On the random max-closure for heavy-tailed random variables

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Abstract. In this paper, we study the random max-closure property for not necessarily identically distributed real-valued random variables $X_1, X_2, \ldots$, which states that, given distributions $F_{X_1}, F_{X_2}, \ldots$ from some class of heavy-tailed distributions, the distribution of the random maximum $X_{(\eta)} := \max \{0, X_1, \ldots, X_\eta\}$ or random maximum $S_{(\eta)} := \max \{0, S_1, \ldots, S_\eta\}$ belongs to the same class of heavy-tailed distributions. Here, $S_n := X_1 + \cdots + X_n$, $n \geq 1$, and $\eta$ is a counting random variable, independent of $\{X_1, X_2, \ldots\}$. We provide the conditions for the random max-closure property in the case of classes $\mathcal{D}$ and $\mathcal{L}$.

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1 Introduction

Let $X_1, X_2, \ldots$ be identically or nonidentically distributed random variables (r.v.s). Let $\eta$ be an integer-valued nonnegative r.v. such that $\Pr(\eta = 0) < 1$; we call $\eta$ a counting r.v. In addition, suppose that $\eta$ is independent of the sequence $\{X_1, X_2, \ldots\}$.

Let now $S_0 := 0$, $S_n := X_1 + \cdots + X_n$, $n \geq 1$, be the partial sums, and let $S_\eta := X_1 + \cdots + X_\eta$ be the random sum of r.v.s $X_1, X_2, \ldots$. Similarly, let $X_{(n)} := \max \{0, X_1, \ldots, X_n\}$, $n \geq 1$, $X_{(0)} := 0$, and let $X_{(\eta)} := \max \{0, X_1, \ldots, X_\eta\}$. Finally, let $S_{(n)} := \max \{S_0, S_1, \ldots, S_n\}$, $n \geq 0$, and let $S_{(\eta)} := \max \{S_0, S_1, \ldots, S_\eta\}$ be the random maximum of sums $S_0, S_1, S_2, \ldots$. For any real $x$, denote

$$F_{S_\eta}(x) := \Pr(S_\eta \leq x) = \sum_{n=0}^{\infty} \Pr(S_n \leq x) \Pr(\eta = n),$$

$$F_{X_{(\eta)}}(x) := \Pr(X_{(\eta)} \leq x) = \sum_{n=0}^{\infty} \Pr(X_{(n)} \leq x) \Pr(\eta = n),$$

$$F_{S_{(\eta)}}(x) := \Pr(S_{(\eta)} \leq x) = \sum_{n=0}^{\infty} \Pr(S_{(n)} \leq x) \Pr(\eta = n).$$
In this paper, we are interested in the closure property of the random maxima \( X_{(n)} \) and \( S_{(n)} \), which states that, given \( F_{X_1}, F_{X_2}, \ldots \) from some class of heavy-tailed distributions, the distributions \( F_{X_{(n)}} \) and \( F_{S_{(n)}} \) belong to the same class of heavy-tailed distributions. In this case, we say that corresponding random variables (or distributions) possess the random max-closure property.

Recall that a r.v. \( X \), together with its distribution function \( F_X \), is said to be heavy-tailed if \( \mathbb{E}e^{\delta \xi} = \infty \) for all \( \delta > 0 \). We further present definitions of three important subclasses of heavy-tailed distributions. A d.f. \( F \) is said to belong to the class \( \mathcal{L} \) of long-tailed d.f.s if, for every \( a > 0 \), its tail \( \bar{F}(x) = 1 - F(x) \) satisfies \( \lim_{x \to \infty} \frac{F(x + a)}{\bar{F}(x)} = 1 \). A d.f. \( F \) belongs to the class \( \mathcal{D} \) (has dominantly varying tail) if, for every \( a \in (0, 1) \), \( \lim_{x \to \infty} \frac{F(xa)}{F(x)} < \infty \). A d.f. \( F \), supported by \( [0, \infty) \), belongs to the class of subexponential d.f.s \( \mathcal{S} \) if \( \lim_{x \to \infty} \frac{F^{*}(x)}{\bar{F}(x)} = 2 \), where \( * \) denotes the convolution of d.f.s. If a d.f. \( F \) is supported by \( \mathbb{R} \), then \( F \in \mathcal{S} \) if \( F^{\ast}(x) := F(x)1_{[0,\infty)}(x) \) belongs to \( \mathcal{S} \). Lemma 1.3.5 and Proposition 1.4.4 of [10] (see also [4, 11] and [14]) imply that

\[
\mathcal{L} \cap \mathcal{D} \subset \mathcal{I} \subset \mathcal{L}, \quad \mathcal{D} \not\subset \mathcal{I}, \quad \mathcal{I} \not\subset \mathcal{D}.
\]

It is appropriate to mention that the subject of the paper is partially motivated by the closure problem of the random convolution. In the case of independent and identically distributed (i.i.d.) r.v.s \( X, X_1, X_2, \ldots \), we say that some class of d.f.s \( \mathcal{H} \) is closed with respect to the random convolution if \( F_X \in \mathcal{H} \) implies \( F_{X_1}, F_{X_2}, \ldots \in \mathcal{H} \). The random closure property for subexponential distributions was studied by Embrechts and Goldie (see [9, Thm. 4.2]) and Cline (see [5, Thm. 2.13]). The random closure results for class \( \mathcal{D} \) can be found in [7, 15] and for class \( \mathcal{L} \) in [1, 15, 20] and [21]. Note that Xu et al. [21] and Danilenko and Šiaulys [7] considered the case where r.v.s \( X_1, X_2, \ldots \) are not necessarily identically distributed. Also, we note that the closure properties for d.f.s \( F_{S_{(n)}} \) and \( F_{S_{(n)}} \) in the case of i.i.d. r.v.s can be derived from asymptotic formulas obtained, for instance, in [8, 13, 16, 18, 22].

Like in [15], we restrict ourselves to the classes \( \mathcal{L} \) and \( \mathcal{D} \). We find minimal conditions for real-valued (not necessarily identically distributed) r.v.s \( X_1, X_2, \ldots \) and a counting r.v. \( \eta \) under which the d.f.s \( F_{X_{(n)}} \) and \( F_{S_{(n)}} \) belong to the corresponding class. It should be noted that the case of nonrandom \( \eta \) is trivial since the implications \( F_{X_k} \in \mathcal{L}, k = 1, \ldots, n \Rightarrow F_{X_{(n)}} \in \mathcal{L} \) and \( F_{X_k} \in \mathcal{D}, k = 1, \ldots, n \Rightarrow F_{X_{(n)}} \in \mathcal{D} \) hold by relation \( \mathbb{P}(X_{(n)} > x) \sim \sum_{k=1}^{n} \mathbb{P}(X_k > x) \).

The rest part of the paper is organized as follows. In Section 2, we present our results. In Section 3, we collect auxiliary lemmas. In Section 4, we give the proofs of main results. Finally, in Section 5, we present some examples.

## 2 Main results

In this section, we formulate the main results of the paper.

### 2.1 Random max-closure for class \( \mathcal{D} \)

**Theorem 1.** Suppose that \( X_1, X_2, \ldots \) are real-valued arbitrarily dependent r.v.s with corresponding d.f.s \( F_{X_1}, F_{X_2}, \ldots \), and let \( \eta \) be a counting r.v., independent of \( \{X_1, X_2, \ldots\} \), such that

\[
\limsup_{x \to \infty} \sup_{n \geq \infty} \frac{1}{\varphi(n)} \sum_{k=1}^{n} \frac{F_{X_k}(x)}{F_{X_{\infty}}(x)} < \infty
\]

for some \( \varphi \in \text{supp}(\eta) := \{n \geq 0: \mathbb{P}(\eta = n) > 0\} \) and some positive sequence \( \{\varphi(n), n \geq 1\} \) with \( \mathbb{E}(\varphi(\eta)1_{[1,\infty)}(\eta)) < \infty \). Then the following two statements are equivalent:

(i) \( F_{X_{(n)}} \in \mathcal{D} \),

(ii) \( F_{X_{\infty}} \in \mathcal{D} \).

The proof of the theorem is given in Section 4. The following two corollaries follow immediately from Theorem 1. The first corollary deals with the case of identically distributed variables.

**Corollary 1.** Let $X_1, X_2, \ldots$ be arbitrarily dependent r.v.s with common d.f. $F_X$, and let $\eta$ be a counting r.v., independent of $\{X_1, X_2, \ldots\}$, such that $E\eta < \infty$. Then $F_{X(n)} \in \mathcal{D}$ if and only if $F_X \in \mathcal{D}$.

The next corollary considers the case of finite number of r.v.s.

**Corollary 2.** Assume that $m \geq 1$ is a nonrandom integer, $X_1, \ldots, X_m$ are arbitrarily dependent r.v.s, and $\eta$ is a counting r.v., independent of $\{X_1, \ldots, X_m\}$, such that $\text{supp}(\eta) \subset \{0, 1, \ldots, m\}$. Let

$$\limsup_{x \to \infty} \frac{F_{X_\alpha}(x)}{F_{X_\alpha}(x)} < \infty, \quad k = 1, \ldots, m,$$

for some $\alpha \in \text{supp}(\eta)$. Then $F_{X(n)} \in \mathcal{D}$ if and only if $F_{X, \alpha} \in \mathcal{D}$.

The following two results give sufficient conditions for the property $F_{S(n)} \in \mathcal{D}$. Recall that a d.f. $F$ belongs to the class $\mathcal{D}$ if and only if the upper Matuszewska index of $F$ is finite (see [2, Thm. 2.1.8] and discussion in Section 2 of [17]):

$$J_F^+ := -\lim_{y \to \infty} \frac{1}{\log y} \log \left( \liminf_{x \to \infty} \frac{F(xy)}{F(x)} \right) < \infty.$$

**Remark 1.** The first three our results, Theorem 1 and Corollaries 1, 2, hold for r.v.s that can be dependent. Whereas the statements below are proved only for independent r.v.s., the structure of the presented proofs show that similar statements will be true for some mild dependence structures like NUOD (negative upper orthant dependence) or WUOD (widely upper orthant dependence). Such dependence structures and their useful properties are described in [19] and references therein.

**Theorem 2.** Let $X_1, X_2 \ldots$ be independent real-valued r.v.s, and let $\eta$ be a counting r.v. independent of $\{X_1, X_2, \ldots\}$. Assume that $F_{X, \alpha} \in \mathcal{D}$ for some $\alpha \in \text{supp}(\eta)$ and $E\eta^{p+1} < \infty$ for some $p > J_{F_{X, \alpha}}^+$. If condition (2.1) with $\varphi(n) = n$ is satisfied, then $F_{S(n)} \in \mathcal{D}$.

The proof of the theorem is presented in Section 4. The following corollary can be easily derived from Theorem 2.

**Corollary 3.** Let $X_1, X_2, \ldots$ be i.i.d. r.v.s with common d.f. $F_X \in \mathcal{D}$, and let $\eta$ be a counting r.v. independent of $\{X_1, X_2, \ldots\}$, such that $E\eta^{p+1} < \infty$ for some $p > J_{F_X}^+$. Then $F_{S(n)} \in \mathcal{D}$.

### 2.2 Random max-closure for class $\mathcal{L}$

**Theorem 3.** Suppose that $X_1, X_2, \ldots$ are independent real-valued r.v.s with corresponding d.f.s $F_{X_1}, F_{X_2}, \ldots$ and $\eta$ is a counting r.v. independent of $\{X_1, X_2, \ldots\}$.

(i) Suppose that there exists $\alpha \in \text{supp}(\eta)$ such that $F_{X, \alpha} \in \mathcal{L}$ for all $k \leq \alpha$, $E(\varphi(\eta)1_{[1, \infty)}(\eta)) < \infty$ for some positive sequence $\{\varphi(n), n \geq 1\}$, and, as $x \to \infty$,

$$\sum_{k=1}^{n} F_{X, \alpha}(x) \sim \varphi(n) F_{X, \alpha}(x)$$

uniformly in $n \geq \alpha$. Then $F_{X(n)} \in \mathcal{L}$.
Lemma 2. Suppose that there exists \( \varepsilon \in \text{supp}(\eta) \) such that
\[
F_{X_k}(x) = o\left(F_{X,n}(x)\right) \quad \text{for all } k < \varepsilon
\]
and (2.2) holds uniformly in \( n \geq \varepsilon \). Then \( F_{X_n} \in \mathcal{L} \iff F_{X(n)} \in \mathcal{L} \).

Corollary 4. Let \( X_1, X_2, \ldots \) be i.i.d. r.v.s with common d.f. \( F_X \), and let \( \eta \) be a counting r.v. independent of \( \{X_1, X_2, \ldots\} \). If \( \mathbb{E}\eta < \infty \), then \( F_X \in \mathcal{L} \iff F_{X(n)} \in \mathcal{L} \).

Consider now conditions for \( F_{S(n)} \in \mathcal{L} \).

Theorem 4. Let \( X_1, X_2, \ldots \) be independent real-valued r.v.s with corresponding d.f.s \( F_{X_1}, F_{X_2}, \ldots \), and let \( \eta \) be a counting r.v. independent of \( \{X_1, X_2, \ldots\} \).

(iii) If \( \text{supp}(\eta) \subset \{0, 1, \ldots, m\} \) (\( m < \infty \)) and \( F_{X_k} \in \mathcal{L} \) for all \( k \in \{1, \ldots, m\} \), then \( F_{S(n)} \in \mathcal{L} \).

(ii) Assume that \( \mathbb{P}(\eta = k) > 0 \) for all \( k \geq 1 \), and let the following conditions hold:
\[
\begin{align*}
\text{(ii.1)} \quad & \limsup_{x \to \infty} \sup_{k \geq 1} \frac{F_{X_k}(x - 1)}{F_{X_k}(x)} = 1, \\
\text{(ii.2)} \quad & \lim_{n \to \infty} \frac{\mathbb{P}(\eta > n)}{\min_{1 \leq k \leq n} \mathbb{P}(\eta = k)} = 0 \quad \text{and} \quad \limsup_{n \to \infty} \frac{\mathbb{P}(\eta \geq n)}{\mathbb{P}(\eta = n)} = 1, \\
\text{(ii.3)} \quad & \mathbb{E}\eta 1_{\{\eta \geq \delta \}} = o\left(F_{X_1}(x)\right) \quad \text{for any } \delta \in (0, 1).
\end{align*}
\]

Then \( F_{S(n)} \in \mathcal{L} \).

Corollary 5. Let \( X_1, X_2, \ldots \) be i.i.d. real-valued r.v.s with common d.f. \( F_X \in \mathcal{L} \), and let \( \eta \) be a counting r.v. independent of \( \{X_1, X_2, \ldots\} \).

(i) If \( \eta \) has a finite support, then \( F_{S(n)} \in \mathcal{L} \);

(ii) If \( \mathbb{P}(\eta = k) > 0 \) for all \( k \geq 1 \) and conditions (ii.2) and (ii.3) of Theorem 4 hold, then \( F_{S(n)} \in \mathcal{L} \).

Corollaries 4 and 5 follow from Theorems 3 and 4, respectively. The proofs of Theorems 3 and 4 are given in Section 4.

3 Auxiliary results

In this section, we present several results used in the proofs of main theorems. The first lemma is well known (see, e.g., [2, Prop. 2.2.1]).

Lemma 1. For a d.f. \( F \in \mathcal{D} \) and any \( p > J_F^+ \), there exist positive constants \( c_1 \) and \( c_2 \) such that
\[
\frac{F(u)}{F(v)} \leq c_1 \left(\frac{v}{u}\right)^p
\]
for all \( v \geq u \geq c_2 \). In addition, \( u^{-p} = o(F(u)) \) for any \( p > J_F^+ \).

The second lemma, which is an inhomogeneous case of Theorem 3 from [6], was proved in [7].

Lemma 2. Suppose that \( \xi_1, \xi_2, \ldots \) are nonnegative independent r.v.s. Let, for some \( \nu \geq 1 \), \( F_{\xi_\nu} \in \mathcal{D} \) and
\[
\limsup_{x \to \infty} \sup_{n \geq \nu} \frac{1}{n} \sum_{k=1}^{n} \frac{F_{\xi_k}(x)}{F_{\xi_\nu}(x)} < \infty.
\]
Then, for every \( p > J_{F_{\nu}}^+ \), there exists a positive constant \( c_3 \) such that \[
\frac{F_{\xi_1} \cdots F_{\xi_m}(x)}{F_{\xi_m}(x)} \leq c_3 n^{p+1} \quad \text{for all } n \geq \nu, \ x \geq 0.
\]

The third lemma is a combination of Theorem 2.1 and Lemma 4.1 from [3]. Denote \( \xi^+ := \xi_1(\xi \geq 0) \).

**Lemma 3.** Let \( \xi_1, \ldots, \xi_m \) be independent real-valued r.v.s such that \( F_{\xi_k} \in \mathcal{L} \) for all \( k = 1, \ldots, m \). Then \[
P \left( \max_{1 \leq k \leq m} \sum_{i=1}^k \xi_i > x \right) \sim P \left( \sum_{i=1}^m \xi_i > x \right) \sim P \left( \sum_{i=1}^m \xi_i^+ > x \right).
\]

The last auxiliary lemma was proved by Embrechts and Goldie (see inequality (2.12) in [9]).

**Lemma 4.** Let \( F \) and \( G \) be two d.f.s such that \( \overline{F}(x) > 0, \overline{G}(x) > 0, x \in \mathbb{R} \). Then \[
\frac{F \ast G(x-t)}{F \ast G(x)} \leq \max \left\{ \sup_{y \geq \nu} \frac{F(y-t)}{F(y)}, \sup_{y \geq \nu+t} \frac{G(y-t)}{G(y)} \right\}
\]

for all \( x \in \mathbb{R}, \ v \in \mathbb{R}, \) and \( t > 0 \).

## 4 Proofs

In this section, we present proofs of Theorems 1–4.

**Proof of Theorem 1.** (ii) \( \Rightarrow \) (i) For each positive \( x \), we have \[
P(X_\eta > x) = \sum_{n=1}^{\infty} P \left( \bigcup_{k=1}^{n} \{X_k > x\} \right) P(\eta = n). \tag{4.1}
\]

Therefore, from \[
P(X_\eta > x) \geq P(X_\infty > x) P(\eta = \infty) \tag{4.2}
\]

we get \[
\frac{\overline{F}_{X_\eta}(x/2)}{\overline{F}_{X_\eta}(x)} \leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\overline{F}_{X_k}(x/2)}{\overline{F}_{X_\infty}(x)} P(\eta = n).
\]

Condition (2.1) implies that, for some \( c_4 > 0 \), for large \( x \), and for all \( n \geq 1 \), we have \[
\sum_{k=1}^{n} \overline{F}_{X_k}(x) \leq c_4 \phi(n) \overline{F}_{X_\infty}(x). \tag{4.3}
\]

Hence, for \( x \) large enough, we have \[
\frac{\overline{F}_{X_\eta}(x/2)}{\overline{F}_{X_\eta}(x)} \leq c_4 \mathbb{E}(\phi(\eta) 1_{[1, \infty)}(\eta)) \frac{\overline{F}_{X_\infty}(x/2)}{\overline{F}_{X_\eta}(x)},
\]

and the desired implication follows.
(i) ⇒ (ii) Similarly, using (4.1), (4.2), and (4.3), we obtain
\[
\frac{\bar{F}_{X_{\eta}}(x/2)}{F_{X_{\eta}}(x)} \geq \frac{c_4 \sum_{n=1}^{\infty} \varphi(n) \bar{F}_{X_{\eta}}(x) P(\eta = n)}{c_1} \geq \frac{c_4 \mathbb{E} \left[ \eta \right] 1_{[0, \infty)}(\eta)}{c_1} \frac{\bar{F}_{X_{\eta}}(x/2)}{F_{X_{\eta}}(x)}
\]
if \( x \) is sufficiently large. Hence,
\[
\lim_{x \to \infty} \frac{\bar{F}_{X_{\eta}}(x/2)}{F_{X_{\eta}}(x)} \leq \frac{c_4 \mathbb{E} \left[ \eta \right] 1_{[0, \infty)}(\eta)}{c_1} \lim_{x \to \infty} \frac{\bar{F}_{X_{\eta}}(x/2)}{F_{X_{\eta}}(x)}
\]
and the implication follows. The theorem is proved. \( \square \)

**Proof of Theorem 2.** If \( x > 0 \), then
\[
\bar{F}_{S_{\eta}}(x) = \sum_{n=1}^{\infty} P(\max \{ S_1, \ldots, S_n \} > x) P(\eta = n)
\]
\[
\leq \sum_{n=1}^{\infty} P(\max \{ S_1^{(+)}, \ldots, S_n^{(+)} \} > x) P(\eta = n)
\]
\[
= \sum_{n=1}^{\infty} P(S_n^{(+)}) P(\eta = n),
\]
where \( S_k^{(+)} := X_1^{(+)} + \cdots + X_k^{(+)} \). Hence, by Lemma 2 we have
\[
\bar{F}_{S_{\eta}}(x) \leq \left( \sum_{n < \infty} + \sum_{n \geq \infty} \right) P(S_n^{(+)}) P(\eta = n)
\]
\[
\leq P(S_n^{(+)}) P(\eta < \infty) + c_5 \sum_{n \geq \infty} n^{p+1} P(\eta = n) \bar{F}_{X_{\eta}}(x)
\]
\[
\leq c_5 \bar{F}_{X_{\eta}}(x) (\infty^{p+1} + \mathbb{E} n^{p+1})
\]
(4.4)
with some positive constant \( c_5 \). On the other hand, for positive \( x \),
\[
\bar{F}_{S_{\eta}}(x) \geq P(\eta = \infty) P(\max \{ S_1, \ldots, S_{\infty} \} > x)
\]
\[
\geq P(\eta = \infty) P(S_{\infty} > x)
\]
\[
\geq P(\eta = \infty) P \left( X_{\eta} > 2x, X_1 > -\frac{x}{\infty}, \ldots, X_{\eta-1} > -\frac{x}{\infty} \right)
\]
\[
= P(\eta = \infty) \bar{F}_{X_{\eta}}(2x) \prod_{i=1}^{\infty} P \left( X_i > -\frac{x}{\infty} \right).
\]
(4.5)

Inequalities (4.4) and (4.5) imply that
\[
\lim_{x \to \infty} \frac{\bar{F}_{S_{\eta}}(x/2)}{\bar{F}_{S_{\eta}}(x)} \leq c_5 (\infty^{p+1} + \mathbb{E} n^{p+1}) P(\eta = \infty) \lim_{x \to \infty} \frac{\bar{F}_{X_{\eta}}(x/2)}{\bar{F}_{X_{\eta}}(2x)} \frac{1}{\lim inf_{x \to \infty} \prod_{i=1}^{\infty} P( X_i > -x/\infty )}
\]
(4.6)
\[
= c_5 (\infty^{p+1} + \mathbb{E} n^{p+1}) P(\eta = \infty) \lim_{x \to \infty} \frac{\bar{F}_{X_{\eta}}(x/2)}{\bar{F}_{X_{\eta}}(2x)} < \infty. \quad \square
\]
Proof of Theorem 3. (i) To prove that $F_{X_0}(x) \in \mathcal{L}$, we have to show that
\[
\limsup_{x \to \infty} \frac{F_{X_0}(x) - 1}{F_{X_0}(x)} \leq 1.
\] (4.6)

Using equality (4.1) we have that, for $x > 0$,
\[
F_{X_0}(x) \leq \sum_{n < \infty} F_{X_n}(x) \mathbb{P}(n \leq \eta < \infty) + \sum_{n \geq \infty} \mathbb{P} \left( \bigcup_{k=1}^{n} \{ X_k > x \} \right) \mathbb{P}(\eta = n)
\]
\[
\leq \sum_{n < \infty} F_{X_n}(x) \mathbb{P}(n \leq \eta < \infty)
\]
\[
+ F_{X_{n\infty}}(x) \sup_{n \geq \infty} \left\{ \frac{1}{\varphi(n)} \sum_{k=1}^{n} \frac{F_{X_k}(x)}{F_{X_{n\infty}}(x)} \right\} \sum_{l \geq \infty} \varphi(l) \mathbb{P}(\eta = l)
\]
\[
=: s_1(x) + s_2(x).
\] (4.7)

For the lower bound, take any integer $K > \infty$ and write, using the Bonferroni inequality,
\[
F_{X_0}(x) \geq \sum_{n \in \kappa} \mathbb{P} \left( \bigcup_{k=1}^{n} \{ X_k > x \} \right) \mathbb{P}(\eta = n) + \sum_{n \geq \kappa} \mathbb{P} \left( \bigcup_{k=1}^{n} \{ X_k > x \} \right) \mathbb{P}(\eta = n)
\]
\[
\geq \sum_{n < \kappa} \left( \sum_{k=1}^{n} F_{X_k}(x) - \sum_{k=1}^{n} \sum_{j=1}^{K} F_{X_k}(x) F_{X_j}(x) \right) \mathbb{P}(\eta = n)
\]
\[
+ \left( 1 - \sum_{j=1}^{K} F_{X_j}(x) \right) \sum_{n \in \kappa} \mathbb{P}(\eta = n) \sum_{k=1}^{n} F_{X_k}(x)
\]
\[
\geq \left( 1 - \sum_{j=1}^{K} F_{X_j}(x) \right) \sum_{n \in \kappa} F_{X_n}(x) \mathbb{P}(n \leq \eta < \infty)
\]
\[
+ \left( 1 - \sum_{j=1}^{K} F_{X_j}(x) \right) F_{X_{\infty}}(x) \inf_{n \in \kappa} \left\{ \frac{1}{\varphi(n)} \sum_{k=1}^{n} \frac{F_{X_k}(x)}{F_{X_{\infty}}(x)} \right\} \sum_{l \geq \infty} \varphi(l) \mathbb{P}(\eta = l)
\]
\[
=: s_3(x) + s_4(x)
\] (4.8)

if $x$ is large enough. Consider now two cases: (a) $\mathbb{P}(\eta < \infty) > 0$; (b) $\mathbb{P}(\eta < \infty) = 0$. In case (a), by (4.7), (4.8), and the inequality
\[
\frac{a_1 + \cdots + a_m}{b_1 + \cdots + b_m} \leq \max \left\{ \frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m} \right\} \quad \text{(for } a_i \geq 0, b_i > 0) \]
(4.9)

we have
\[
\frac{F_{X_0}(x) - 1}{F_{X_0}(x)} \leq \max \left\{ \frac{s_1(x) - 1}{s_3(x)}, \frac{s_2(x) - 1}{s_4(x)} \right\}.
\]
From the conditions of the theorem and (4.9) we get that

\[
\limsup_{x \to \infty} \frac{s_1(x) - 1}{s_3(x)} \leq \limsup_{x \to \infty} \frac{\sum_{n < \kappa} \mathcal{F}_{X_n}(x - 1) \mathbb{P}(n \leq \eta < \kappa)}{\sum_{n < \kappa} \mathcal{F}_{X_n}(x) \mathbb{P}(n \leq \eta < \kappa)} \\
\leq \max_{n < \kappa} \limsup_{x \to \infty} \frac{\mathcal{F}_{X_n}(x - 1)}{\mathcal{F}_{X_n}(x)} = 1,
\]

(4.10)

\[
\limsup_{x \to \infty} \frac{s_2(x) - 1}{s_4(x)} \leq \limsup_{x \to \infty} \frac{\mathcal{F}_{X_n}(x - 1)}{\mathcal{F}_{X_n}(x)} \times \limsup_{x \to \infty} \sup_{n \geq \kappa} \left\{ \frac{1}{\varphi(n)} \sum_{k=1}^{n} \frac{\mathcal{F}_{X_k}(x - 1)}{\mathcal{F}_{X_k}(x)} \right\} \mathbb{E}[\varphi(\eta)1_{\{\eta \geq \kappa\}}] \\
\times \liminf_{x \to \infty} \inf_{n \geq \kappa} \left\{ \frac{1}{\varphi(n)} \sum_{k=1}^{n} \frac{\mathcal{F}_{X_k}(x)}{\mathcal{F}_{X_k}(x)} \right\} \mathbb{E}[\varphi(\eta)1_{\{\kappa \leq \eta \leq \kappa\}}] = \frac{\mathbb{E}[\varphi(\eta)1_{\{\eta \geq \kappa\}}]}{\mathbb{E}[\varphi(\eta)1_{\{\kappa \leq \eta \leq \kappa\}}]},
\]

(4.11)

Hence,

\[
\limsup_{x \to \infty} \frac{\mathcal{F}_{X_n}(x - 1)}{\mathcal{F}_{X_n}(x)} \leq \max \left\{ 1, \frac{\mathbb{E}[\varphi(\eta)1_{\{\eta \geq \kappa\}}]}{\mathbb{E}[\varphi(\eta)1_{\{\kappa \leq \eta \leq \kappa\}}]} \right\}
\]

for every \( K > \kappa \). Letting \( K \) tend to infinity, we get inequality (4.6). Case (b) is obvious.

(ii) First, we show that, under the conditions of the theorem, \( F_{X(n)} \in \mathcal{L} \Rightarrow F_{X(n)} \in \mathcal{L} \).

From (4.7) we have that, for \( x > 0 \),

\[
\mathcal{F}_{X_n}(x) \sup_{n \geq \kappa} \left\{ \frac{1}{\varphi(n)} \sum_{k=1}^{n} \frac{\mathcal{F}_{X_k}(x)}{\mathcal{F}_{X_k}(x)} \right\} \mathbb{E}[\varphi(\eta)1_{\{\eta \geq \kappa\}}] \geq \mathcal{F}_{X_n}(x) - \sum_{n < \kappa} \mathcal{F}_{X_n}(x) \mathbb{P}(n \leq \eta < \kappa).
\]

Similarly, estimate (4.8) implies that, for \( K > \kappa \) and sufficiently large \( x \),

\[
\mathcal{F}_{X_n}(x) \inf_{n \geq \kappa} \left\{ \frac{1}{\varphi(n)} \sum_{k=1}^{n} \frac{\mathcal{F}_{X_k}(x)}{\mathcal{F}_{X_k}(x)} \right\} \mathbb{E}[\varphi(\eta)1_{\{\kappa \leq \eta \leq \kappa\}}] \leq \frac{\mathcal{F}_{X_n}(x)}{1 - \sum_{j=1}^{K} \mathcal{F}_{X_n}(x)}.
\]

The last two estimates give

\[
\frac{\mathcal{F}_{X_n}(x - 1)}{\mathcal{F}_{X_n}(x)} \leq \frac{\mathcal{F}_{X(n)}(x - 1)/(1 - \sum_{j=1}^{K} \mathcal{F}_{X_n}(x - 1))}{\mathcal{F}_{X(n)}(x) - \sum_{n < \kappa} \mathcal{F}_{X_n}(x) \mathbb{P}(n \leq \eta < \kappa)} \times \frac{\sup_{n \geq \kappa} \left\{ \frac{1}{\varphi(n)} \sum_{k=1}^{n} \frac{\mathcal{F}_{X_k}(x)}{\mathcal{F}_{X_k}(x)} \right\} \mathbb{E}[\varphi(\eta)1_{\{\eta \geq \kappa\}}]}{\inf_{n \geq \kappa} \left\{ \frac{1}{\varphi(n)} \sum_{k=1}^{n} \frac{\mathcal{F}_{X_k}(x - 1)}{\mathcal{F}_{X_k}(x - 1)} \right\} \mathbb{E}[\varphi(\eta)1_{\{\kappa \leq \eta \leq \kappa\}}]}.
\]
By the assumption of the theorem and (4.2) we have that $\bar{F}_{X_n}(x) = o(\bar{F}_{X_{(n)}}(x))$ for all $n \to \infty$. Thus,

$$\limsup_{x \to \infty} \frac{\bar{F}_{X_n}(x-1)}{\bar{F}_{X_n}(x)} \leq \limsup_{x \to \infty} \frac{\bar{F}_{X_{(n)}}(x-1)}{\bar{F}_{X_{(n)}}(x)} \times \frac{1}{1 - \limsup_{x \to \infty} \sum_{n \leq \infty} \frac{\bar{F}_{X_n}(x)}{\bar{F}_{X_{(n)}}(x)} P(n \leq \eta < \infty)} \frac{E\varphi(\eta)1_{\{\eta \geq \infty\}}}{E\varphi(\eta)1_{\{\eta \leq \infty \leq K\}}}$$

for every $K > \infty$. Letting $K \to \infty$, we get

$$\limsup_{x \to \infty} \frac{\bar{F}_{X_n}(x-1)}{\bar{F}_{X_n}(x)} \leq 1,$$

proving that $F_{X_n} \in \mathcal{L}$. Next, we show that $F_{X_n} \in \mathcal{L} \Rightarrow F_{X_{(n)}} \in \mathcal{L}$. As in (4.7), write $\bar{F}_{X_{(n)}}(x) \leq s_1(x) + s_2(x)$. Then, for large $x$,

$$\frac{\bar{F}_{X_{(n)}}(x-1)}{\bar{F}_{X_{(n)}}(x)} \leq \frac{s_1(x-1)}{\bar{F}_{X_{(n)}}(x)} + \frac{s_2(x-1)}{\bar{F}_{X_{(n)}}(x)} \leq \frac{s_1(x-1)}{s_3'(x)} + \frac{s_2(x-1)}{s_4'(x)},$$

where $s_3'(x) := \bar{F}_{X_n}(x)P(\eta = \infty)$; see (4.2) and (4.8). Here,

$$\limsup_{x \to \infty} \frac{s_1(x-1)}{s_3'(x)} = \limsup_{x \to \infty} \sum_{n \leq \infty} \frac{\bar{F}_{X_n}(x-1)}{\bar{F}_{X_n}(x)} P(n \leq \eta < \infty) = 0$$

by condition (2.3) and the assumption $F_{X_n} \in \mathcal{L}$, and

$$\limsup_{x \to \infty} \frac{s_2(x-1)}{s_4'(x)} \leq \frac{E\varphi(\eta)1_{\{\eta \geq \infty\}}}{E\varphi(\eta)1_{\{\eta \leq \infty \leq K\}}}$$

by (4.11). Thus,

$$\limsup_{x \to \infty} \frac{\bar{F}_{X_{(n)}}(x-1)}{\bar{F}_{X_{(n)}}(x)} \leq \frac{E\varphi(\eta)1_{\{\eta \geq \infty\}}}{E\varphi(\eta)1_{\{\eta \leq \infty \leq K\}}}.$$

Letting $K \to \infty$, we get

$$\limsup_{x \to \infty} \frac{\bar{F}_{X_{(n)}}(x-1)}{\bar{F}_{X_{(n)}}(x)} \leq 1. \quad \Box$$

**Proof of Theorem 4.** (i) If $\eta$ has finite support, supp$(\eta) \subset \{0, 1, \ldots, m\}$, then

$$\bar{F}_{S_{(n)}}(x) = \sum_{n=1}^{m} P\left( \max_{1 \leq k \leq m} \sum_{i=1}^{k} X_i > x \right) P(\eta = n).$$
Thus, by inequality (4.9) and Lemma 3,
\[
\limsup_{x \to \infty} \frac{F_{S(n)}(x-1)}{F_{S(n)}(x)} \leq \max_{1 \leq n \leq K} \left\{ \limsup_{x \to \infty} \frac{p(n \leq k \leq n) \sum_{i=1}^{k} X_i > x - 1}{p(n \leq k \leq n) \sum_{i=1}^{k} X_i > x} \right\} = 1.
\]

(ii) For any \( K > 1 \) and \( x \geq K^2 \), we have
\[
F_{S(n)}(x) = \left( \sum_{1 \leq n \leq K} + \sum_{K < n \leq x/K} + \sum_{n > x/K} \right) P(S(n) > x)P(\eta = n)
= p_1(x) + p_2(x) + p_3(x).
\]
Write now \( p_3(x) \leq P(\eta > x/K) =: p_3'(x) \) and
\[
p_2(x) \leq \sum_{K < n \leq x/K} \sum_{k=1}^{n} P(S_k > x)P(\eta = n)
= P(k < \eta \leq x/K) \sum_{1 \leq k \leq K} P(S_k > x) + P(S_k > x)P(k \leq \eta \leq x/K)
\leq P(\eta > K) \sum_{1 \leq k \leq K} P(S_k > x) + \sum_{K < k \leq x/L} P(S_k > x)P(k \leq \eta \leq x/K)
+ \sum_{x/L < k \leq x/K} P(S_k > x)P(k \leq \eta \leq x/K)
= p_1(x) + p_2(x) + p_3(x),
\]
where \( L > K \) and \( x \geq L^2 \). We also need the following lower bound:
\[
\overline{F}_{S(n)}(x) \geq p_1(x) + p_2(x)
\geq \sum_{1 \leq n \leq K} P(S(n) > x)P(\eta = n) + \sum_{K < n \leq x/L} P(S_n > x)P(\eta = n)
= p_1(x) + p_2'(x).
\]
Now we can write, using inequality (4.9),
\[
\overline{F}_{S(n)}(x-1) \leq \max_{1 \leq n \leq K} \left\{ \frac{p(n \leq x - 1)}{p(n \leq x)} \right\}
+ \frac{p_2(x - 1)}{p_2(x)} + \frac{p_3(x - 1)}{p_3(x)} \quad (4.12)
\]
if \( K > 1, L > K \), and \( x - 1 \geq L^2 \).

By Lemma 3, \( F_{S(n)} \in \mathcal{L} \) for each \( n \). Hence, by (4.9),
\[
\limsup_{x \to \infty} \frac{p(n \leq x - 1)}{p(n \leq x)} \leq \max_{1 \leq n \leq K} \left\{ \limsup_{x \to \infty} \frac{\overline{F}_{S(n)}(x-1)}{\overline{F}_{S(n)}(x)} \right\} = 1. \quad (4.13)
\]
Next, we consider the bound for \( p_{22}(x - 1)/p'_2(x) \). Let \( \epsilon \) be an arbitrary positive number, and let w.l.o.g. \( K > 1 \) and \( L > 1 \) be integers. By Lemma 3 the d.f. \( F_{S_K}(x) \) is long tailed. Hence,

\[
\frac{F_{S_K}(x - 1)}{F_{S_K}(x)} \leq 1 + \epsilon
\]  (4.14)

for \( x \geq M/2 \), where \( M = M(K) \) is sufficiently large. Also, condition (ii.1) of the theorem implies

\[
\frac{F_{X_k}(x - 1)}{F_{X_k}(x)} \leq 1 + \epsilon
\]  (4.15)

for all \( x \geq L/2 \), sufficiently large \( L > M \), and all \( k \geq 1 \).

By Lemma 4,

\[
\frac{F_{S_{K+1}}(x - 1)}{F_{S_{K+1}}(x)} \leq \max \left\{ \sup_{y \geq L/2} \frac{F_{X_{K+2}}(y - 1)}{F_{X_{K+2}}(y)}, \sup_{y \geq x - L/2 + 1} \frac{F_{S_K}(y - 1)}{F_{S_K}(y)} \right\},
\]

implying, due to estimates (4.14) and (4.15),

\[
\sup_{x \geq L} \frac{F_{S_{K+1}}(x - 1)}{F_{S_{K+1}}(x)} \leq 1 + \epsilon.
\]  (4.16)

Next, apply Lemma 4 again for the sum \( S_{K+2} = S_{K+1} + X_{K+2} \) with \( v = x/2 + 1/2 \):

\[
\frac{F_{S_{K+2}}(x - 1)}{F_{S_{K+2}}(x)} \leq \max \left\{ \sup_{y \geq x/2 + 1/2} \frac{F_{S_{K+1}}(y - 1)}{F_{S_{K+1}}(y)}, \sup_{y \geq x/2 + 1/2} \frac{F_{X_{K+2}}(y - 1)}{F_{X_{K+2}}(y)} \right\}.
\]

If \( x \geq 2(L - 1) + 1 \), then \( x/2 + 1/2 \geq L \), so that

\[
\sup_{x \geq 2(L - 1) + 1} \frac{F_{S_{K+2}}(x - 1)}{F_{S_{K+2}}(x)} \leq 1 + \epsilon
\]  (4.17)

due to estimates (4.15) and (4.16). Applying Lemma 4 once again with \( v = 2x/3 + 1/3 \), we get

\[
\frac{F_{S_{K+3}}(x - 1)}{F_{S_{K+3}}(x)} \leq \max \left\{ \sup_{y \geq 2x/3 + 1/3} \frac{F_{S_{K+2}}(y - 1)}{F_{S_{K+2}}(y)}, \sup_{y \geq x/3 + 2/3} \frac{F_{X_{K+3}}(y - 1)}{F_{X_{K+3}}(y)} \right\}.
\]

If \( x \geq 3(L - 1) + 1 \), then \( 2x/3 + 1/3 \geq 2(L - 1) + 1 \) and \( x/3 + 2/3 \geq L \). Hence, the last inequality, together with estimates (4.15) and (4.17), implies

\[
\sup_{x \geq 3(L - 1) + 1} \frac{F_{S_{K+3}}(x - 1)}{F_{S_{K+3}}(x)} \leq 1 + \epsilon.
\]

Continuing, we obtain that

\[
\sup_{x \geq k(L - 1) + 1} \frac{F_{S_{K+k}}(x - 1)}{F_{S_{K+k}}(x)} \leq 1 + \epsilon
\]  (4.18)

for each \( k \geq 1 \).
Since \( |(x - 1)/L| \geq L > K \) for \( x - 1 \geq L^2 \), the set of indices \( k \) in \( \sum_{k < L \leq (x - 1)/L} \) is nonempty, and we have by (4.18) and (4.9) that
\[
\frac{p_{22}(x - 1)}{p_2(x)} \leq (1 + \varepsilon) \frac{\sum_{k < L \leq (x - 1)/L} \mathbb{P}(S_k > x) \mathbb{P}(\eta \geq k)}{\sum_{k < L \leq (x - 1)/L} \mathbb{P}(S_k > x) \mathbb{P}(\eta = k)} \leq (1 + \varepsilon) \sup_{k > K} \frac{\mathbb{P}(\eta \geq k)}{\mathbb{P}(\eta = k)}
\] (4.19)
for \( x \geq L^2 + 1, L > \max\{K, M\} \).

Next, we estimate \( p_{23}(x - 1)/F_{S_1}(x) \). Write
\[
\frac{p_{23}(x - 1)}{F_{S_1}(x)} \leq \frac{\sum_{k > (x - 1)/L} \mathbb{P}(\eta \geq k)}{\mathbb{P}(\eta = 1)F_{X_1}(x)} \leq \frac{1}{\mathbb{P}(\eta = 1)} \sum_{k > (x - 1)/L} \mathbb{P}(\eta \geq (x - 1)/K) \mathbb{E}\eta 1_{\{\eta \geq (x - 1)/L\}} / F_{X_1}(x).
\]
Thus, condition (ii.3) and the definition of class \( \mathcal{L} \) imply
\[
\limsup_{x \to \infty} \frac{p_{23}(x - 1)}{F_{S_1}(x)} = 0.
\] (4.20)
Similarly,
\[
\limsup_{x \to \infty} \frac{p_2'(x - 1)}{F_{S_1}(x)} \leq \limsup_{x \to \infty} \frac{\mathbb{P}(\eta > (x - 1)/K)}{\mathbb{P}(\eta = 1)F_{X_1}(x)} \leq \frac{1}{\mathbb{P}(\eta = 1)} \limsup_{x \to \infty} \frac{K}{x - 1} \sum_{k > (x - 1)/L} \mathbb{P}(\eta \geq (x - 1)/K) \mathbb{E}\eta 1_{\{\eta \geq (x - 1)/L\}} / F_{X_1}(x) = 0.
\] (4.21)
Finally, for the term \( p_{21}(x - 1)/F_{S_1}(x) \), by Lemma 3 we have
\[
\limsup_{x \to \infty} \frac{p_{21}(x - 1)}{F_{S_1}(x)} \leq \mathbb{P}(\eta > K) \limsup_{x \to \infty} \frac{\sum_{1 \leq k \leq K} \mathbb{P}(S_k(x - 1))}{\sum_{1 \leq k \leq K} \mathbb{P}(S_k(x)) \mathbb{P}(\eta = k)} \leq \frac{\mathbb{P}(\eta > K)}{\min_{1 \leq j \leq K} \mathbb{P}(\eta = j)} \max_{1 \leq k \leq K} \left\{ \limsup_{x \to \infty} \frac{\mathbb{P}(S_k(x - 1))}{\mathbb{P}(S_k(x))} \right\} \leq \frac{\mathbb{P}(\eta > K)}{\min_{1 \leq j \leq K} \mathbb{P}(\eta = j)}.
\] (4.22)
Relations (4.12), (4.13), (4.19), (4.20), (4.21), and (4.22) imply that
\[
\limsup_{x \to \infty} \frac{\mathbb{P}(S_k(x - 1))}{\mathbb{P}(S_k(x))} \leq \max \left\{ 1, (1 + \varepsilon) \sup_{k > K} \frac{\mathbb{P}(\eta \geq k)}{\mathbb{P}(\eta = k)} \right\} + \frac{\mathbb{P}(\eta > K)}{\min_{1 \leq j \leq K} \mathbb{P}(\eta = j)}
\]
for all \( \varepsilon > 0 \) and all \( K > 1 \). Hence, letting \( K \) tend to infinity, by condition (ii.2) of the theorem we get
\[
\limsup_{x \to \infty} \frac{\mathbb{P}(S_k(x - 1))}{\mathbb{P}(S_k(x))} \leq 1 + \varepsilon
\]
for arbitrary \( \varepsilon > 0 \). This proves that \( F_{S_1} \in \mathcal{L} \). \( \square \)

5 Examples

Example 1. Let \(X_1, X_2, \ldots\) be independent r.v.s such that \(X_k\) is exponentially distributed for odd \(k\), that is,
\[
F_{X_k}(x) = e^{-x}, \quad x \geq 0, \ k \in \{1, 3, 5, \ldots\},
\]
and \(X_2, X_4, \ldots\) are distributed according to the Peter and Paul law, that is,
\[
F_{X_k}(x) = \sum_{l \geq 1: 2^l \geq x} \frac{1}{2^l} = 2^{-\lfloor \log x / \log 2 \rfloor}, \quad x \geq 1, \ k \in \{2, 4, \ldots\}.
\]
Assume that \(\eta\) is a counting r.v., independent of \(\{X_1, X_2, \ldots\}\), with probabilities
\[
P(\eta = n) = \frac{1}{\zeta(5)} \frac{1}{(n+1)^5}, \ n \in \{0, 1, \ldots\},
\]
where \(\zeta(s)\) denotes the Riemann zeta function.

It is easy to see that \(F_{X_k} \in \mathcal{L}\) (although \(F_{X_2} \notin \mathcal{L}\); see [12]), \(J_{F_{X_k}}^+ = 1\), \(\lim_{x \to \infty} \sup_{n \geq 2} n^{-1} \times \sum_{k=1}^n F_{X_k}(x)/F_{X_1}(x) < 1\), and \(E\eta^3 < \infty\). Theorems 1 and 2 imply that \(F_{X_{(n)}} \in \mathcal{D}\) and \(F_{S_{(n)}} \in \mathcal{D}\).

Example 2. Let \(X_1, X_2, \ldots\) be independent r.v.s such that
\[
F_{X_k}(x) = \frac{1}{k} e^{-\sqrt{x}}, \quad x \geq 0, \ k \in \{1, 2, \ldots\},
\]
and let \(\eta\) be a Poisson r.v. with parameter \(\lambda > 0\), independent of \(\{X_1, X_2, \ldots\}\). Then we have \(F_{X_k} \in \mathcal{L}\) for \(k \geq 1\) and:
\[
\begin{align*}
E\varphi(\eta)1_{\{\eta \geq 1\}} & \leq E(1 + \log \eta)1_{\{\eta \geq 1\}} < \infty, \\
\sum_{k=1}^n \frac{F_{X_k}(x)}{F_{X_1}(x)} &= \sum_{k=1}^n \frac{1}{k} \equiv \varphi(n) \quad \text{(condition (2.2) of Theorem 3(i))}, \\
\sup_{k \geq 1} \frac{F_{X_k}(x-1)}{F_{X_k}(x)} &= e^{\sqrt{x} - \sqrt{x-1}} \xrightarrow{x \to \infty} 1 \\
P(\eta > n) &= \sum_{k>n} \frac{\lambda^k/k!}{\lambda^n/n!} \leq \lambda e^{\lambda} \frac{1}{n+1} \xrightarrow{n \to \infty} 0 \quad \text{(assumption (ii.2) of Theorem 4)}, \\
\frac{P(\eta \geq n)}{P(\eta = n)} &= 1 + \frac{\lambda}{n+1} + \frac{\lambda^2}{(n+1)(n+2)} + \cdots \xrightarrow{n \to \infty} 1 \quad \text{(assumption (ii.2) of Theorem 4)}, \\
E\eta 1_{\{\eta \geq \delta x\}} & \leq e^{-\delta x} E\eta e^{\eta} = o(e^{-\sqrt{x}}) \quad \text{for all} \ \delta > 0 \quad \text{(assumption (ii.3) of Theorem 4)}.
\end{align*}
\]
Theorems 3 and 4 imply that \(F_{X_{(n)}} \in \mathcal{L}\), \(F_{S_{(n)}} \in \mathcal{L}\).

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References


