Littlewood polynomials with prescribed number of zeros inside the unit disk

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Definitions

▶ **Littlewood polynomials:**

\[ p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \]

where \( a_j \in \{-1, 1\}, 0 \leq j \leq n. \)

▶ **Inversion:**

\[ p^*(z) := z^{\deg p} p(1/z) \]

for any \( p(z) \in \mathbb{R}[z]. \)

▶ **Self-reciprocal polynomials:** \( p^*(z) = \pm p(z). \)

▶ **Skew-symmetric polynomials:** If \( p^*(z) = \pm p(-z). \)
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- Let $N(p)$ denote the number of roots of $p(z)$ in $D$ (counted with multiplicities).
- Let $U(p)$ denote the number of roots of $p(z)$ in $\partial D$ (counted with multiplicities). Such roots are called *unimodular*. 
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Problem 1: Let \((n, k)\) be a pair of integers such that
\[1 \leq k \leq n - 1.\] For such a pair, does there always exist a
Littlewood polynomial \(p(z)\) of degree \(n\) with precisely \(k\) roots inside the unit disk and no unimodular roots, that
is, \(N(p) = k, \ U(p) = 0\)?

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polynomial of degree \(n\). If the sign of the coefficient of
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Motivation: Littlewood’s problem from 1960’s

Take \( z := e^{it}, \ t \in [0, 2\pi) \) and consider the real part of \( p(e^{it}) \), the trigonometric polynomial

\[
T(t) = \Re \left( \sum_{j=1}^{n} a_j e^{jt} \right) = \sum_{j=1}^{n} a_j \cos(jt).
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Question: What is the lower bound on the number of real zeros in the period \([0, 2\pi)\) of \( T(t) \)?

Usually, such \( T(t) \) oscillate a lot. Thus, one may expect lots of zeros.
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- **P. Borwein, T. Erdélyi (2007):** If $A \subset \mathbb{R}$ non-empty and finite, then, for any sequence $(a_j)_{j=1}^{\infty} \in A^\mathbb{N}$ that is not eventually zero, # of zeros of $T_n(t) = \sum_{j=1}^{n} a_j \cos(jt)$ in the period $\to \infty$ as $n \to \infty$.

- **Conjecture:** Probably, # of zeros of $T(t)$ in $[0, 2\pi)$ goes $\to \infty$ together with the degree of $T(t)$ regardless if we select the coefficients from a fixed sequence or not.

- What it has to do with the initial problem on zeros of $p(z)$ in $\mathbb{D}$?

- By argument principle, if $N(p) = k$, then # of zeros of $T(t)$ in $[0, 2\pi)$ is at least $2k$.

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- **P. Borwein, S. Choi (1999), S. Akhtari, S. Choi (2008):** In case $N(p) = 0$, $p(z)$ must be a product of cyclotomic polynomials; there is a formula that describes all these polynomials.

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  \[ p(z) = z^n - z^{n-1} - \cdots - z - 1. \]

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Main difficulties …

- We do not yet understand the relation between the patterns of the coefficients of $p(z)$ and numbers $N(p)$, $U(p)$.
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Our approach

- Start with a geometric progression polynomial

\[ z^n + z^{n-1} + \cdots + z + 1 = \frac{z^{n+1} - 1}{z - 1} \]

and replace some + with −.

(M. Mossinghoff, C. Pinner, J. Vaaler (2008)).

- Apply Boyd’s formula (1977):

1) calculate auxiliary polynomial \( q(z) = p(z) + \varepsilon p^*(z) \), \( \varepsilon \in \{-1, 1\} \);
2) find \( E(p, q) \) – the number of exit points of the algebraic curve \( q(z, t) = p(z) + \varepsilon tp^*(z), t \in [0, 1] \), by evaluating the sign of \( \varepsilon z^{1-\deg p} p(z) q'(z) \) at the unimodular roots of \( q(z) \);
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Main results

Theorem 1
Suppose that $n$ and $k$ are two positive integers $1 \leq k \leq n - 1$. We assume that
\[
gcd(k, n + 1) = 1, \quad \text{if } n > 2k,
\]
and
\[
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Let $n + 1$ be an odd prime. Then, for any $k$ in the range $1 \leq k \leq n - 1$, there exists a Littlewood polynomial $p(z)$ of degree $n$ with $N(p) = k$ and $U(p) = 0$. 
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The trick: Littlewood polynomials with one sign change:
Up to sign \( \pm \),

\[
p(z) = z^n + z^{n-1} + \cdots + z^k - \underbrace{z^{k-1} - \cdots - z - 1}_{k \text{ negative terms}} = \]

\[
= \frac{z^n - 2z^k + 1}{z - 1},
\]

for some integers \( n \geq k \geq 1 \).
Main results

Littlewood polynomial of degree $n \geq 2$ with one negative term:

$$p(z) = z^n + \cdots + z^{k+1} - z^k + z^{k-1} + \cdots + 1 = \frac{z^{n+1} - 1}{z - 1} - 2z^k.$$  

Basic cases:

a) Case 1: $p(z) = p^*(z)$, (the central term is negative).

b) Case 2: $p(z) \neq p^*(z)$. This occurs if $n \neq 2k$. There are two sub-cases:

a) $p(z)$ has no unimodular roots on the unit circle when $n \not\equiv 2 \pmod{6}$ or $k \not\equiv 1 \pmod{6}$.

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**Theorem 3**

Let \( p(z) \) be a self-reciprocal Littlewood polynomial of degree \( n \geq 2 \) with one negative coefficient. Then

\[
U(p) = 4 \left\lfloor \frac{n - 2}{12} \right\rfloor + 2,
\]

\[
N(p) = N(p^*) = \frac{n}{2} - 2 \left\lfloor \frac{n - 2}{12} \right\rfloor - 1,
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where \( \lfloor x \rfloor \) is the floor function of \( x \). All roots of \( p(z) \) are simple. In particular, both \( U(p) \) and \( N(p) \) \( \sim n/3 \), as \( n \to \infty \).
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Main results

**Theorem 4**

Let the polynomial $p(z)$ be as in Case 2a and $l = |n - 2k|$. If $n > 2k$, then

$$k + 1 \leq N(p) \leq k + 2 \left\lceil l/6 \right\rceil - 1.$$

where $\left\lceil x \right\rceil$ is the ceiling function of $x$. The lower bound is attained when $k \equiv 0 \pmod{l}$ and the upper bound is attained when $k \equiv 1 \pmod{l}$.

If $n < 2k$, then

$$k - 2 \left\lceil l/6 \right\rceil + 1 \leq N(p) \leq k - 1.$$

The lower bound is attained when $k \equiv 1 \pmod{l}$ and the upper bound is attained when $k \equiv 0 \pmod{l}$. 
Main results

Theorem 5
Let the polynomial \( p(z) \) be as in Case 2b and \( l = |n − 2k| \).
If \( n > 2k \), then

\[
k + 1 \leq N(p) \leq \frac{n + k}{3} - 1.
\]

The lower bound is attained when \( k \equiv 0 \pmod{l} \) and the upper bound is attained when \( k \equiv 1 \pmod{l} \).
If \( n < 2k \), then

\[
\frac{n + k}{3} - 1 \leq N(p) \leq k - 3.
\]

The lower bound is attained when \( k \equiv 1 \pmod{l} \) and the upper bound is attained when \( k \equiv 0 \pmod{l} \).
Main results

A situation when the negative term occurs close to the middle term.

**Corollary 6**

Let \( p(z) \) be in Case 2. We have

(i) If \( \lim_{n \to \infty} k/n = 1/2 \), then \( \lim_{n \to \infty} N(p)/n = 1/2 \).

(ii) If \( 0 < n - 2k \leq 6 \), then \( N(p) = k + 1 \).

(iii) If \( 0 < 2k - n \leq 6 \), then

\[
N(p) = \begin{cases} 
    k - 1 & \text{if } p(z) \text{ is in Case 2a}, \\
    k - 3 & \text{if } p(z) \text{ is in Case 2b}.
\end{cases}
\]
Main results

Definition 7
Let $k \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$ be non-negative. Define the set $\mathcal{D}_k(\alpha)$ as the subset of the interval $[0, 1]$ where the scaled Dirichlet kernel $D_k(2\pi t)$ of degree $k$ takes values greater than $\alpha$:

$$\mathcal{D}_k(\alpha) := \{ t \in [0, 1] : D_k(2\pi t) > \alpha \}.$$
Main results

**Theorem 8**

Let \( p(z) \) be a Littlewood polynomial of degree \( n \) with one negative term \( z^k \). If \( k \) is fixed, then

\[
\lim_{n \to \infty} \frac{N(p)}{n} = \text{meas}(\mathcal{D}_k(2)),
\]

where \( \text{meas}(\mathcal{D}_k(2)) \) denotes the Lebesgue measure of the set \( \mathcal{D}_k(2) \). If \( k \) and \( n \) varies in such a way that the difference \( m = n - k \) is fixed, then

\[
\lim_{n \to \infty} \frac{N(p)}{n} = \text{meas}(\mathcal{D}_m^c(2)).
\]

Here, \( \mathcal{D}_m^c(2) := [0, 1] \setminus \mathcal{D}_m(2) \).
Values of $\text{meas}(\mathcal{D}_k(2))$

**Table:** Table of measures of the set $\mathcal{D}_k(2)$ for $1 \leq k \leq 15$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\text{meas}(\mathcal{D}_k(2))$</th>
<th>$k$</th>
<th>$\text{meas}(\mathcal{D}_k(2))$</th>
<th>$k$</th>
<th>$\text{meas}(\mathcal{D}_k(2))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/3</td>
<td>6</td>
<td>0.132914</td>
<td>11</td>
<td>0.126664</td>
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<tr>
<td>2</td>
<td>0.274187</td>
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<td>0.127286</td>
<td>12</td>
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<td>0.218549</td>
<td>8</td>
<td>0.141399</td>
<td>13</td>
<td>0.117483</td>
</tr>
<tr>
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<td>0.180278</td>
<td>9</td>
<td>0.138567</td>
<td>14</td>
<td>0.124823</td>
</tr>
<tr>
<td>5</td>
<td>0.153086</td>
<td>10</td>
<td>0.132949</td>
<td>15</td>
<td>0.124141</td>
</tr>
</tbody>
</table>
Open questions

- Is there a more general theorem that gives sufficient conditions for the existence of a limit $\lim_{n \to \infty} N(p)/n$? for some sequence of polynomials $p(z)$?
- Are there any other constructions that modify the number of roots of $p(z)$ in a predictable way, without using the cyclotomic polynomials $p(z)$?
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