On a class of polynomials related to Barker sequences

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2011
Unimodular sequences and autocorrelations

- A unimodular sequence:

\[
a_0 \ a_1 \ a_2 \ \ldots \ \ldots \ a_n
\]

\[
a_j \in \mathbb{C}, \quad |a_j| = 1.
\]

- For such a sequence, the autocorrelation coefficients \( c_k, \ k = 0, 1, \ldots, n \) are defined as a dot product of the sequence and a shift of the same sequence:

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c_k := \sum_{j=0}^{n-k} a_j \overline{a}_{j+k}, \quad \text{for } k = 0, 1, \ldots, n.
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▶ In particular, the coefficient $c_0$ is called a central autocorrelation coefficient:

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▶ A simple example: a sequence of length four:

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A finite unimodular sequence $a_0, a_1, \ldots, a_n$ is called a **Barker sequence**, if

1. All numbers $a_j$ in this sequence are equal to $-1$ or $1$;
2. The sequence has minimal possible autocorrelations: $c_k \in \{-1, 0, 1\}$.

**Applications:** Barker sequences are of considerable importance in the signal processing theory. In particular, Barker sequences are **optimal** sequences for the phase-modulated pulse compression in radar design.
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- Restrictions on the patterns of coefficients and possible lengths of Barker sequences were obtained by:
  - Turyn (1965),
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- Computations show that no Barker sequences small even length exist. Current computer record belongs to Mossinghoff (2009): lengths up to $2 \cdot 10^{30}$ (with a possible exception 189260468001034441522766781604).

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A conjecture

To prove the non-existence of long Barker sequences of even length, we consider a two part conjecture:

**Conjecture 2**

1. *If* \( p(z) \in \mathbb{C}[z] \) *is a Barker polynomial of odd degree* \( n \), *then* \( M(p) \) *is extremely close to its* \( L_2 \) *norm:*

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\lim_{n \to \infty} \left( M(p) - \sqrt{n + 1} \right) = 0.
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2. *If a polynomial* \( p(z) \) *has all coefficients equal to* \(-1\) *or* \( 1 \), *then Mahler measure* \( M(p) \) *is bounded away from its* \( L_2 \) *norm:*

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New results

Suppose that $p(z)$ is a Barker polynomial of odd degree $n$. Then the product $p(z)p(1/z)$ takes the form

$$P(z) = (n + 1) + \sum_{\substack{k=1 \kern-1.5ex \atop k - \text{odd}}}^{n} c_k \left( z^k + \frac{1}{z^k} \right),$$

where $c_k \in \{-1, 1\}$.

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For a polynomial $R_n \in \mathcal{LP}_n$,

$$M(R_n) > n - \frac{2}{\pi} \log n + O(1).$$

Theorem 2
The polynomials $R_n(z)$ and $R_n(-z)$ have minimal Mahler measures in $\mathcal{LP}_n$, namely, for any $P \in \mathcal{LP}_n$

$$M(P) \geq M(R_n).$$

Theorem 3
For $s < 1$, the polynomials $R_n(\pm z)$ have minimal $L_s$ norms in the class $\mathcal{LP}_n$. In the other hand, $R_n$ have maximal $L_s$ norms in $\mathcal{LP}_n$ for $s \in [2j - 1, 2j]$, $j \in \mathbb{N}$, and also for all $s$ which are sufficiently large: $s > s_0(n)$. 
New results

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Corollary 4

If \( p(z) \) is Barker polynomial of odd degree \( n \), then

\[
M(p) > \sqrt{n - 2/\pi \log n} + O(1).
\]

*This proves the first part of Conjecture 2.*

However, we still need a considerable progress on the second part of the Conjecture 2.