On Littlewood and Newman multiples of Borwein polynomials

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July 2017

Papers

- *On Littlewood and Newman polynomial multiples of Borwein polynomials*, Math. of Comp., (to appear).
- *On certain multiples of Littlewood and Newman polynomials* (in progress, with G. Junevičius, L.Klebonas)

Borwein, Newman and Littlewood polynomials

- Let $d \in \mathbb{N}$ and let $P(X)$ be a polynomial
$$P(X) = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0$$
with integer coefficients $a_j \in \mathbb{Z}$. To avoid trivialities: $a_d \cdot a_0 \neq 0$.
- **Borwein**[1] **polynomials**: $a_j \in \{-1, 0, 1\}$, eg.
$$P(X) = X^5 - X^2 + 1.$$
- **Littlewood polynomials**: $a_j \in \{-1, 1\}$, eg.
$$P(X) = X^4 + X^3 - X^2 + X - 1.$$
- **Newman polynomials**: $a_j \in \{0, 1\}$, eg.
$$P(X) = X^7 + X + 1.$$

[1] In honor of Peter Borwein

Sets $\mathcal{B}$, $\mathcal{L}$, $\mathcal{N}$

- $\mathcal{B}_d$ – Borwein polynomials of degree d.
- $\mathcal{L}_d$ – Littlewood polynomials of degree d.
- $\mathcal{N}_d$ – Newman polynomials of degree d.

$$\mathcal{N}_{\leqslant d} = \bigcup_{j=0}^{d} \mathcal{N}_j, \qquad \mathcal{L}_{\leqslant d} = \bigcup_{j=0}^{d} \mathcal{L}_j, \qquad \mathcal{B}_{\leqslant d} = \bigcup_{j=0}^{d} \mathcal{B}_j.$$

Multiples of $P(X) \in \mathcal{B}, \mathcal{L}, \mathcal{N}$

Let $\mathcal{A} \subset \mathbb{Z}[X]$

- $\mathcal{L}(\mathcal{A})$ – polynomials from $\mathcal{A}$ that have multiple in $\mathcal{L}$.
- $\mathcal{N}(\mathcal{A})$ – polynomials from $\mathcal{A}$ that have multiple in $\mathcal{N}$.

For instance:

- $\mathcal{L}(\mathcal{B})$ – Borwein polynomials that divide some $Q(X) \in \mathcal{L}$.
- $\mathcal{N}(\mathcal{B})$ - Borwein polynomials that divide some $Q(X) \in \mathcal{N}$.

Partition of the set $\mathcal{B}$

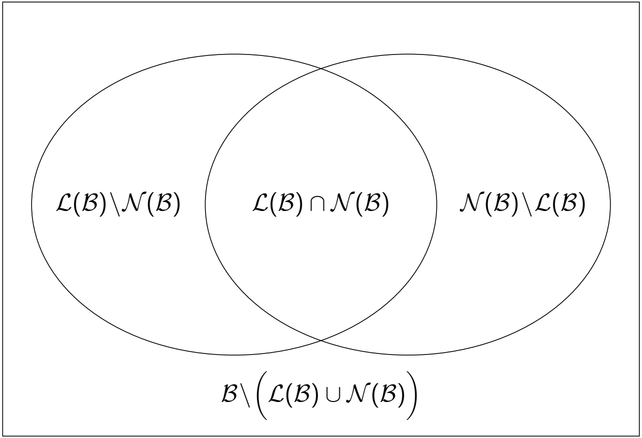


Figure: Decomposition of the set of Borwein polynomials.

Finding a multiple of $P(X)$ in $\mathcal{D}[X]$

Let $\mathcal{D} \subset \mathbb{Z}$ be a finite digit set.

Question

Given a monic polynomial $P(X) \in \mathbb{Z}[X]$, does there exist a nonzero polynomial with coefficients in $\mathcal{D}$ which is divisible by $P(X)$?

Answer

There exists an algorithm which decides whether a given $P(X) \in \mathbb{Z}[X]$ without zeros $|\alpha| = 1$ has a multiple in $\mathcal{D}[X]$.

Previous work

[1.] C. Frougny, *Representation of numbers and finite automata*, Math. Systems Theory 25 (1992).

[2.] K.-S. Lau, *Dimension of a family of singular Bernoulli convolutions*, J. Funct. Anal. 116, (1993).

[3.] P. Borwein, K. G. Hare, *Some computations on the spectra of Pisot and Salem numbers*, Math. Comp. 71 (238) (2002).

[4.] A. Dubickas, J. Jankauskas, *On Newman polynomials that divide no Littlewood polynomial*, Math. Comp., 78(265)(2009).

[5.] D. Stankov, *On spectra of neither Pisot nor Salem algebraic integers*, Monatsh. Math. 159 (2010), 115–131.

[6.] K. G. Hare, M. J. Mossinghoff, *Negative Pisot and Salem numbers as roots of Newman polynomials*, Rocky Mountain J. Math. 44 (1) (2014).

[7.] S. Akiyama, J. M. Thuswaldner, T. Zaimi, *Comments on the height reducing property II*, Indag. Math. 26(1)(2015).

The algorithm

- First: $R(x) = b_0 \in \mathcal{D}$, iteration:

$$R(X) \mapsto X \cdot R(X) + d \pmod{P(X)}, d \in \mathcal{D},$$

- with

$$|R^{(k)}(z)| \leqslant \frac{k! \cdot \max_{d \in \mathcal{D}} |d|}{||z| - 1|^{k+1}}, \quad 0 \leqslant k \leqslant m - 1,$$

at every root $z = \alpha$, $|z| > 1$ of $P(X)$ of multiplicity m.

- The search graph $\mathcal{G}$: paths $b_0 \to \cdots \to b_n \to 0$:

$$P(X) \mid Q(X) = b_0 X^n + \cdots + b_{n-1} X + b_n.$$

$P(X)$ with zeros on the unit circle

When $P(\alpha) = 0$ at the root $|\alpha| = 1$, the search graph $\mathcal{G}$ might be infinite (unbounded the reminder set). Ways to fix it:

- If $0 \in \mathcal{D}$, or $\mathcal{D} = -\mathcal{D}$, one can remove safely all cyclotomic factors from $P(X)$.
- Discard roots $|\alpha| = 1$ when evaluating the bounds: if algorithm terminates, then it still produces a correct result.

Computations

We run the Algorithm to verify the statements

$$P(X) \in \mathcal{L}(\mathcal{B}) \text{ and } P(X) \in \mathcal{N}(\mathcal{B})$$

for all $P(X) \in \mathcal{B}_{\leqslant 9}$ and calculated the following numbers:

- $\#\Big(\mathcal{L}(\mathcal{B}_d) \backslash \mathcal{N}(\mathcal{B})\Big)$,
- $\#\Big(\mathcal{N}(\mathcal{B}_d) \backslash \mathcal{L}(\mathcal{B})\Big)$,
- $\#\Big(\mathcal{L}(\mathcal{B}_d) \cap \mathcal{N}(\mathcal{B}_d)\Big)$

for every $d \in \{1, 2, \ldots, 9\}$.

d	$\#\big(\mathcal{L}(\mathcal{B}_d)\backslash\mathcal{N}(\mathcal{B})\big)$	$\#\big(\mathcal{N}(\mathcal{B}_d)\backslash\mathcal{L}(\mathcal{B})\big)$	$\#\big(\mathcal{L}(\mathcal{B}_d)\cap\mathcal{N}(\mathcal{B}_d)\big)$
1	2	0	2
2	6	0	6
3	24	0	12
4	72	0	32
5	224	0	68
6	612	0	164
7	1518	0	342
8	3610	0	822
9	8564	60	1596

Theorem 1

Every Borwein polynomial of degree $\leqslant 8$ which divides some Newman polynomial divides some Littlewood polynomial as well.

From the the partition diagram of $\mathcal{B}$ one obtains:

d	$\#\Big(\mathcal{B}_d \setminus \mathcal{L}(\mathcal{B})\Big)$	$\#\Big(\mathcal{B}_d \setminus \mathcal{N}(\mathcal{B})\Big)$	$\#\Big(\mathcal{B}_d \setminus (\mathcal{L}(\mathcal{B}) \cup \mathcal{N}(\mathcal{B}))\Big)$
1	0	2	0
2	0	6	0
3	0	24	0
4	4	76	4
5	32	256	32
6	196	808	196
7	1056	2574	1056
8	4316	7926	4316
9	16084	24588	16024

Polynomials in $\mathcal{B} \setminus \mathcal{L}(\mathcal{B})$

Proposition 2

The smallest degree Borwein polynomial which does not divide any Littlewood polynomial is $P(X) = X^4 + X^3 - X + 1$*. Moreover,*

$$\mathcal{B}_{\leqslant 4} \setminus \mathcal{L}(\mathcal{B}) = \{\pm P(X), \pm P^*(X)\}.$$

Polynomials in $\mathcal{B} \setminus \mathcal{N}(\mathcal{B})$

Newman polynomial has **no nonnegative** real roots.

$\mathcal{A}^-$ – polynomials $P(X) \in \mathcal{A}$ without nonnegative roots.

Proposition 3

The smallest degree Borwein polynomial without nonnegative real roots and no Newman multiple is

$$P(X) = X^3 + X^2 - X + 1,$$

and

$$\mathcal{B}^-_{\leqslant 3} \setminus \mathcal{N}(\mathcal{B}) = \{\pm P(X), \pm P^*(X)\}.$$

Polynomials in $\mathcal{N}_9 \setminus \mathcal{L}(\mathcal{N})$

The following result completes the classification of $\mathcal{N}_9 \setminus \mathcal{L}(\mathcal{N})$ started by Dubickas and J. [3]

Table: The complete set $\mathcal{N}_9 \setminus \mathcal{L}(\mathcal{N})$ (reciprocals omitted).

#	Polynomial $P(X)$
1	$X^9 + X^6 + X^2 + X + 1$
2	$X^9 + X^7 + X^6 + X^2 + 1$
3	$X^9 + X^7 + X^6 + X^4 + 1$
4	$X^9 + X^8 + X^6 + X^5 + X^2 + 1$
5	$X^9 + X^8 + X^7 + X^5 + X^3 + 1$
6	$X^9 + X^8 + X^7 + X^5 + X^2 + X + 1$
7	$X^9 + X^8 + X^5 + X^3 + X^2 + X + 1$
8	$X^9 + X^7 + X^6 + X^3 + X^2 + X + 1$
9	$X^9 + X^8 + X^5 + X^4 + X^3 + X^2 + 1$

Polynomials in $\mathcal{N} \setminus \mathcal{L}(\mathcal{N})$

The sets

$$\mathcal{L}(\mathcal{N}_d) \quad \text{and} \quad \mathcal{N}_d \setminus \mathcal{L}(\mathcal{N})$$

have been completely determined for $d = 10$ and $d = 11$.

In particular,

$$\#\mathcal{N}_{10} \setminus \mathcal{L}(\mathcal{N}) = 36,$$

$$\#\mathcal{N}_{11} \setminus \mathcal{L}(\mathcal{N}) = 174.$$

Symmetric problem: polynomials in $\mathcal{N}(\mathcal{L}_{\leqslant 12})$

Theorem 4 (G. Junevičius)

There are exactly nine monic polynomials $P \in \mathcal{L}_{\leqslant 12}$ *which are not products of cyclotomic polynomials and divide some Newman polynomial.*

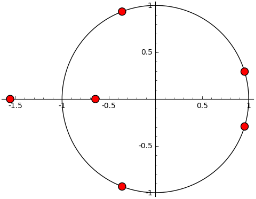
Corollary 5 (G. Junevičius)

Every nontrivial polynomial $P \in \mathcal{N}(\mathcal{L}_{\leqslant 12})$ *has a root on* $|z| = 1$ *which is not a root of unity.*

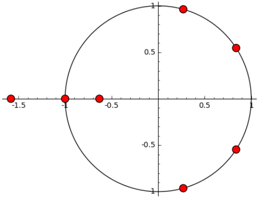
Polynomials in $\mathcal{N}(\mathcal{L}_{\leqslant 12})$

k	$P_k(X)$
1	$X^6 + X^5 - X^4 - X^3 - X^2 + X + 1$
2	$X^7 + X^6 - X^5 + X^4 + X^3 - X^2 + X + 1$
3	$X^8 + X^7 - X^6 - X^5 + X^4 - X^3 - X^2 + X + 1$
4	$X^9 + X^8 + X^7 - X^6 - X^5 - X^4 - X^3 + X^2 + X + 1$
5	$X^{10} - X^9 + X^8 + X^7 - X^6 + X^5 - X^4 + X^3 + X^2 - X + 1$
6	$X^{10} + X^9 + X^8 - X^7 - X^6 - X^5 - X^4 - X^3 + X^2 + X + 1$
7	$X^{12} + X^{11} - X^{10} - X^9 - X^8 + X^7 + X^6 + X^5 - X^4 - X^3 - X^2 + X + 1$
8	$X^{12} - X^{11} + X^{10} + X^9 - X^8 + X^7 + X^6 + X^5 - X^4 + X^3 + X^2 - X + 1$
9	$X^{12} + X^{11} + X^{10} - X^9 - X^8 - X^7 + X^6 - X^5 - X^4 - X^3 + X^2 + X + 1$

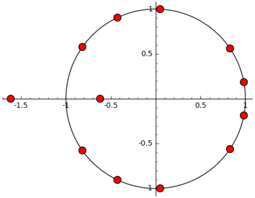
Roots of $P(X) \in \mathcal{N}(\mathcal{L}_{\leqslant 12})$



$P_1(x)$

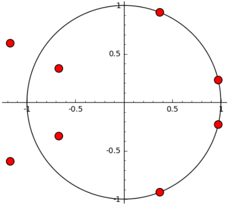


$P_2(x)$

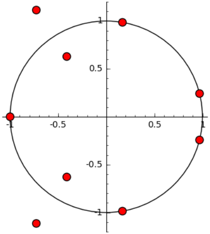


$P_7(x)$

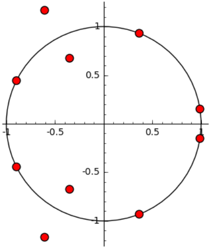
Roots of $P(X) \in \mathcal{N}(\mathcal{L}_{\leqslant 12})$



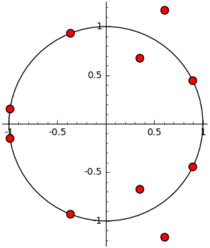
$P_3(x)$



$P_4(x)$

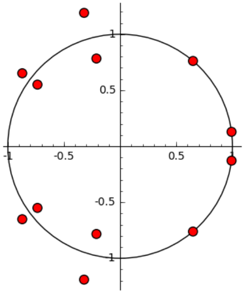


$P_5(x)$

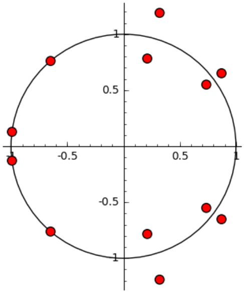


$P_6(x)$

Roots of $P(X) \in \mathcal{N}(\mathcal{L}_{\leqslant 12})$



$P_8(x)$



$P_9(x)$

Trinomials and quadrinomials in $\mathcal{L}(\mathcal{B})$

Theorem 6 (A. Dubickas and J.J., 2009)

Every trinomial

$$x^b \pm x^a \pm 1, \qquad a, b \in \mathbb{N}, \quad a < b,$$

has a Littlewood multiple.

Question (2009)

What about quadrinomials?

Quadrinomials in $\mathcal{B} \setminus \mathcal{L}(\mathcal{B})$

Table: Monic quadrinomials in $\mathcal{B}_{\leqslant 9} \setminus \mathcal{L}(\mathcal{B})$.

$X^4 + X^3 - X + 1$
$X^4 - X^3 + X + 1$
$X^6 - X^5 - X - 1$
$X^6 + X^5 + X - 1$
$X^8 - X^5 + X^3 + 1$
$X^8 + X^5 - X^3 + 1$
$X^8 + X^7 - X + 1$
$X^8 - X^7 + X + 1$
$X^8 + X^6 - X^2 + 1$
$X^8 - X^6 + X^2 + 1$

Question (2009)

Does there exist a quadrinomial $P(X) \in \mathcal{N} \setminus \mathcal{L}(\mathcal{N})$?

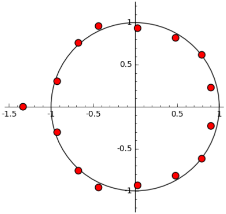
Quadrinomials in $\mathcal{L}(\mathcal{N})$

Theorem 7 (L. Klebonas)

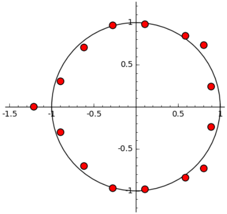
Every quadrinomial $Q(X) \in \mathcal{N}_{\leqslant 15}$, except possibly for those given in the table below, divides some Littlewood polynomial. Moreover, every quadrinomial $Q(X) \in \mathcal{N}_{\leqslant 14}$ possesses a Littlewood multiple of smallest possible degree $\deg_2 \widetilde{Q}$.

Quadrinomial $Q(X)$	$\deg_2 \widetilde{Q}$
$X^{15} + X^{14} + X^{10} + 1$	10922
$X^{15} + X^{5} + X + 1$	10922
$X^{15} + X^{12} + X^{10} + 1$	32766
$X^{15} + X^{5} + X^{3} + 1$	32766
$X^{15} + X^{12} + X^{4} + 1$	31682
$X^{15} + X^{11} + X^{3} + 1$	31682
$X^{15} + X^{9} + X^{7} + 1$	32766
$X^{15} + X^{8} + X^{6} + 1$	32766

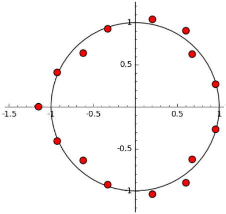
Quadrinomials in $\mathcal{L}(\mathcal{N})$



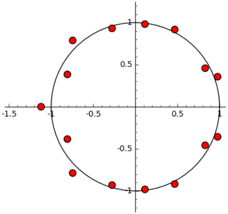
$X^{15} + X^{14} + X^{10} + 1.$



$X^{15} + X^{12} + X^{10} + 1.$



$X^{15} + X^{14} + X^{4} + 1$



$X^{15} + X^{8} + X^{6} + 1$

Examples with special factors I

If $P, Q \in \mathcal{L}(\mathcal{B})$, then not necessarily $P \cdot Q \in \mathcal{L}(\mathcal{B})$.

Example 8

$P(X) = X^4 + X^3 + 1$ and $Q(X) = X^5 - X^4 + X^3 - X + 1$ belong to $\mathcal{L}(\mathcal{B})$. However, $P \cdot Q \notin \mathcal{L}(\mathbb{Z}[X])$. Consequently,

$$R(X) = X^{11} + X^{10} + X^9 + X^8 + X^7 + X^5 + X^4 + X^3 + 1 =$$

$$= (X^2 + X + 1) \cdot P \cdot Q$$

also has no Littlewood multiple.

Examples with special factors II

If $P \in \mathcal{N}(\mathcal{B})$, then not necessarily $P \cdot P^* \in \mathcal{N}(\mathcal{B})$.

Example 9

Let $P(X) = X^3 - X + 1$. *Then* $P, P^* \in \mathcal{N}(\mathcal{B})$, *eg.* $P \mid X^5 + X^4 + 1$. *However,*

$$P(X)P^*(X) = X^6 - X^5 - X^4 + 3X^3 - X^2 - X + 1$$

has no Newman multiple. In contrast, $p(X)p^*(X)$ *divides Borwein polynomial*

$$X^8 - X^6 + X^5 + X^4 + X^3 - X^2 + 1$$

that, in turn, has its own Littlewood multiple.

Examples with special factors III

Our program can deal with polynomials with repeated roots.

Example 10

$P(X) = X^3 - X + 1$ has a Newman multiple, but P^2 does not. P^2 has a multiple $L(X) \in \mathcal{L}_{195}$, while $P^3 \notin \mathcal{L}(\mathbb{Z}[X])$. Consequently, the Borwein multiple

$$Q(X) = (X^2 + X + 1)P(X)^2 = X^8 + X^7 - X^6 + X^4 + X^3 - X + 1$$

of P^2 divides no Newman polynomial. Q still has a Littlewood multiple, namely the polynomial $L(X)\Phi_3(X^{196})$.

Computational complexity

- To decide that

$$P(X) = X^9 + X^8 - X^7 - X^5 + X^3 + X^2 - 1 \in \mathcal{B} \backslash \mathcal{L}(\mathcal{B})$$

$t = 119$ min. $\#\mathcal{G}(P, \mathcal{D}) = 1\,428\,848$, depth $= 471$.

- To find a Littlewood multiple for

$$P(X) = X^9 - X^8 + X^7 + X^6 - X^5 + X^4 - X^3 + X - 1.$$

$t = 92$ min., $\#\mathcal{G}(P, \mathcal{D}) = 9\,372\,425$, depth $= 43554$.

Thank you!