On the Intersection of Infinite Geometric and Arithmetic Progressions

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Basic notation

- Let $u > 0$ and $q > 1$ be real numbers. The *geometric progression*

$$u, \quad uq^2, \quad uq^3, \quad \ldots, \quad uq^n, \quad \ldots$$

  will be denoted by $\mathcal{G} = \mathcal{G}(u, q)$.

- Let v and D be real numbers, $0 \leqslant v < D$. The *arithmetic progression*

$$v, \quad v + D, \quad v + 2D, \quad \ldots, \quad v + kD, \quad \ldots$$

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Main problem

How *large* is the intersection of sets

$$|\mathcal{G} \cap \mathcal{A}| \quad = \quad ?$$

How many elements the sets $\mathcal{G}$ and $\mathcal{A}$ have *in common*?

Example 1

Let

$$\mathcal{G} = 1, \quad 2, \quad 4, \quad \ldots, \quad 2^n, \quad \ldots$$

and

$$\mathcal{A} = \mathbb{N} = 1, \quad 2, \quad 3, \quad \ldots, \quad k, \quad \ldots.$$

Clearly, in this case

$$\mathcal{G} \subset \mathcal{A} \qquad \text{and} \qquad |\mathcal{G} \cap \mathcal{A}| = \infty.$$

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Example 2

Let $q = u = 3/2$ so that

$$\mathcal{G} \quad = \quad \frac{3}{2}, \quad \frac{9}{4}, \quad \ldots, \quad \left(\frac{3}{2}\right)^n, \quad \ldots$$

and $v = 0$, $D = 1/2^r$

$$\mathcal{A} \quad = \quad \frac{1}{2^r}, \quad \frac{2}{2^r}, \quad \frac{3}{2^r}, \quad \ldots, \quad \frac{k}{2^r}, \quad \ldots.$$

Only elements of $\mathcal{G}$ with denominators at most 2^r belong to $\mathcal{A}$ so

$$|\mathcal{G} \cap \mathcal{A}| \;=\; r.$$

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More examples

If one changes the $q = 2$ to $\sqrt[d]{2}$ in Example 1 or $q = 3/2$ to $\sqrt[d]{3/2}$ in Example 2, the picture does not change too much.

However, if the ratio q is not the root of a rational number, the things get complicated. Consider the geometric progression with

$$u = 1, \qquad q = \frac{1+\sqrt{5}}{2}.$$

Does for any positive integer r there exist $\mathcal{A}$ with

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Fractional Parts of Powers

Let $\xi > 0$ and $\alpha > 1$ be real numbers. The elements of the sequence

$$\{\xi\alpha^n\}, \qquad n = 1, \quad 2, \ldots$$

are called *fractional parts of powers* of α.
If one has an element in $\mathcal{G} \cap \mathcal{A}$:

$$uq^j = v + kD$$

$$\frac{u}{qD} q^{j+1} = \frac{v}{D} + k.$$

Set

$$\xi = u/qD, \qquad \alpha = q, \quad t = v/D.$$

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Equivalent problem

The size of $\mathcal{G} \cap \mathcal{A}$ is equal to the number of times the fractional parts of powers take value t:

$$\{\xi\alpha^n\} = t$$

For the special case $\xi = 1$ - this is a classical problem.

- Vijayaraghavan (around 1940) conjectured: $\{\alpha^a\} = \{\alpha^b\} = \{\alpha^c\}$ holds if and only if

$$\alpha = m^{1/d}, m \in \mathbb{N}, d \in \mathbb{N}.$$

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Special case: A notable result of A. Schinzel

W. Sierpiński (1954): How long are the arithmetic progressions, contained in the geometric progression

$$\mathcal{G}(1, q) = 1, q, q^2, \ldots,$$

where the number q is irrational?
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Results

Theorem 1
For any geometric progression $\mathcal{G}$ with ratio $q > 1$, we have $|\mathcal{G} \cap \mathcal{A}| = \infty$ for some arithmetic progression $\mathcal{A}$ if and only if $q = m^{1/d}$ for some integers $m \geqslant 2$ and $d \geqslant 1$.

Theorem 2
Suppose that $r > 1$ is a rational non-integer number, $d \in \mathbb{N}$ and $q = r^{1/d}$. Then, for every nonnegative integer s, there is a geometric progression $\mathcal{G}$ with ratio q which contains exactly s positive integers, so that $|\mathcal{G} \cap \mathbb{N}| = s$.

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Main result

Theorem 3

Suppose that the ratio $q > 1$ is not of the form $\beta^{1/d}$, where $d \in \mathbb{N}$ and where $\beta > 1$ is either a rational number or a cubic algebraic number with two nonreal conjugates over $\mathbb{Q}$ of moduli distinct from β. Then $|\mathcal{G} \cap \mathcal{A}| \leqslant 3$ for each $\mathcal{G} = \mathcal{G}(u, q)$ and each $\mathcal{A}$.

Moreover, for every integer $s \geqslant 2$, there exist an algebraic number $q > 1$ of degree s (satisfying $q \neq \beta^{1/d}$) and a positive real number $u \in \mathbb{Q}(q)$ such that $|\mathcal{G} \cap \mathcal{A}| = 3$ for the geometric progression $\mathcal{G} = \mathcal{G}(u, q)$ and some arithmetic progression $\mathcal{A}$.

Theorem 4

For $q = \beta^{1/d}$, where $d \in \mathbb{N}$ and $\beta > 1$ is a cubic algebraic number with two nonreal conjugates of moduli distinct from β, we have $|\mathcal{G} \cap \mathcal{A}| \leqslant 6$ for each $\mathcal{G} = \mathcal{G}(u, q)$ and each $\mathcal{A}$.

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The second version of main result

Theorem 5

Let $\xi > 0$ and $\alpha > 1$ be arbitrary real numbers. Then the fractional parts $\{\xi\alpha^n\}$, $n = 1, 2, 3, \ldots$, take a fixed value $t \in [0, 1)$ at most 3 times, with two possible exceptions. The first exception occurs for $\alpha = r^{1/d}$, where $r \in \mathbb{Q}$, $r > 1$ and $d \in \mathbb{N}$. The other (possible) exception may occur for $\alpha = \beta^{1/d}$, where $\beta > 1$ is a real cubic algebraic number with two nonreal conjugates of moduli distinct from β. In the cubic case, the sequence of fractional parts can take the same value at most 6 times.

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Conjecture

We conjecture that one can replace the bound 6 with 3 in the cubic case in Theorem 4 and Theorem 5.