ such that 1 and ζ are on one of its sides. Then, by Lemma 5, we find that $|1+\zeta| = 1/(2\sin(\pi/10)) = (1+\sqrt{5})/2 = 1.61803\dots$.

Similarly, taking $\zeta = \exp(9\pi i/7)$ to be one of the roots of $z^{14} - 1 = 0$ and $n = 14$, one can choose ℓ to be the imaginary axis. Then one of the extremal Newman polynomials will be $f(z) = 1 + z^3 + z^5 + z^6 + z^8 + z^9 + z^{11}$, because $0, 3, \dots, 11$ are the only powers of ζ that are on the right hand side of ℓ. Lemma 5 and Theorem 3 gives $f(\zeta) = 1/\sin(\pi/14) = 4.49395\dots$.

Take $\zeta = i$ and $n = 5$. By Theorem 1, there are four possible quadrants for the location of $\mathbf{s}$. The maximum for $|f(i)|$ is attained by Newman polynomials $1 + z + z^4$ and $1 + z^3 + z^4$, giving $\mathbf{s} = 2 \pm i$. Hence $S(i, 5) = \sqrt{5}$. Note that the maximal vectors $2 \pm i$ do not lie on an axis of symmetry of the square with vertices $1, i, -1, -i$. So Lemma 5 does not hold, because there is one 'double' vector $1 = i^4$.

It seems likely that when ζ is not a root of unity one cannot expect any simple formulae for $S(\zeta, n)$. For example, for ζ satisfying $\zeta^2 - \zeta/2 + 1 = 0$, we calculated the value $S(\zeta, 100) = 31.8928\dots$. It is easy to see that $S(\zeta, 100)/100 = 0.31892\dots$ is quite close to the limit value $1/\pi = 0.31830\dots$, given by Theorem 4. The value $S(\zeta, 100)$ is attained by the polynomial $f(z) = z^{97} + z^{96} + z^{95} + z^{92} + z^{91} + z^{90} + z^{87} + z^{86} + z^{82} + z^{81} + z^{78} + z^{77} + z^{76} + z^{73} + z^{72} + z^{71} + z^{68} + z^{67} + z^{63} + z^{62} + z^{58} + z^{57} + z^{54} + z^{53} + z^{52} + z^{49} + z^{48} + z^{44} + z^{43} + z^{39} + z^{38} + z^{35} + z^{34} + z^{33} + z^{30} + z^{29} + z^{28} + z^{25} + z^{24} + z^{20} + z^{19} + z^{16} + z^{15} + z^{14} + z^{11} + z^{10} + z^9 + z^6 + z^5 + z + 1$.

Finally, we remark that the results of this paper may be applied to polynomials whose coefficients lie in any real interval $[a, b]$. In this case, if $\zeta \neq 1$, the constant factor $b - a$ will appear on the right hand side of the formulas established by Theorems 3 and 4. Indeed, any polynomial $f(z) = \sum_{j=0}^{n-1} c_j z^j$ with coefficients $c_j \in [a, b]$ can be written as

$$f(z) = (b - a)g(z) + ah(z),$$

where $g(z) = \sum_{j=0}^{n-1} ((c_j - a)/(b - a))z^j$ is a polynomial with coefficients in $[0, 1]$ and $h(z) = 1 + \dots + z^{n-1} = (z^n - 1)/(z - 1)$. Now, $h(\zeta) = 0$ if $\zeta \neq 1$ is an nth root of unity. Furthermore, $|h(\zeta)|$ is bounded by an absolute constant depending on ζ only if $|\zeta| \leqslant 1$ and $\zeta \neq 1$, so that $|h(\zeta)|/n \to 0$ as $n \to \infty$. Taking $n = d$, Theorem 3 may be applied immediately to $g(z)$. To obtain a corresponding limit in Theorem 4, one can divide the equality by n, and then let $n \to \infty$.

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