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On Newman polynomials, which divide no Littlewood polynomial

Mathematics of Computation, **78** (2009).

Definitions

A polynomial $P(x)$ with all coefficients from the set $\{0, 1\}$, and $P(0) \neq 0$ is called a *Newman* polynomial.

Polynomials $P(x)$ which have all coefficients $\{-1, 1\}$, are called *Littlewood* polynomials.

$V_{\mathcal{N}}$ – The set of complex zeros of Newman polynomials, the *Newman numbers*.

$V_{\mathcal{L}}$ – The set of roots of Littlewood polynomials - the *Littlewood numbers*.

V – The set of complex zeros of $\{$-1, 0, 1$\}$ polynomials with $P(0) \neq 0$.

$H(P)$ – Height of the polynomial P.

Number theory

Special case of *Siegel's lemma*: if *Mahler's measure* $M(P)$ of the polynomial P satisfies $M(P) < 2$, then P divides some polynomial with the coefficients in the set $\{-1, 0, 1\}$.

The Mahler measure of polynomial P is given by the formulae:

$$M(P) = |a_d| \prod_{j=1}^{d} \max\{1, |\alpha_j|\},$$

where $\alpha_1, \ldots, \alpha_d$ are all the roots of the polynomial $P(x)$, not necessarily distinct.

The problem of smallest Mahler measure for Littlewod polynomials is settled – P. Borwein, E. Dobrowolski and M. J. Mossinghoff, *"Lehmer's problem for polynomials with odd coefficients"* (2007).

Combinatorial problem

P. Borwein and M. Mossinghoff in *"Newman polynomials with prescribed vanishing and integer sets with distinct subset sums"*(2003) performed large scale computations on Newman polynomials.

The set $\mathcal{D} = \{e_1, \ldots, e_d\} \subset \mathbb{N}$ has distinct subset sums if and only if

$$P(x) = \prod_{j=1}^{d} (1 + x^{e_j})$$

is a Newman polynomial.

Sets of roots

The set $V_{\mathcal{N}}$ was investigated by A.M. Odlyzko and B. Poonen in the article *"Zeros of polynomials with* $0, 1$ *coefficients"* (1993).

P. Borwein, C. Pinner, *"Polynomials with* $\{0, +1, -1\}$ *coefficients and a root close to a given point"* (1997).

F. Beacoup, P. Borwein, D. W. Boyd and C. Pinner, *"Multiple roots of* $[-1, 1]$ *power series"* and *"Power series with restricted coefficients and a root on a given ray"* (1998).

Main problem

Artūras Dubickas, West Coast Number Theory Conference (2006):

Problem 1. *Does there exist an algebraic number* α, *which is a root of some Newman polynomial, but is not a root of any Littlewood polynomial?*

Dubickas, Drungilas, "*Roots of polynomials of bounded height*". The minimal polynomial

$$P(x) = x^4 + x^3 + 2x^2 - x + 1$$

of the number $\theta = (-1 + i\sqrt{3})(1 + \sqrt{5})/4$ does not divide any polynomial of height 1.

Main results

Theorem 1. *For every trinomial $P(x)$ with coefficients $\{-1, 0, 1\}$ and $P(0) \neq 0$, there exists a polynomial $Q(x)$ with coefficients $\{-1, 0, 1\}$, such that the product PQ is a Littlewood polynomial.*

Theorem 2. *Every Newman polynomial of degree at most 8 divides some Littlewood polynomial.*

Four exceptional Newman polynomials of degree 8 with $H(L/P) \geqslant 2$.

	Table 1: Polynomial $P(x)$	$H(Q)$	$\deg Q$
1.	$1 + x + x^4 + x^6 + x^8$	2	208
2.	$1 + x^2 + x^4 + x^7 + x^8$	2	208
3.	$1 + x + x^2 + x^5 + x^6 + x^8$	2	47
4.	$1 + x^2 + x^3 + x^6 + x^7 + x^8$	2	47

Examples

Some Newman polynomials of degree 9 which have no Littlewood multiple.

	Table 2: Polynomial $P(x)$
1.	$1 + x^4 + x^6 + x^7 + x^9$
2.	$1 + x^2 + x^3 + x^5 + x^9$
3.	$1 + x^3 + x^7 + x^8 + x^9$
4.	$1 + x + x^2 + x^6 + x^9$
5.	$1 + x + x^2 + x^4 + x^6 + x^9$
6.	$1 + x^3 + x^5 + x^7 + x^8 + x^9$
7.	$1 + x + x^4 + x^5 + x^6 + x^7 + x^9$
8.	$1 + x^2 + x^3 + x^4 + x^5 + x^8 + x^9$

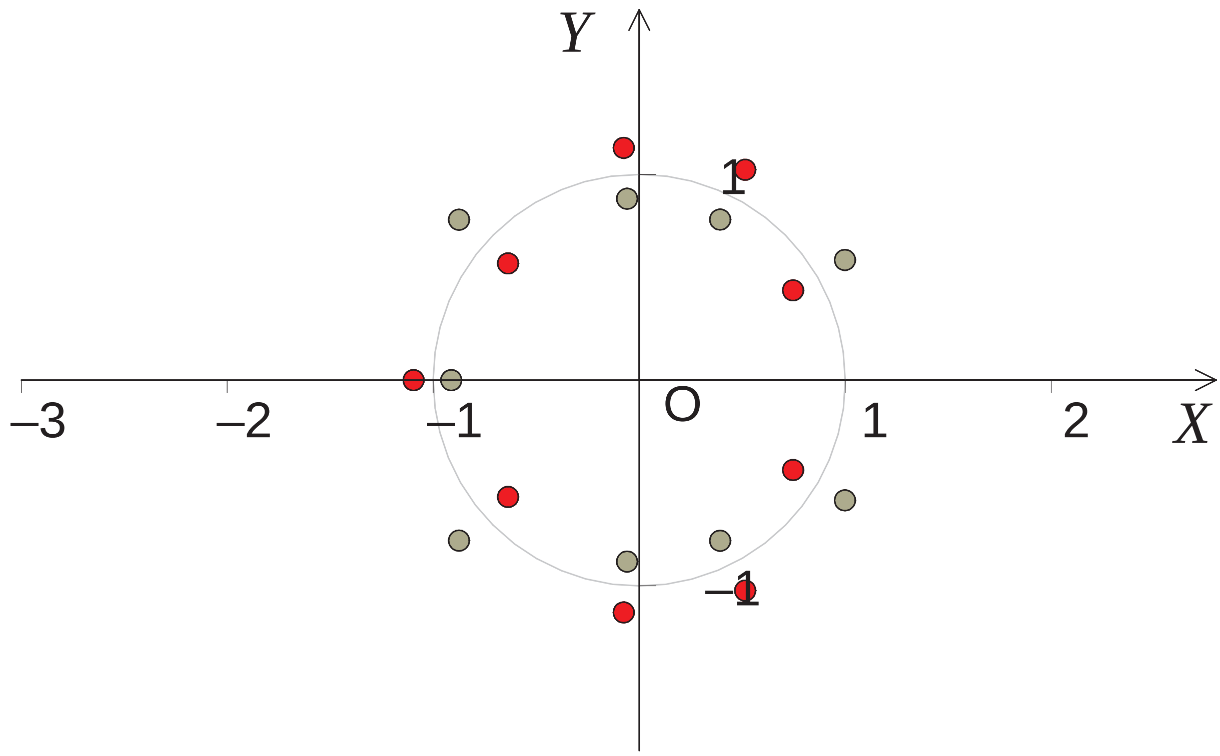
Theorem 3. *There exist infinitely many primitive Newman polynomials which do not divide any Littlewood polynomial.*

Primitive: $P(x) \neq P_1(x^k), k = 2, 3, \ldots$.

It turns out that they may be chosen of the form:

$$x^n P(x) + R(x).$$

Figure: The roots of the polynomials $P(x) = 1 + x^4 + x^6 + x^7 + x^9$ and $x^9 P(1/x)$, colored in red and grey, respectively.



New questions

Problem 2. *Does there exist a Newman quadrinomial (a polynomial with four non–zero terms), which does not divide any Littlewood polynomial?*

Problem 3. *Does there exist an irreducible polynomial with precisely one root outside the unit circle, which divides a Newman but not Littlewood polynomial?*