Simple linear relations between conjugate algebraic numbers

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Papers

- A. Dubickas, K. G. Hare, J. Jankauskas, *No two non-real conjugates of a Pisot number have the same imaginary part,* (submitted), `http://arxiv.org/abs/1410.1600`
- A. Dubickas, J. Jankauskas, *Simple linear relations with algebraic numbers of small degree,* J. Ramanujan Math. Soc. **30** (2) (2015), 219–235.

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Pisot numbers

- **Pisot number:**
  A real algebraic integer $\alpha > 1$ whose algeraic conjugates over $\mathbb{Q}$ $\alpha' \neq \alpha$ satisfy $|\alpha'| < 1$.

- **Example:** Golden ratio

$$\alpha = \frac{1+\sqrt{5}}{2} = 1.61803\ldots, \quad \alpha' = \frac{1-\sqrt{5}}{2} = -0.61803\ldots$$

Minimal polynomial:

$$f(x) = x^2 - x - 1.$$

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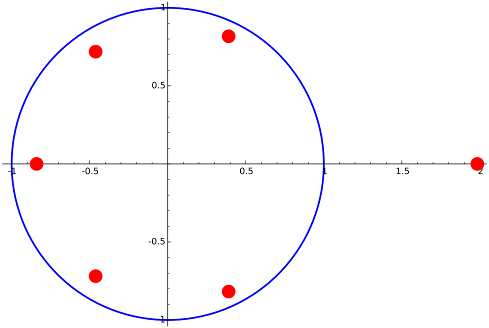

Figure: Roots of $f(x) = x^6 - x^5 - x^4 - x^3 - x^2 - x - 1$

- **Importance:** They behave like *almost integers*. They appear in dynamical systems, fractals, crystallography etc.

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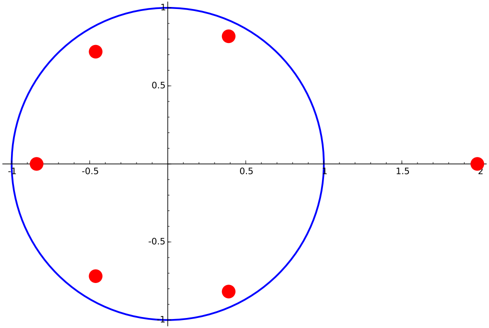

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Arithmetical neighbours: Salem numbers

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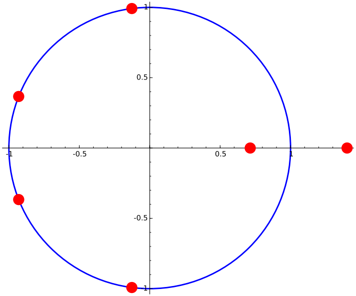
Figure: Roots of $f(x) = x^6 - x^4 - 2x^3 - x^2 + 1$



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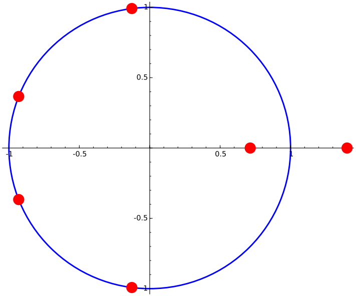

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A. Dubickas and C. J. Smyth investigated the configurations of lines in $\mathbb{C}$ joining the pairs of conjugates of an algebraic number.

- Theorem 1
  No three conjugates of a Salem number α lie on a line.
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  No two lines that pass through two distinct conjugates of a Salem number are parallel, apart from $d/2-1$ lines parallel to the imaginary axis passing through complex conjugate pairs.
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- Lemma 3 (Mignotte, 1984)

  The equality $\alpha_1^{k_1}\alpha_2^{k_2}\dots\alpha_d^{k_d} = 1$ with algebraic numbers $\alpha_1, \alpha_2, \dots, \alpha_d$ that are conjugates of a Pisot number α of degree d over $\mathbb{Q}$ and $k_1, k_2, \dots, k_d \in \mathbb{Z}$ can only hold if $k_1 = k_2 = \dots = k_d$.

- Corollary 4

  No two non-real conjugates of a Pisot number have the same argument.

  - This follows by applying Mignotte's result to the multiplicative relation $\alpha_1\overline{\alpha_2}\alpha_2^{-1}\overline{\alpha_1}^{-1} = 1$
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Simpler question

- Problem 5 (A. Dubickas, C. J. Smyth, 2008)

  Let $\mathcal{L}$ be a line that is parallel to the real or imaginary axis. How many conjugates of a Pisot number lay on $\mathcal{L}$?

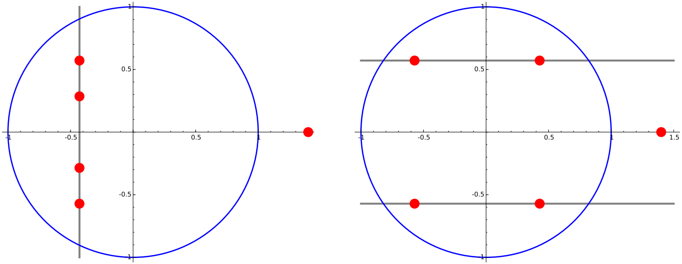

Figure: Conjugates of Pisot numbers on vertical and horizontal lines

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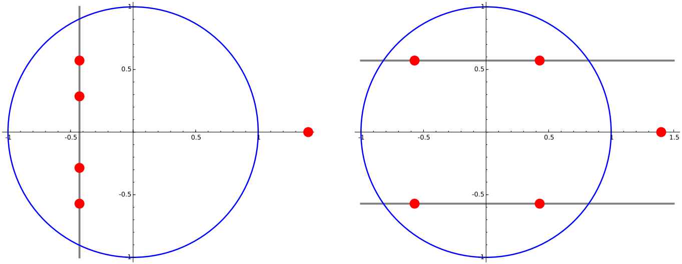

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Linear equations for conjugates of Pisot numbers

- Problem 6

  Does there exist a Pisot number α with conjugates α_1, α_2, α_3, such that

  $$\alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \textit{or} \quad \alpha_1 = \alpha_2 + \alpha_3$$

  holds?

- Problem 7

  Does there exist a Pisot number α with conjugates α_1, α_2, α_3, α_4 that satisfy at least one of the equations

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Connection to the geometrical problem

- **Vertical lines:** If

$$\Re(\alpha') = \Re(\alpha''), \qquad \alpha'' \neq \overline{\alpha'},$$

then $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4$ holds for

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Main results, part I: four term equations

- Theorem 8

  If α is a Pisot number of degree $d \geqslant 4$ whose four distinct conjugates over $\mathbb{Q}$ satisfy the relation

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  then

  $$\alpha = \frac{1 + \sqrt{3 + 2\sqrt{5}}}{2}.$$

  Moreover, there exists no Pisot number α whose four distinct conjugates satisfy the linear relation

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Main results, part II: geometrical consequences

- The Pisot number given in Theorem 8 is

$$\alpha = \alpha_1 = (1 + \sqrt{3 + 2\sqrt{5}})/2 = 1.8667\dots$$

has a minimal polynomial

$$f(x) = x^4 - 2x^3 + x - 1.$$

It has another real conjugate

$$\alpha_2 = (1 - \sqrt{3 + \sqrt{5}})/2,$$

and two non-real conjugates

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- Corollary 10

  At most two conjugates of a Pisot number can have the same real part, in which case they are complex-conjugate to each other.

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  If α is a Pisot number having at least three conjugates over $\mathbb{Q}$ satisfying the relation

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then α is Siegel's number $\theta = 1.32471\ldots$ (the root of $x^3 - x - 1 = 0$). Furthermore, there does not exist a Pisot number α whose three conjugates satisfy the relation

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Demonstration for the equation $\pm\alpha_1 = \alpha_2 + \alpha_3$

- We can assume $\alpha = \alpha_1$ is a Pisot number. From $\pm\alpha_1 = \alpha_2 + \alpha_3$ we obtain $\alpha < 2$.

- **Lemma 12 (Beukers and Zagier, 1997)**

Let $\beta_1, \dots, \beta_r$ be non-zero algebraic numbers such that their sum $N = \beta_1 + \cdots + \beta_r$ is a rational integer. If

$$\beta_1^{-1} + \cdots + \beta_r^{-1} \neq N \tag{1}$$

then $h(\beta_1) + \cdots + h(\beta_r) \geqslant \frac{1}{2}\log\left(\frac{1+\sqrt{5}}{2}\right)$. Here, $h(\gamma)$ stands for the Weill height of an algebraic number γ.

- If γ or $-\gamma$ is algebraically conjugate to a Pisot number α, then $h(\gamma) = \log\alpha/d$, where $d = \deg(\alpha)$

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- Beukers–Zagier inequality gives
$$\frac{3\log\alpha}{d} = 3h(\alpha) = h(\pm\alpha_1) + h(\alpha_2) + h(\alpha_3) =$$
$$= h(\beta_1) + h(\beta_2) + h(\beta_3) \geqslant \frac{1}{2}\log\left(\frac{1+\sqrt{5}}{2}\right),$$
or
$$\frac{3\log\alpha}{d} \geqslant \frac{1}{2}\log\left(\frac{1+\sqrt{5}}{2}\right).$$
- Since $1 < \alpha < 2$,
$$d \leq \frac{6\log 2}{\log\left(\frac{1+\sqrt{5}}{2}\right)} = 8.64252\ldots.$$

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or

$$\frac{3\log\alpha}{d} \geqslant \frac{1}{2}\log\left(\frac{1+\sqrt{5}}{2}\right).$$

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- Consequently, d can be only $d = 3, 4, 5, \ldots, 8$. Plugging these d back, one gets refined intervals:

$$\left(\frac{1+\sqrt{5}}{2}\right)^{d/6} \leqslant \alpha < 2, \qquad 3 \leqslant d \leqslant 8.$$

- We computed all Pisot numbers in these intervals up to degree 8 by using *Boyd's algorithm*. In total, 78 Pisot numbers were found.
- After testing, it was found that the only solution was produced by Siegel's polynomial $x^3 - x - 1 = 0$.

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Computations for the four term equations

- Similar methods can be used to show that all Pisot numbers whose conjugates satisfy any of the four therm linear relation must lie in the intervals

$$\left(\frac{1+\sqrt{5}}{2}\right)^{d/8} \leqslant \alpha < 3, \qquad 4 \leqslant d \leqslant 18.$$

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- The initial searches for small degrees (up to $d \leqslant 17$) were done on a single RedHat Linux server equipped with two Intel Xeon X5672 series 3.20GHz 12MB Cache 1333MHz 95W CPUs and 96735MB of RAM at the University of Waterloo.

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Computations (continued)

Table: Single Intel Xeon 3.4GHz machine search timings

deg α	Search interval	CPU time
10	$[1, 3]$	25 sec.
11	$[1, 3]$	2 min. 13 sec.
12	$[1, 3]$	11 min. 7
13	$[1, 3]$	54 min.
14	$[1, 3]$	4 h. 13 min.
15	$[1, 3]$	18 h. 47 min.
16	$[610/233, 3]$	3 days 11 h.
17	$[367/132, 3]$	13 days 17 h.
18	$[437/148, 3]$	$\leqslant$ 60 days (estimated)

Computations (continued)

- Consequently, the computations were distributed on a large collection of 2 Intel 5272 series 3.4Ghz/6M/1600Mhz 80W Dual Core Xeon Processor machines, allowing up to 120 simultaneous searches to be done. This was done by partitioning the search interval into 2868 subintervals.
- Distributed computations took 13.64 CPU days. In total, 1 955 183 Pisot numbers were found. They are are counted in Table 2.

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Table: # of Pisot numbers $\alpha \in (\tau^{\deg(\alpha)/8}, 3)$

$\deg \alpha$	# of α's	$\deg \alpha$	# of α's	$\deg \alpha$	# of α's
4	43	9	5 555	14	140 587
5	162	10	9 937	15	273 851
6	353	11	23 410	16	402 209
7	1 075	12	40 812	17	630 025
8	2 069	13	85 979	18	339 116

- After that, the minimal polynomials of Pisot numbers were tested for possible linear relations among roots (numerically and by using the resultants). There was only one solution detected, namely, a Pisot number with the minimal polynomial $f(x) = x^4 - 2x^3 + x - 1$.

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More on 3-term equations

Note: Assume that α can be *arbitrary algebraic number* (not necessarilly Pisot!).

Theorem 13

Let d be an integer in the range $3 \leqslant d \leqslant 8$ and let α be an algebraic number of degree d over $\mathbb{Q}$. Then some three of its conjugates α_1, α_2, α_3 satisfy the relation

$$\alpha_1 = \alpha_2 + \alpha_3$$

if and only if $d = 6$ and the minimal polynomial of α over $\mathbb{Q}$ is an irreducible polynomial of the form

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$$f(x) = x^6 + 2ax^4 + 2bx^3 + (a^2 - c^2t)x^2 + 2(ab - cet)x + b^2 - e^2t$$

for some rational numbers $a, b, c, e \in \mathbb{Q}$ and some square-free integer $t \in \mathbb{Z}$.

Basic steps for $\pm\alpha_1 = \alpha_2 + \alpha_3$ up to $d \leqslant 8$:

- Low degree cases $d = 3$, $d = 4$ and prime degree cases $d = 5$, $d = 7$ are eliminated by elementary considerations and Kurbatov's lemma.
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- Case $d = 8$, $\alpha_1 = \alpha_2 + \alpha_3$ is most complex: use combinatorics and evaluate group determinants for the Galois group of α.
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