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# On the fractional parts of powers of Pisot numbers of length at most 4



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## ABSTRACT

The aim of this paper is twofold. We first give a list of all Pisot polynomials of length at most 4. It contains seven polynomials of degree at most 5, and two infinite series of polynomials with unbounded degree. Then, for Pisot numbers  $\alpha$  of length 3 and 4, we find explicitly the largest positive number  $\mathcal{L}(\alpha)$  such that for some  $\xi = \xi(\alpha) \in \mathbb{R}$  the limit points of the sequence of fractional parts  $\{\xi\alpha^n\}_{n=1}^{\infty}$  all lie in the interval  $[\mathcal{L}(\alpha), 1 - \mathcal{L}(\alpha)]$ .

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## 1. Introduction

Let  $\|x\|$  be the distance from  $x \in \mathbb{R}$  to the nearest integer. For a real number  $\alpha > 1$ , we put

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$$\mathcal{L}(\alpha) := \sup_{\xi \in \mathbb{R}} \liminf_{n \rightarrow \infty} \|\xi \alpha^n\|. \tag{1.1}$$

In other words,  $\mathcal{L}(\alpha)$  is the largest positive number for which there exists some  $\xi = \xi(\alpha) \in \mathbb{R}$  such that the limit points of the sequence of fractional parts  $\{\xi \alpha^n\}_{n=1}^\infty$  all lie in the interval  $[\mathcal{L}(\alpha), 1 - \mathcal{L}(\alpha)]$ . These kinds of problems have been studied in [15,21] from the metrical point of view. Then, Mahler raised a (still unsolved) problem on the powers of  $3/2$  in [16] and some estimates have been obtained in [1,2,7,8,12,14,19]. Some special cases with Pisot and Salem numbers  $\alpha$  have been considered in [9,13,20,23]. A similar quantity

$$\mathcal{L}(M) := \sup_{\xi \in \mathbb{R}} \min_{m \in M} \|\xi m\|,$$

where  $M$  is a finite set of integers, was earlier considered by Wills [22] and, independently, by Cusick [6]. Their famous conjecture, which nowadays is usually called the *lonely runner conjecture*, asserts that  $\mathcal{L}(M) \geq 1/(1 + |M|)$ . This conjecture was proved for  $M$  satisfying  $1 \leq |M| \leq 6$  and also for some very special sets  $M$  (see, e.g., a recent paper of Pandey [17]).

However, all known results, except for rational integers  $\alpha$ , were just estimates for  $\mathcal{L}(\alpha)$ . The first exact value of the quantity defined in (1.1) was found by Zhuravleva in [24], where she showed that, for the golden section number  $\alpha_0 = (1 + \sqrt{5})/2$ , the equality  $\mathcal{L}(\alpha_0) = 1/5$  holds. Then, a much simpler proof of the same equality was found by the first named author in [10], who also showed that  $\mathcal{L}(\alpha) = \alpha/(2\alpha + 2)$  for an even rational integer  $\alpha \geq 2$ . Moreover, in [10] it was shown that  $\mathcal{L}(\alpha) = 1/2$  for every Pisot number  $\alpha$  whose minimal polynomial  $f \in \mathbb{Z}[x]$  at  $x = 1$  is an even integer, namely,  $f(1) \in 2\mathbb{Z}$ . Recall that  $\alpha > 1$  is called a *Pisot* number if it is an algebraic integer such that the other roots of its minimal polynomial over  $\mathbb{Q}$  (if there are such roots) all lie in the unit disc  $|z| < 1$ . In particular, this implies that  $\mathcal{L}(\alpha) = 1/2$  for every odd rational integer  $\alpha \geq 3$ . Furthermore, in [10] the value of  $\mathcal{L}(\alpha)$  was calculated explicitly for all quadratic Pisot numbers  $\alpha$  (clearly, the golden section number is one of them). Finally, in [25] two smallest Pisot numbers that are roots of  $x^3 - x - 1$  (say,  $\alpha_1 = 1.32471\dots$ ) and  $x^4 - x^3 - 1$  (say,  $\alpha_3 = 1.38027\dots$ ) have been considered. (The fact that these are two smallest Pisot numbers was proved by Siegel in [18].) A computational proof of  $\mathcal{L}(\alpha_1) = 1/5$  has been sketched in [25] and the equality  $\mathcal{L}(\alpha_3) = 3/17$  announced without proof.

It is quite natural, therefore, to look for some other Pisot numbers  $\alpha$  of small length and to investigate whether it is possible to calculate  $\mathcal{L}(\alpha)$  explicitly. Recall that the length of the polynomial

$$f(x) = a_d x^d + \dots + a_0 \in \mathbb{C}[x]$$

is defined by  $L(f) = |a_d| + \dots + |a_0|$ . We call a polynomial  $f \in \mathbb{Z}[x]$  a *Pisot polynomial*, if  $f(x)$  is the minimal polynomial of a Pisot number  $\alpha$ .

Although a lot of work was done in order to find small Pisot numbers (see, e.g., [11] and Boyd’s papers [3–5]), it seems that the Pisot polynomials of small length have not yet been classified. Our next theorem describes all Pisot polynomials of length  $L(f) \leq 4$ .

**Theorem 1.1.** *Any Pisot polynomial of length  $L(f) \leq 4$  must be one of the following:*

*binomials*

$$x - 2, \quad x - 3,$$

*trinomials*

$$x^2 - x - 1, \quad x^3 - x - 1, \quad x^3 - x^2 - 1, \quad x^4 - x^3 - 1, \\ x^d - 2x^{d-1} - 1 \quad (d \geq 2),$$

*or quadrinomials*

$$x^5 - x^4 - x^2 - 1, \quad x^{d-2}(x^2 - x - 1) - 1 \quad (d \geq 3, d\text{-odd}).$$

From what we already said above, it follows that  $\mathcal{L}(2) = 1/3$  and  $\mathcal{L}(3) = 1/2$ . If  $f$  is the trinomial  $x^d - 2x^{d-1} - 1$ , where  $d \geq 2$ , the quadrinomial  $x^5 - x^4 - x^2 - 1$  or the quadrinomial  $x^d - x^{d-1} - x^{d-2} - 1$ , where  $d \geq 3$  is odd, then  $f(1) = -2$  is even. So,  $\mathcal{L}(\alpha) = 1/2$  for each of these Pisot numbers. By Theorem 1.1, there are no other Pisot numbers of length 4, so we have  $\mathcal{L}(\alpha) = 1/2$  for each Pisot number  $\alpha$  of length 4.

Pisot numbers that are roots of three out of the four remaining polynomials, namely,  $x^2 - x - 1$ ,  $x^3 - x - 1$  and  $x^4 - x^3 - 1$  have been treated in [10,24,25]. We establish the value of  $\mathcal{L}(\alpha_2)$  for the root  $\alpha_2 = 1.46557\dots$  of  $x^3 - x^2 - 1$  and, at the same time, give a simple combinatorial treatment of this problem for  $\mathcal{L}(\alpha_0)$ ,  $\mathcal{L}(\alpha_1)$  and  $\mathcal{L}(\alpha_3)$ . (In [24] and [25], the upper bounds  $1/5$  for  $\mathcal{L}(\alpha_0)$  and  $\mathcal{L}(\alpha_1)$  rely heavily on computer calculations, whereas the value  $\mathcal{L}(\alpha_3) = 3/17$  is announced without proof.)

**Theorem 1.2.** *Let*

$$\alpha_0 = 1.61803\dots, \quad \alpha_1 = 1.32471\dots, \quad \alpha_2 = 1.46557\dots, \quad \alpha_3 = 1.38028\dots$$

*be Pisot numbers that are the roots of polynomials*

$$x^2 - x - 1, \quad x^3 - x - 1, \quad x^3 - x^2 - 1, \quad x^4 - x^3 - 1,$$

*respectively. Then*

$$\mathcal{L}(\alpha_0) = \mathcal{L}(\alpha_1) = 1/5, \quad \mathcal{L}(\alpha_2) = 1/3, \quad \mathcal{L}(\alpha_3) = 3/17.$$

In the next section we prove [Theorem 1.1](#). Then, in [Section 3](#) we prove [Theorem 1.2](#) for numbers  $\alpha_0, \alpha_1$  and  $\alpha_2$ . The proof of the equality  $\mathcal{L}(\alpha_3) = 3/17$  is carried out in [Section 4](#).

**2. Proof of [Theorem 1.1](#)**

Our strategy in the proof of [Theorem 1.1](#) is to reduce the problem to two cases. One case is where  $\alpha$  is less than the number  $\alpha_0 = (1 + \sqrt{5})/2$  or is just slightly greater than  $\alpha_0$ . Then we can apply the following classical result of Dufresnoy and Pisot [[11](#)].

**Proposition 2.1.** *Define polynomials*

$$\begin{aligned}
 p(x) &:= x^2 - x - 1, \\
 p_{2n}^+(x) &:= \frac{x^{2n}p(x) + 1}{x - 1}, \quad n \geq 1, & p_{2n}^-(x) &:= \frac{x^{2n}p(x) - 1}{x + 1}, \quad n \geq 0, \\
 p_{2n+1}^+(x) &:= \frac{x^{2n+1}p(x) + 1}{x^2 - 1}, \quad n \geq 1, & p_{2n+1}^-(x) &:= x^{2n+1}p(x) - 1, \quad n \geq 0, \\
 q_n^+(x) &:= x^n p(x) + x^2 - 1, \quad n \geq 1, & q_n^-(x) &:= x^n p(x) - x^2 + 1, \quad n \geq 1, \\
 r(x) &:= x^6 - 2x^5 + x^4 - x^2 + x - 1.
 \end{aligned}$$

Then the minimal polynomial of each Pisot number that is less or equal to the root  $\theta_{15}^- = 1.61836\dots$  of the polynomial  $p_{15}^-(x)$  must coincide with one of the polynomials

$$p_n^+(x), \quad p_n^-(x), \quad q_n^+(x), \quad q_n^-(x), \quad r(x)$$

listed above.

In the second case, namely  $\alpha > \theta_{15}^-$ , it will be shown that the minimal polynomial  $f(x)$  of  $\alpha$  is either a trinomial of a very simple form, or belongs to a certain small class of quadrinomials that will be checked by computer. Before we proceed to the proof of [Theorem 1.1](#), we state three elementary lemmas.

**Lemma 2.2.** *Let  $k \geq 1$  be a fixed integer and let*

$$f_n(x) := x^n(x^k - x^{k-1} - \dots - x - 1) - 1.$$

Then each polynomial  $f_n(x)$  has a single positive root, say,  $\beta_n > 1$  and the sequence  $(\beta_n)_{n=1}^\infty$  is decreasing.

**Proof.** Consider the reciprocal polynomial

$$f_n^*(x) = x^{n+k} f(1/x) = 1 - x - \dots - x^k - x^{n+k}.$$

From  $f(0)f(1) < 0$  and the fact that the derivative  $f_n^*(x)' < 0$  for all  $x > 0$  we see that  $f_n^*(x)$  has a single root in the interval  $(0, 1)$ . Therefore,  $f(x)$  has a single positive root, say,  $\beta_n$  in the interval  $(1, \infty)$ . Observe that  $f_n(\beta_n) = 0$  implies that

$$\beta_n^n (\beta_n^k - \beta_n^{k-1} - \dots - \beta_n - 1) = 1.$$

Thus, the value of  $f_{n+1}(x)$  at  $x = \beta_n$  is

$$f_{n+1}(\beta_n) = \beta_n^{n+1} (\beta_n^k - \beta_n^{k-1} - \dots - \beta_n - 1) - 1 = \beta_n - 1 > 0.$$

Hence,  $\beta_{n+1} < \beta_n$ .  $\square$

**Lemma 2.3.** *If  $\alpha > 1$  is a root of a  $k$ -nomial*

$$f(x) = x^{d_{k-1}} \pm x^{d_{k-2}} \pm \dots \pm x^{d_1} \pm 1$$

( $k \geq 2$ ) for some integers

$$1 \leq d_1 < \dots < d_{k-1} =: d$$

and arbitrary choice of signs  $+$  and  $-$ , then  $\alpha$  is less or equal to the single positive root of the polynomial

$$h(x) = x^d - x^{d-1} - \dots - x^{d-k+2} - 1.$$

**Proof.** We have

$$f(\alpha) = \alpha^{d_{k-1}} \pm \alpha^{d_{k-2}} \pm \dots \pm \alpha^{d_1} \pm 1 = 0.$$

Note that replacing  $+$  signs with  $-$  signs and increasing powers in terms with negative coefficients only makes the value of  $f(\alpha) = 0$  smaller. Therefore,

$$0 = f(\alpha) \geq \alpha^d - \alpha^{d-1} - \dots - \alpha^{d-k+2} - 1 = h(\alpha).$$

From Lemma 2.2,  $h(x)$  has a single positive root, say,  $\beta$ . Therefore,  $h(\alpha) \leq 0$  implies that  $\alpha \leq \beta$ .  $\square$

**Lemma 2.4.** *Let  $\theta_n^-$  be the single positive root of the polynomial  $p_n^-(x)$  defined in Proposition 2.1. Then  $\theta_n^- > \alpha_0$  and  $\theta_n^- \downarrow \alpha_0$  as  $n \rightarrow \infty$ .*

**Proof.** From Lemma 2.2 applied to  $k = 2$ , we already know that each of  $p_n^-(x)$  has a single positive root  $\theta_n^- > 1$ . The sequence  $(\theta_n^-)_{n=1}^\infty$  is bounded and decreasing, therefore, it converges. Observe that the polynomial  $p(x) = x^2 - x - 1$  defined in Proposition 2.1 is the minimal polynomial of  $\alpha_0$ . Since  $p_{2n+1}^-(x) = x^{2n+1}p(x) - 1$  and  $p_{2n}^-(x) = (x^{2n}p(x) - 1)/(x + 1)$ , we see that the equality  $p_n^-(\theta_n^-) = 0$  yields  $p(\theta_n^-) = (\theta_n^-)^{-n} > 0$ . Hence,  $\theta_n^- > \alpha_0$ . By a simple limit calculation,

$$p\left(\lim_{n \rightarrow \infty} \theta_n^-\right) = \lim_{n \rightarrow \infty} p(\theta_n^-) = \lim_{n \rightarrow \infty} (\theta_n^-)^{-n} = 0.$$

Hence,  $\theta_n^- \downarrow \alpha_0$  as  $n \rightarrow \infty$ .  $\square$

**Proof of Theorem 1.1.** Since the length of a Pisot polynomial  $f(x)$  is at most 4, it must consist of at least 2 and at most 4 non-zero terms. The proof naturally splits into three parts (binomials, trinomials and quadrinomials).

The part concerning binomials is trivial. By simple considerations, it is easy to see that among all possible monic integer binomials of length  $L(f) \leq 4$ , only  $x - 2$  and  $x - 3$  each have a single root of modulus  $> 1$ .

Suppose that  $f(x)$  has three non-zero terms. If  $L(f) = 3$ , then  $f(x) = x^d \pm x^c \pm 1$  for some integers  $1 \leq c < d$  and some choice of signs  $\pm$ . By Lemma 2.3 (with  $k = 3$ ), a Pisot number  $\alpha$  that is a root of  $f(x)$  is less than or equal to a single positive root  $\beta$  of  $x^d - x^{d-1} - 1$ . By Lemma 2.2 (with  $k = 1$  and  $n = d - 1$ ), one has  $\beta \leq \alpha_0$ . This, in turn, is less than  $\theta_{15}^-$  by Lemma 2.4. According to the Dufresnoy–Pisot classification given in Proposition 2.1,  $f(x) = p(x)$  or  $f(x)$  belongs to one of the sequences  $p_n^+(x)$  or  $p_{2n}^-(x)$ . By a direct inspection, we find only trinomials  $p(x) = x^2 - x - 1$ ,  $p_2^+(x) = q_1^+(x) = x^3 - x - 1$ ,  $p_3^+(x) = x^3 - x^2 - 1$  and  $q_2^+(x) = x^4 - x^3 - 1$ . There are no more Pisot trinomials of length 3, since all polynomials listed in Proposition 2.1 have 4 or more non-zero terms if  $n \geq 3$ .

If  $f(x)$  is a trinomial with  $L(f) = 4$ , then  $f(x)$  must be of the form

$$\text{either } (i) f(x) = x^d \pm x^c \pm 2 \quad \text{or} \quad (ii) f(x) = x^d \pm 2x^c \pm 1.$$

The Pisot polynomial  $f(x)$  cannot take the form (i). Indeed, for a complex number  $|z| < 1$ , we have

$$|f(z)| = |z^d \pm z^c \pm 2| \geq 2 - |z|^d - |z|^c > 2 - 1 - 1 = 0.$$

Therefore, the Pisot number  $\alpha$  whose minimal polynomial is  $f(x)$  in (i) cannot have any other conjugates over  $\mathbb{Q}$  than itself, so it must be  $\alpha = 2$ , which is impossible. Thus,  $f(x)$  with  $L(f) = 4$  must be as in (ii). Taking into account the possible choices of signs we find the following polynomials:  $f(x) = x^d + 2x^c + 1$ ,  $f(x) = x^d + 2x^c - 1$ ,  $f(x) = x^d - 2x^c + 1$  and  $f(x) = x^d - 2x^c - 1$ . In first two cases, we always have  $f(x) > 0$  for  $x > 1$ . In the third case,  $f(1) = 0$  so  $f(x)$  is reducible in  $\mathbb{Z}[x]$ . Hence  $f(x) = x^d - 2x^c - 1$  is the only

remaining possibility. Assuming that  $f(x)$  is a Pisot polynomial, we will show that one must have  $c = d - 1$ .

Indeed, for  $\lambda > 1$ , consider the following auxiliary polynomial

$$h_\lambda(x) := x^d + 2\lambda x^{d-c} - 1.$$

By the theorem of Rouché, it has  $d - c$  roots inside the unit circle. By taking  $\lambda \downarrow 1$  and using the continuity of the roots of  $f(x)$  with respect to  $\lambda$ , we see that  $h_1(x) := x^d + 2x^{d-c} - 1 = -f^*(x)$  has at least  $d - c$  roots of absolute value  $|z| \leq 1$  (and, by continuity, at most  $d - c$  roots of modulus  $|z| < 1$ ). Therefore,  $f(x)$  has at least  $d - c$  roots of modulus  $|z| \geq 1$  (at most  $d - c$  in the region  $|z| > 1$ ). None of these roots can be on the unit circle  $|z| = 1$ , for otherwise,  $f(x)$  would have a non-constant greatest common divisor with  $f^*(x)$ . This is impossible, since  $f(x)$  is irreducible.

It follows that  $f(x)$  has precisely  $d - c$  roots  $|z| > 1$ . Since  $f(x)$  is a Pisot polynomial, we must have  $d - c = 1$ , so that  $f(x) = x^d - 2x^{d-1} - 1$ . To verify that this is a Pisot polynomial, it suffices to check that  $f(x)$  is irreducible which is equivalent to checking that  $f(x)$  has no roots on the circle  $|z| = 1$ . If  $f(z) = 0$  for a complex number  $z$  of modulus 1, then the equation  $z^d - 1 = 2z^{d-1}$  holds only with  $z^d = z^{d-1} = -1$ . It follows that  $z = 1$  and  $z^d = z^{d-1} = 1$ , a contradiction. This completes the proof of the theorem for trinomials.

Now assume that  $f(x)$  has 4 non-zero terms. Then, as  $L(f) = 4$ , we have

$$f(x) = x^d \pm x^c \pm x^b \pm 1 \tag{2.1}$$

for some three positive integers  $d > c > b$  and unknown choice of signs  $\pm$ . We argue exactly as in the case of trinomials. By Lemma 2.3 (with  $k = 4$ ), the Pisot number  $\alpha$  that is a root of a polynomial  $f(x)$  of this form must be less than or equal to a single positive root  $\theta_{d-2}^-$  of the polynomial

$$h_d(x) = x^d - x^{d-1} - x^{d-2} - 1 = x^{d-2}(x^2 - x - 1) - 1.$$

According to Lemma 2.2 (with  $k = 2$  and  $n = d - 2$ ), for degrees  $d \geq 17$  one has  $\alpha \leq \theta_{15}^-$ , where  $\theta_{15}^-$  denotes the root of the polynomial  $p_{15}^-(x) = x^{15}(x^2 - x - 1) - 1$ . Therefore, polynomials  $f(x)$  in (2.1) of degree  $d \geq 17$  are covered by the Dufresnoy–Pisot classification. By direct inspection, we find that all Pisot quadrinomials  $f(x)$  in Proposition 2.1 are given by the sequence  $p_{2n+1}^-(x)$  for  $d = 2n + 3$ ,  $n \geq 7$ .

It remains to find all Pisot quadrinomials  $f(x) = x^d \pm x^c \pm x^b \pm 1$  of degree  $d \leq 16$ . A simple computer search on SAGE for irreducible quadrinomials  $f(x)$  of this shape with precisely 1 root outside the unit circle gives only polynomials from the sequence  $p_{2n+1}^-(x)$ ,  $1 \leq n \leq 6$ , and one more polynomial  $f(x) = x^5 - x^4 - x^2 - 1$  that already appears in the list of Proposition 2.1 as the polynomial  $p_5^+(x)$ . This exhausts all possibilities and concludes the proof of Theorem 1.1 for quadrinomials.  $\square$

**3. Proof of Theorem 1.2 for the numbers  $\alpha_0, \alpha_1$  and  $\alpha_2$**

For the lower bounds we shall use the next lemma which was proved in [10]:

**Lemma 3.1.** *Let  $m \geq 2$  be an integer, and let  $f(x) = x^d + a_{d-1}x^{d-1} + \dots + a_0$  be the minimal polynomial of a Pisot number  $\alpha$ . Suppose a sequence  $B = (b_n)_{n=1}^\infty$  with  $b_n \in \{0, 1, \dots, m - 1\}$  satisfies*

$$b_{n+d} + a_{d-1}b_{n+d-1} + \dots + a_0b_n = 0, \quad n = 0, 1, 2, \dots, \tag{3.1}$$

modulo  $m$ , where  $B$  is a purely periodic sequence with period  $b_1, \dots, b_t$  of length  $t$ . Then,

$$\mathcal{L}(\alpha) \geq \min\{M_{\min}/m, 1 - M_{\max}/m\}, \tag{3.2}$$

where

$$M_{\min} := \min\{b_1, \dots, b_t\} \quad \text{and} \quad M_{\max} := \max\{b_1, \dots, b_t\}.$$

Since  $\alpha_0$  is the root of  $x^2 - x - 1$ , the linear recurrence (3.1) is  $b_{n+2} - b_{n+1} - b_n = 0$ ,  $n = 1, 2, 3, \dots$ . Select  $m = 5$  and  $b_1 = 1, b_2 = 3$ . Then,  $B$  is the sequence  $1, 3, 4, 2, 1, 3, 4, 2, \dots$  which is purely periodic with period  $1, 3, 4, 2$ . We see that  $M_{\min} = 1, M_{\max} = 4$ , so that  $M_{\min}/m = 1/5, M_{\max}/m = 4/5$ . This gives the value  $1/5$  for the right hand side of (3.2). Thus, Lemma 3.1 implies  $\mathcal{L}(\alpha_0) \geq 1/5$ .

Next, as  $\alpha_1$  is the root of  $x^3 - x - 1$ , the linear recurrence (3.1) is  $b_{n+3} - b_{n+1} - b_n = 0, n = 1, 2, 3, \dots$ . Select  $m = 5$  and  $b_1 = 1, b_2 = 2, b_3 = 4$ . Then  $B$  is the sequence  $1, 2, 4, 3, 1, 2, 4, 3, \dots$  which is purely periodic with period  $1, 2, 4, 3$ . As above, we find that  $M_{\min}/m = 1/5, M_{\max}/m = 4/5$ . So, Lemma 3.1 implies  $\mathcal{L}(\alpha_1) \geq 1/5$  again.

Finally,  $\alpha_2$  is the root of  $x^3 - x^2 - 1$ , so the linear recurrence (3.1) is  $b_{n+3} - b_{n+2} - b_n = 0, n = 1, 2, 3, \dots$ . Selecting  $m = 3$  and  $b_1 = 1, b_2 = 2, b_3 = 1$ , we find that  $B$  is the sequence  $1, 2, 1, 2, 1, 2, \dots$  which is purely periodic with period  $1, 2$ . This time,  $M_{\min} = 1, M_{\max} = 2$ , so that  $M_{\min}/m = 1/3, M_{\max}/m = 2/3$ , giving the constant  $1/3$  on the right hand side of (3.2). Now, Lemma 3.1 implies  $\mathcal{L}(\alpha_2) \geq 1/3$ .

It remains to prove the upper bounds

$$\mathcal{L}(\alpha_0) \leq 1/5, \quad \mathcal{L}(\alpha_1) \leq 1/5 \quad \text{and} \quad \mathcal{L}(\alpha_2) \leq 1/3. \tag{3.3}$$

For this, we first write  $u_n := [\xi\alpha^n]$  and  $w_n := \{\xi\alpha^n\} - 1/2$ , where  $\alpha$  is the root of  $f(x) = x^d + a_{d-1}x^{d-1} + \dots + a_0$ . As  $\xi\alpha^n = u_n + w_n + 1/2$ , we find that

$$u_{n+d} + w_{n+d} + 1/2 + \sum_{j=0}^{d-1} a_j(u_{n+j} + w_{n+j} + 1/2) = 0.$$



Hence

$$w_{n+d} + \sum_{j=0}^{d-1} a_j w_{n+j} = -\frac{f(1)}{2} - u_{n+d} - \sum_{j=0}^{d-1} a_j u_{n-j}. \tag{3.4}$$

Since  $-1/2 \leq w_n < 1/2$  for each  $n \in \mathbb{N}$ ,  $u_n \in \mathbb{Z}$  and in all four cases ( $x^2 - x - 1$ ,  $x^3 - x - 1$ ,  $x^3 - x^2 - 1$ ,  $x^4 - x^3 - 1$ ) we have  $f(1) = -1$ , the left hand side of (3.4) is less than  $3/2$  in absolute value, whereas the right hand side of (3.4) is of the form  $\mathbb{Z} + 1/2$ . Thus,

$$\delta_n := w_{n+d} + \sum_{j=0}^{d-1} a_j w_{n+j} \in \{-1/2, 1/2\}$$

for each  $n \in \mathbb{N}$ . Putting  $v_n := 2w_n$  and  $\Delta_n := 2\delta_n$ , we further rewrite this in the form

$$\Delta_n := v_{n+d} + \sum_{j=0}^{d-1} a_j v_{n+j} \in \{-1, 1\} \tag{3.5}$$

for each  $n \in \mathbb{N}$ .

Contrary to what we claim in (3.3), assume that  $\mathcal{L}(\alpha_0) > 1/5$ . Then all the fractional parts  $\{\xi\alpha^n\}$  lie in a subinterval of  $(1/5, 4/5)$  starting from a certain  $n = n_0$ . Thus, as  $v_n = 2w_n = 2\{\xi\alpha_0^n\} - 1$ , there is a positive number  $\varepsilon$  and an integer  $n_0$  such that

$$|v_n| \leq 3/5 - \varepsilon \tag{3.6}$$

for each  $n \geq n_0$ . We will show that this contradicts (3.5), which is

$$\Delta_n := v_{n+2} - v_{n+1} - v_n \in \{-1, 1\} \tag{3.7}$$

for  $n \in \mathbb{N}$ , since  $\alpha_0$  is the root of  $x^2 - x - 1$ .

Indeed, consider the sequence  $(\Delta_n)_{n=1}^\infty$  consisting of  $-1$  and  $1$ . If we have infinitely many patterns  $1, 1, 1$  in this sequence, then, by adding  $v_{n+2} - v_{n+1} - v_n = 1$ ,  $v_{n+3} - v_{n+2} - v_{n+1} = 1$  and  $v_{n+4} - v_{n+3} - v_{n+2} = 1$ , we find that  $v_{n+4} - v_{n+2} - 2v_{n+1} - v_n = 3$  for infinitely many  $n$ 's. Thus,

$$3 \leq (1 + 2 + 1 + 1) \max\{|v_n|, |v_{n+1}|, |v_{n+2}|, |v_{n+4}|\}$$

and at least of the numbers  $v_{n+4}$ ,  $v_{n+2}$ ,  $v_{n+1}$ ,  $v_n$  must be greater than  $3/5$  in absolute value, contrary to (3.6). The same is true when the sequence  $(\Delta_n)_{n=1}^\infty$  has infinitely many patterns  $-1, -1, -1$ . So, starting from certain place, the sequence  $(\Delta_n)_{n=1}^\infty$  contains at most two equal elements in a row.

Assume there are infinitely patterns of the form  $-1, 1, 1$ . Then, by adding  $v_{n+2} - v_{n+1} - v_n = -1$  (with weight  $-1$ ),  $v_{n+3} - v_{n+2} - v_{n+1} = 1$  and  $v_{n+4} - v_{n+3} - v_{n+2} = 1$ , we find that  $v_{n+4} - 3v_{n+2} + v_n = 3$  for infinitely many  $n$ 's. This leads to the same contradiction as above, since  $1 + 3 + 1 = 5$ . In the same way, one again obtains the same contradiction in the symmetric case, when  $(\Delta_n)_{n=1}^\infty$  contains infinitely many patterns  $1, -1, -1$ . Thus,

$(\Delta_n)_{n=1}^\infty$  cannot contain two equal elements in a row. This means that  $(\Delta_n)_{n=1}^\infty$ , starting from certain place, is  $1, -1, 1, -1, 1, -1, \dots$ . Then, adding  $\Delta_n = v_{n+2} - v_{n+1} - v_n = 1$ ,  $\Delta_{n+2} = v_{n+4} - v_{n+3} - v_{n+2} = 1$  and  $\Delta_{n+4} = v_{n+6} - v_{n+5} - v_{n+4} = 1$ , we obtain  $v_{n+6} - v_{n+5} - v_{n+3} - v_{n+1} - v_n = 3$ . Thus, at least one of the numbers  $v_{n+6}, v_{n+5}, v_{n+3}, v_{n+1}, v_n$  must be greater than  $3/5$  in absolute value, contrary to (3.6). This shows that  $\mathcal{L}(\alpha_0) \leq 1/5$  and so completes the proof of Theorem 1.2 for  $\alpha_0$ .

To get the required bound for  $\mathcal{L}(\alpha_1)$  as in (3.3), assume again that  $\mathcal{L}(\alpha_1) > 1/5$ . Now, as  $\alpha_1$  is the root of  $x^3 - x - 1$ , we have (3.6) as above, but instead of (3.7), the sequence  $\Delta_n$  is given by

$$\Delta_n := v_{n+3} - v_{n+1} - v_n \in \{-1, 1\}$$

for  $n \in \mathbb{N}$ . It is easy to see that we must have at least one of the following four possibilities. The sequence  $(\Delta_n)_{n=1}^\infty$ , starting from certain place, is  $1, 1, 1, 1, \dots$  (let us call this first case),  $-1, -1, -1, -1, \dots$  (second case), it contains infinitely many patterns  $1, -1, ?, 1$ , where ? stands for an arbitrary element (third case), or it contains infinitely many patterns  $1, -1, ?, -1$  (fourth case). In the first two cases we obtain the same contradiction as above, by adding  $\Delta_n, \Delta_{n+1}$  and  $\Delta_{n+3}$  to obtain  $v_{n+6} - v_{n+2} - 2v_{n+1} - v_n = \pm 3$ . In the third case, we add  $v_{n+3} - v_{n+1} - v_n = 1, -v_{n+4} + v_{n+2} + v_{n+1} = 1$  and  $v_{n+6} - v_{n+4} - v_{n+3} = 1$ . This gives  $v_{n+6} - 2v_{n+4} + v_{n+2} - v_n = 3$ . Now, as above, at least one number of  $v_{n+6}, v_{n+4}, v_{n+2}, v_n$  must be greater than  $3/5$  in absolute value, contrary to (3.6). Finally, in the fourth case, we obtain exactly the same contradiction by adding  $v_{n+3} - v_{n+1} - v_n = 1, -v_{n+4} + v_{n+2} + v_{n+1} = 1$  and  $-v_{n+6} + v_{n+4} + v_{n+3} = 1$ . This completes the proof of Theorem 1.2 for  $\alpha_1$ .

It remains to prove the required bound  $\mathcal{L}(\alpha_2) \leq 1/3$  as in (3.3). Now, as  $\alpha_2$  is the root of  $x^3 - x^2 - 1$ , we have

$$\Delta_n = v_{n+3} - v_{n+2} - v_n \in \{-1, 1\}$$

for each  $n \in \mathbb{N}$ . Of course, modulus considerations immediately imply that  $\max(|v_{n+3}|, |v_{n+2}|, |v_n|) \geq 1/3$ . Hence,

$$|2\{\xi\alpha_2^n\} - 1| = |v_n| \geq 1/3$$

for infinitely many  $n$ 's. This shows that, for each  $\xi \in \mathbb{R}$ , we must have  $\{\xi\alpha_2^n\} \in [0, 1/3] \cup [2/3, 1)$  for infinitely many positive integers  $n$ . Thus  $\mathcal{L}(\alpha_2) \leq 1/3$ .

#### 4. Proof of Theorem 1.2 for $\mathcal{L}(\alpha_3)$

For the lower bound we use Lemma 3.1 again. Consider the linear recurrence  $b_{n+4} = b_{n+3} + b_n, n \geq 1$ , modulo  $m = 17$  with  $b_1 = 3, b_2 = 10, b_3 = 5, b_4 = 11$  that corresponds to the minimal polynomial  $x^4 - x^3 - 1$  of  $\alpha_3$ . This linear recurrence produces a periodic sequence  $B = 3, 10, 5, 11, 14, 7, 12, 6, \dots$  with period of length 8. One has  $M_{\min} = 3, M_{\max} = 14$  in Lemma 3.1, hence  $\mathcal{L}(\alpha_3) \geq 3/17$ .

To prove the upper bound, we proceed as in Section 3. Consider the sequence  $v_n = 2w_n = 2\{\xi\alpha^n\} - 1$  and the sequence

$$\Delta_n := v_{n+4} - v_{n+3} - v_n, \tag{4.1}$$

associated with the minimal polynomial  $x^4 - x^3 - 1$  of the number  $\alpha_3$ . By Eqs. (3.4), (3.5) of Section 3, one has  $\Delta_n \in \{-1, 1\}$ .

Let us assume that  $\mathcal{L}(\alpha_3) > 3/17$ , so that all the fractional parts  $\{\xi\alpha^n\}$  belong to some proper subinterval of  $(3/17, 14/17)$  starting from some  $n \geq n_0$ . Then,  $|v_n| < 11/17$  for every  $n \geq n_0$ . Our goal is to show that this is impossible.

We say that a pattern  $\mathbf{\Delta} = \Delta_n, \dots, \Delta_{n+k} \in \{-1, 1\}$  is *not admissible* in the sequence  $(\Delta_n)_{n=n_0}^\infty$  if there exists an integer vector

$$\mathbf{s} = \mathbf{s}(\mathbf{\Delta}) = (s_1, s_2, \dots, s_k) \in \mathbb{Z}^k$$

such that the sums of the left and right hand sides of Eq. (4.1) for the numbers  $\Delta_{n+j}$ ,  $0 \leq j \leq k - 1$  in (4.1) scaled by  $s_1, s_2, \dots, s_k$ , respectively, produce a linear relation of the form

$$s_1\Delta_n + \dots + s_k\Delta_{n+k-1} = t_1v_{n+k+3} + \dots + t_{k+4}v_n, \tag{4.2}$$

where the vector  $\mathbf{t} := (t_1, t_2, \dots, t_{k+4}) \in \mathbb{Z}^{k+4}$  makes the ratio

$$r = r(\mathbf{\Delta}, \mathbf{s}, \mathbf{t}) := \frac{|s_1\Delta_n + s_2\Delta_{n+1} + \dots + s_k\Delta_{n+k-1}|}{|t_1| + |t_2| + \dots + |t_{k+4}|}$$

greater or equal to  $11/17 = 0.647058\dots$

Of course, this implies that the absolute value of at least one of numbers  $v_n, v_{n+1}, \dots, v_{n+k+3}$  that appear on the right hand side of (4.2) must be greater or equal to  $r \geq 11/17$ . We will refer to vectors  $\mathbf{s}$  and  $\mathbf{t}$  as *the weight vector* and *the vector of coefficients*, respectively.

For instance, the pattern  $\mathbf{\Delta} = \Delta_n, \Delta_{n+1}, \Delta_{n+2}, \Delta_{n+3} = 1, 1, 1, 1$  is not admissible. Indeed, by adding the corresponding sides of the equations

$$\begin{array}{rccccccc} \Delta_n & = & & & v_{n+4} & -v_{n+3} & & -v_n, \\ \Delta_{n+1} & = & & & v_{n+5} & -v_{n+4} & & -v_{n+1}, \\ \Delta_{n+2} & = & & v_{n+6} & -v_{n+5} & & -v_{n+2}, & \\ \Delta_{n+3} & = & v_{n+7} & -v_{n+6} & & -v_{n+3}, & & \end{array}$$

one obtains

$$\Delta_n + \Delta_{n+1} + \Delta_{n+2} + \Delta_{n+3} = v_{n+7} - 2v_{n+3} - v_{n+2} - v_{n+1} - v_n.$$

In this case,  $\mathbf{s} = (1, 1, 1, 1)$ ,  $\mathbf{t} = (1, 0, 0, 0, -2, -1, -1, -1)$ . Since  $\Delta_n = \Delta_{n+1} = \Delta_{n+2} = \Delta_{n+3} = 1$ , the ratio  $r$  is equal to

$$r = \frac{|1 + 1 + 1 + 1|}{|1| + |0| + |0| + |0| + |2| + |-1| + |-1| + |-1|} = \frac{2}{3} > \frac{11}{17},$$

hence the pattern 1, 1, 1, 1 is not admissible.

Likewise, the pattern 1, 1, 1, -1 is also not admissible, since the sum of  $\Delta_n, \Delta_{n+1}, \Delta_{n+2}, \Delta_{n+3}$  with weights  $\mathbf{s} = (1, 1, 1, -1)$  results in the relation  $\Delta_n + \Delta_{n+1} + \Delta_{n+2} - \Delta_{n+3} = -v_{n+7} + 2v_{n+3} - v_{n+2} - v_{n+1} - v_n$  with coefficients  $\mathbf{t} = (-1, 2, 0, 0, 0, -1, -1, -1)$  and the ratio  $r = 2/3$ .

It is important to note that, if the pattern  $\Delta$  is not admissible, then pattern  $-\Delta$  that is obtained from  $\Delta$  by changing all the signs is not admissible either. This follows from the fact that replacing  $\Delta$  by  $-\Delta$  in the left hand side of (4.2) (with the same weight vector  $\mathbf{s}$ ) produces the same vector  $\mathbf{t}$  and the same ratio  $r(\Delta, \mathbf{s}, \mathbf{t})$ . In the proof, we will make frequent use of this symmetry property.

**Lemma 4.1.** *The following eleven patterns are non-admissible (Table 1).*

**Table 1**  
Non-admissible patterns.

#	$\Delta = \Delta_n, \Delta_{n+1}, \dots, \Delta_{n+k-1}$ :
1	1, 1, 1, 1
2	1, 1, 1, -1
3	1, 1, -1, -1
4	1, -1, 1, -1, 1, -1, 1
5	1, 1, -1, 1, -1, 1, 1
6	1, 1, -1, 1, -1, 1, -1, 1
7	1, 1, -1, 1, -1, 1, -1, -1
8	1, 1, -1, 1, -1, -1, 1, -1, -1
9	1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1
10	1, 1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1, 1
11	1, 1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1, -1, -1

**Proof of Lemma 4.1.** Consider the weights listed in Table 2.

**Table 2**  
Weights.

#	$\mathbf{s} = (s_1, s_2, \dots, s_k)$	Ratios $r$
1	(1, 1, 1, 1)	2/3
2	(1, 1, 1, -1)	2/3
3	(1, 1, -1, -1)	2/3
4	(1, 0, 0, -1, 1, 0, 1)	2/3
5	(1, 2, 0, -1, -1, 2, 2)	7/9
6	(1, 3, 0, -1, -2, 3, 1, 1)	2/3
7	(1, 0, -3, -1, 1, 3, -2, -2)	9/13
8	(0, 1, 0, 0, -1, 0, 0, -1, -1)	2/3
9	(7, 3, 0, 5, 4, 3, 2, 1, 1, 1, 1)	2/3
10	(0, 1, 0, 0, -1, -1, 0, 1, 0, -1, -1, 1, 1)	2/3
11	(0, 1, 0, 0, -1, -3, 0, 1, 2, 3, -1, -1, -1, -1)	11/17

Let  $\Delta$  be the pattern located in the  $j$ -th row of Table 1. Add the numbers  $\Delta_n, \Delta_{n+1}, \dots, \Delta_{n+k-1}$  produced by (4.1) with weights  $s_1, s_2, \dots, s_k$  specified in the  $j$ -th row of Table 2. This yields a relation of the form (4.2) with coefficients  $\mathbf{t} = (t_1, t_2, \dots, t_{k+3})$ . Evaluate the ratio  $r = r(\Delta, \mathbf{s}, \mathbf{t})$  (all such ratios are all listed in Table 2) and verify that  $r \geq 11/17$ .  $\square$

From Lemma 4.1, we see that both patterns  $1, 1, 1, 1$  and  $1, 1, 1, -1$  are non-admissible, hence, the pattern  $1, 1, 1$  cannot appear in the sequence  $(\Delta_n)_{n=n_0}^\infty$ . By symmetry,  $-1, -1, -1$  cannot appear in that sequence either. Moreover, by Lemma 4.1, we see that  $1, 1, -1, -1$ , and, by symmetry,  $-1, -1, 1, 1$  are also non-admissible.

Let us now represent six possible patterns of length 3 that appear in the sequence as the six vertices of a directed graph depicted in Fig. 1. The vertices are labeled  $A, B, C, D, E$  and  $F$ . We join two vertices, say,  $X = x_1x_2x_3$  and  $Y = y_1y_2y_3$  with a directed edge, if  $x_2x_3 = y_1y_2$ , that is, the tail of the pattern  $X$  is the head of the pattern  $Y$ , provided that the pattern  $x_1x_2x_3y_3$  is admissible, i.e. it does not appear in Table 1 of Lemma 4.1, nor it is symmetric to one of the patterns in Table 1. (In view of this, the vertices  $A$  and  $E$  and also the vertices  $F$  and  $C$  are not connected by an edge.) We label the edge  $XY$  with  $+$  or  $-$  according to the sign of the last digit  $y_3$  that needs to be appended to  $X$  to obtain  $Y$ . The sequence  $(\Delta_n)_{n=n_0}^\infty$  corresponds to the infinite walk in the graph that produces no pattern given in Table 1.

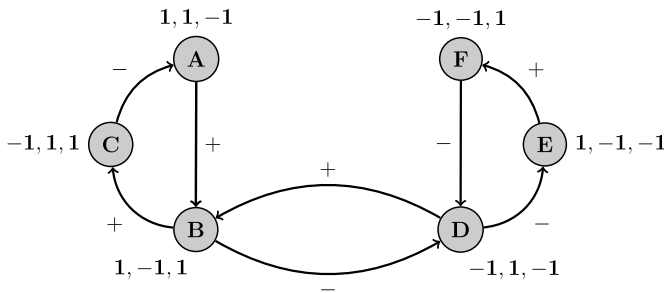


Fig. 1. A graphical representation of admissible adjacent blocks of  $\pm 1$  of length 3 in the sequence  $(\Delta_n)_{n=n_0}^\infty$ .

From the discussion on the admissible patterns of length 3 and 4 given above, we see that no three consecutive terms  $\Delta_n, \Delta_{n+1}, \Delta_{n+2}$  can have the same sign. Also,  $(\Delta_n)_{n=n_0}^\infty$  cannot contain arbitrarily long alternating patterns  $(1, -1)^k$ , since the pattern no. 4 in Table 1 is not admissible. Hence,  $(\Delta_n)_{n=n_0}^\infty$  must contain infinitely many patterns  $1, 1, -1$  or  $-1, -1, 1$ .

By symmetry, without restriction of generality we may suppose that the sequence  $\Delta_{n_0}, \Delta_{n_0+1}, \Delta_{n_0+2}, \dots$  starts with  $1, 1, -1$  at the vertex  $A$ . One cannot run into the infinite cycle  $(ABC)^\infty$ , since the cycle  $(ABC)^3$  produces a non-admissible sequence no. 9 in Table 1. Therefore, at some point one must move from  $A$  to  $D$  through  $B$ . The path  $ABD$  corresponds to the pattern

$$1, 1, -1, 1, -1.$$

We claim that one cannot turn back from  $D$  to  $B$ , that is, the walk  $ABDB$  is impossible. Indeed, if one returns to  $B$ , producing the pattern  $1, 1, -1, 1, -1, 1$ , then there are two choices: move to  $C$  or back to  $D$ . The first choice  $ABDBC$  yields the pattern no. 5 in Table 1:  $1, 1, -1, 1, -1, 1, 1$  that is not admissible. The second choice  $ABDBD$  yields a the pattern  $1, 1, -1, 1, -1, 1, -1$ . An attempt to move from  $D$  back to  $B$  again yields  $ABDBDB$  with a non-admissible pattern no. 6:  $1, 1, -1, 1, -1, 1, -1, 1$ . The move from  $D$  to  $E$  produces  $ABDBDE$  with a non-admissible pattern no. 7:  $1, 1, -1, 1, -1, 1, -1, -1$ .

Therefore, the claim is true, so the walk  $ABDB$  is impossible. Hence, from  $D$  one should move to  $E$ , then to  $F$  and then return back to  $D$ . The walk  $ABDEFD$  corresponds to the pattern

$$1, 1, -1, 1, -1, -1, 1, -1.$$

At this point, one cannot go back to  $E$ , since this results in the non-admissible pattern no. 8:  $1, 1, -1, 1, -1, -1, 1, -1, -1$ . Thus one must move to  $B$ . The walk  $ABDEFDB$  corresponds to the pattern

$$1, 1, -1, 1, -1, -1, 1, -1, 1.$$

After reaching this point, one cannot turn back to  $D$ , since the last four steps  $FDBD$  in the walk  $ABDEFDBD$  yield the pattern symmetric to the one produced by the walk  $ABDB$  and we already established that such a pattern cannot occur. Consequently, one goes from  $B$  to  $C$ , then to  $A$  and then back to  $B$ . The walk  $ABDEFDBCAB$  corresponds to the pattern

$$1, 1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1.$$

There are two possibilities here. If one moves back to  $C$ , then the non-admissible pattern no. 10:  $1, 1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1, 1$  from Table 1 occurs. Thus, one must go to  $D$ :  $ABDEFDBCABD$ . From there, we cannot turn back to  $B$ , since the walk  $ABDB$  leads to non-admissible patterns. Hence, the only remaining possibility is to move to  $E$ . However, this last choice yields  $ABDEFDBCABDE$  with the non-admissible pattern no. 11:

$$1, 1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1, -1.$$

There are no more choices left! So our analysis shows that one always runs into some non-admissible pattern in the sequence  $(\Delta_n)_{n=n_0}^\infty$ . This contradicts our initial assumption made in the beginning of Section 4, so there must be infinitely many  $n \in \mathbb{N}$  for which  $|v_n| \geq 11/17$ . From this, it follows that  $\mathcal{L}(\alpha_3) \leq 3/17$ . Now, combining with the lower bound  $\mathcal{L}(\alpha_3) \geq 3/17$  which was proved earlier, we obtain  $\mathcal{L}(\alpha_3) = 3/17$ .

The proof of Theorem 1.2 is now complete.

## Acknowledgments

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