Closure property and maximum of randomly weighted sums with heavy-tailed increments

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Abstract In this paper, we consider the randomly weighted sum \( S_2^\Theta = \Theta_1 X_1 + \Theta_2 X_2 \), where the two primary random summands \( X_1 \) and \( X_2 \) are real-valued and dependent with long or dominatedly varying tails, and the random weights \( \Theta_1 \) and \( \Theta_2 \) are positive, with values in \([a,b], 0 < a \leq b < \infty\), and arbitrarily dependent, but independent of \( X_1 \) and \( X_2 \). Under some dependence structure between \( X_1 \) and \( X_2 \), we show that \( S_2^\Theta \) has a long or dominatedly varying tail as well, and obtain the corresponding (weak) equivalence results between the tails of \( S_2^\Theta \) and \( M_2^\Theta = \max\{\Theta_1 X_1, \Theta_1 X_1 + \Theta_2 X_2\} \). As corollaries, we establish the asymptotic (weak) equivalence formulas for the tail probabilities of randomly weighted sums of even number of long-tailed or dominatedly varying-tailed random variables.

Keywords: Randomly weighted sum; Long tail; Dominatedly varying tail; Dependence.

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1 Introduction

Throughout this paper, all limit relationships hold for \( x \) tending to \( \infty \) unless stated otherwise. For two positive functions \( u(x) \) and \( v(x) \), we write \( u(x) \sim v(x) \) if \( \lim u(x)/v(x) = 1 \); write \( u(x) \lesssim v(x) \) if \( \limsup u(x)/v(x) \leq 1 \); write \( u(x) \gtrsim v(x) \) if \( \liminf u(x)/v(x) \geq 1 \). For a real number \( x \), write \( x^+ = \max\{x,0\} \). The indicator function of an event \( A \) is denoted by \( 1_A \). For any distribution \( V \), we assume that its tail distribution \( \overline{V}(x) = 1 - V(x) > 0 \) for all \( x \). The Lebesgue–Stieltjes integrals \( \int_{[u,v]} f \) and \( \int_{[u,v]}^\circ \) where \( -\infty \leq u < v < \infty \), are denoted by \( \int_u^v, \int_u^v \), respectively.

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A distribution $V$ is called long-tailed, denoted by $V \in \mathcal{L}$, if $\nabla (x + y) \sim \nabla (x)$ holds for every fixed $y$, and is called dominatedly varying-tailed, denoted by $V \in \mathcal{D}$, if $\limsup_{y \to \infty} \nabla (x+y)/\nabla (x) < \infty$ for any $y \in (0,1)$.

Let $X_1$ and $X_2$ be two dependent real-valued random variables (r.v.s) with distributions $F_1$ and $F_2$, respectively; let $\Theta_1, \Theta_2$ be two arbitrarily dependent r.v.s., independent of $X_1$ and $X_2$, and there exist some constants $0 < a \leq b < \infty$ such that $P(a \leq \Theta_k \leq b) = 1$, $k = 1, 2$. Denote the randomly weighted sum and its maximum, respectively, by

$$S^\Theta_2 := \Theta_1 X_1 + \Theta_2 X_2 \quad \text{and} \quad M^\Theta_2 := \max \{ \Theta_1 X_1, \Theta_1 X_1 + \Theta_2 X_2 \}. \quad (1.1)$$

In this paper, we focus on two main questions. Firstly, we are interested in the closure property of the sum $S^\Theta_2$, which states that, given $F_1$ and $F_2$ from some heavy-tailed distribution class, the distribution function (d.f.) of the sum $S^\Theta_2$ belongs to the same class. Secondly, we are interested in the (weak) equivalence result for the tail probability of $M^\Theta_2$. In Theorem 2.1 below we prove the aforementioned closure property and corresponding equivalence result

$$P(M^\Theta_2 > x) \sim P(S^\Theta_2 > x) \quad (1.2)$$
in the case of long-tailed distributions and dependence structure between $X_1$ and $X_2$ given by Assumption A below. In Theorem 2.2, we prove analogous result in the case of dominatedly varying-tailed distributions.

Historically, the above questions have been considered mostly for $\Theta_1 = \Theta_2 = 1$ and independent variables case, see Embrechts and Goldie (1980), Leslie (1989), Sgibnev (1996), Ng et al. (2002), Tang and Tsitsiashvili (2003a), Cai and Tang (2004), Geluk and Ng (2006), Foss et al. (2009) among others. In particular, Embrechts and Goldie (1980) proved the convolution closure of the long-tailed distributions, whereas the closure of the dominatedly varying-tailed distributions was proved in Cai and Tang (2004), Watanabe and Yamamuro (2010). In fact, in the case when $F_k \in \mathcal{D}$, $k = 1, 2$, are supported on $[0,\infty)$, the closure property is valid for any (not necessarily independent) r.v.s $X_1$ and $X_2$ (see the proof of Proposition 2.1 in Cai and Tang (2004)). Note that, for independent variables, the closure property is linked to the so-called max-sum equivalence relation $P(X_1 + X_2 > x) \sim P(\max\{X_1, X_2\} > x)$. In the case of subexponential distributions on $[0,\infty)$, Embrechts and Goldie (1980) showed that these two properties are equivalent. Sgibnev (1996) related the tail $P(M^\Theta_2 > x)$ to the probability $P(X_1 > x)$ in the case where $F_1 \equiv F_2$ belongs to the generalized subexponential class $\mathcal{S}(\gamma)$, Ng et al. (2002) gave a generalization to the subexponential class $\mathcal{S} = \mathcal{S}(0)$, whereas Geluk and Ng (2006) extended this relation to the class of long-tailed distributions.

The mentioned above closure property and tail-equivalence (1.2) can be extended by induction to the case of $n > 2$ randomly weighted variables $\Theta_1 X_1, \ldots, \Theta_n X_n$, when $X_1, \ldots, X_n$ are independent r.v.s and $\Theta_1, \ldots, \Theta_n$ are nonnegative bounded r.v.s., independent of $X_k$’s, see Tang and Tsitsiashvili (2003b), Chen and Yuen (2009), Gao and Wang (2010), Tang et al. (2011), Chen et al. (2011), Yang et al. (2012), etc. The motivation for these studies come mainly from the insurance risk theory, studying the so-called discrete-time stochastic risk model with insurance risk and financial risks, introduced by Tang and Tsitsiashvili (2003a). In such a model, each $X_k$ is interpreted as the net loss (the total claim amount minus the total premium income) of an insurance company during period $k$, $\Theta_k$ is the corresponding stochastic discount factor to the origin, $S^\Theta_n := \sum_{k=1}^n \Theta_k X_k$ and $M^\Theta_n := \max_{1 \leq k \leq n} S^\Theta_k$ represent the stochastic present value of the aggregate net losses and the maximal net loss during the first $n$ periods.

Recently, Chen et al. (2011) proved that in the case where the independent but not necessarily identically distributed r.v.s $X_1, \ldots, X_n$ are long-tailed, and there exist some constants $0 < a \leq b < \infty$ such that $P(a \leq \Theta_k \leq b) = 1$ for each $1 \leq k \leq n$, it holds that

$$P(M^\Theta_n > x) \sim P(S^\Theta_n > x) \sim P(S^\Theta_n^+ > x), \quad (1.3)$$
where $S_{n}^{\Theta+} := \sum_{k=1}^{n} \Theta_k X_k^+$. Relations (1.3) are not only of theoretical interest but also have some practical implications. For instance, if we need to calculate the distribution tail of maximum $M_n^{\Theta}$ for large $x$, we can reduce the calculation to that of sum $S_{n}^{\Theta}$ or $S_{n}^{\Theta+}$.

In this paper, we consider the case when the r.v.s $X_1$ and $X_2$ in (1.1) are such that the d.f.s $F_k \in \mathcal{L}$ or $F_k \in \mathcal{D}$, $k = 1, 2$, together with the following dependence structure between $X_1$ and $X_2$:

$$
\begin{align*}
\mathbb{P}(X_2 > x | X_1 = y) & \sim h_1(y) F_2(x), \\
\mathbb{P}(X_1 > x | X_2 = y) & \sim h_2(y) F_1(x),
\end{align*}
$$

(1.4)

uniformly for all $y \in \mathbb{R}$, where $h_k(\cdot) : \mathbb{R} \to \mathbb{R}_+ := (0, \infty)$, $k = 1, 2$ are measurable functions and the uniformity is understood as

$$
\lim_{x \to \infty} \sup_{y \in \mathbb{R}} \left| \frac{\mathbb{P}(X_i > x | X_j = y)}{h_j(y) F_i(x)} - 1 \right| = 0, \quad i, j = 1, 2, \quad i \neq j.
$$

When $y$ is not a possible value of $X_j$, the conditional probability in (1.4) is understood as unconditional and therefore $h_j(y) = 1$ for such $y$. Clearly, the uniformity in (1.4) implies $\mathbb{E}h_1(X_1) = \mathbb{E}h_2(X_2) = 1$. If $X_1$ and $X_2$ are independent, then $h_1(y) = h_2(y) \equiv 1$. Dependence structure in (1.4) was proposed by Asimit and Badescu (2010). Some examples of distributions $\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2) = C(F_1(x_1), F_2(x_2))$ (with continuous $F_1$ and $F_2$) and corresponding functions $h_1(\cdot)$, $h_2(\cdot)$ satisfying (1.4) can be found in Asimit and Badescu (2010) and Li et al. (2010). Relations (1.4) are easy to verify for some well-known bivariate copulas, allowing both positive and negative dependence, and are convenient tools when dealing with the tail behavior of the sum of two dependent r.v.s. In the case of Ali-Mikhail-Haq copula of the form

$$
C(u, v) = \frac{uv}{1 - r(1-u)(1-v)}, \quad r \in (-1, 1),
$$

and Farlie-Gumbel-Morgenstern copula of the form

$$
C(u, v) = uv + ruv(1-u)(1-v), \quad r \in (-1, 1),
$$

we have $h_k(y) = 1 + r(1 - 2F_k(y))$, $k = 1, 2$; in the case of Frank copula of the form

$$
C(u, v) = -\frac{1}{r} \log \left( 1 + \frac{(e^{-ru} - 1)(e^{-rv} - 1)}{e^{-r} - 1} \right), \quad r \neq 0,
$$

we have $h_k(y) = r(e^r - 1)^{-1} \exp \{ r(1 - F_k(y)) \}$, $k = 1, 2$.

The rest of the paper is organized as follows. Section 2 presents our two main results and the corollaries. Their proofs are given in Section 3.

2 Main results

In this section, we formulate two main results and corollaries. Our generic assumption on the underlying r.v.s is the following.

Assumption A. $X_1$ and $X_2$ are real-valued r.v.s with distributions $F_1$ and $F_2$, respectively, and satisfying relation (1.4); $\Theta_1$ and $\Theta_2$ are positive and arbitrarily dependent r.v.s, which are independent of $X_1$ and $X_2$ and such that $\mathbb{P}(a \leq \Theta_k \leq b) = 1$, $k = 1, 2$, with some positive constants $0 < a \leq b < \infty$.

Our first result deals with a closure property and equivalence relation (1.2) for the long-tailed r.v.s.
**Theorem 2.1.** Let Assumption A be satisfied. If $F_k \in \mathcal{L}$, $k = 1, 2$, then the distribution of $S^\Theta_2$ is in $\mathcal{L}$ and, moreover,

$$P(M^\Theta_2 > x) \sim P(S^\Theta_2 > x) \sim P(S^{\Theta+}_2 > x). \quad (2.1)$$

As a corollary to Theorem 2.1, we can establish an asymptotic equivalence formula for the tail probability of some weighted sums of even number of r.v.s.

**Corollary 2.1.** Let $\{(X_{2k-1}, X_{2k}), \ k \geq 1\}$ be a sequence of independent random vectors with corresponding marginal distributions $F_{2k-1}, F_{2k}, k \geq 1$, and let $\{(\Theta_{2k-1}, \Theta_{2k}), k \geq 1\}$ be a sequence of independent random vectors such that $P(a \leq \Theta_k \leq b)$ for all $k \geq 1$ and some positive constants $a \leq b$. Let sequences $\{(X_{2k-1}, X_{2k}), k \geq 1\}$ and $\{(\Theta_{2k-1}, \Theta_{2k}), k \geq 1\}$ be independent. Assume that for each $k \geq 1$ there exists some measurable function $h_k : \mathbb{R} \mapsto \mathbb{R}_+$ such that

$$P(X_{2k-1} > x | X_{2k} = y) \sim h_{2k}(y)F_{2k-1}(x)$$

and

$$P(X_{2k} > x | X_{2k-1} = y) \sim h_{2k-1}(y)F_{2k}(x)$$
uniformly in $y \in \mathbb{R}$. If $F_k \in \mathcal{L}$, $k \geq 1$, then for any fixed $n \geq 1$

$$P(M^\Theta_{2n} > x) \sim P(S^\Theta_{2n} > x) \sim P(S^{\Theta+}_{2n} > x).$$

Corollary 2.1, to some extend, generalizes the corresponding result in Ng et al. (2002), whose Theorem 2.1 shows that relation

$$P(M_n > x) \sim P(S_n > x), \quad S_n := \sum_{k=1}^n X_k, \quad M_n := \max_{1 \leq k \leq n} S_k,$$

holds for independent $X_1, \ldots, X_n$ and $F_k \in \mathcal{L}$ for each $k = 1, \ldots, n$.

Our second main result deals with a closure property and weak tail-equivalence in the case of dominatedly varying-tailed r.v.s.

**Theorem 2.2.** Let Assumption A be satisfied. If $F_k \in \mathcal{D}$, $k = 1, 2$, then the distribution of $S^\Theta_2$ is in $\mathcal{D}$. Moreover, for some $c > 0$,

$$cP(S^{\Theta+}_2 > x) \lesssim P(S^\Theta_2 > x) \leq P(M^\Theta_2 > x) \leq P(S^{\Theta+}_2 > x). \quad (2.2)$$

**Remark 2.1.** Theorems 2.1 and 2.2 imply that if $F_k \in \mathcal{L} \cap \mathcal{D}$, $k = 1, 2$, then the distribution of $S^\Theta_2$ is in $\mathcal{L} \cap \mathcal{D}$ as well. In the case of some different dependence structure, under $\Theta_1 = \Theta_2 = 1$, the analogous closure result in class $\mathcal{L} \cap \mathcal{D}$ was proved by Geluk and Tang (2009).

**Remark 2.2.** Using Theorem 2.2 with $h_1 = h_2 \equiv 1$, $\Theta_1 = \Theta_2 = 1$ and applying induction, we get that $\mathcal{D}$ is closed under convolution, i.e. if $F_k \in \mathcal{D}$, $k \geq 1$, then $F_1 \ast \cdots \ast F_n \in \mathcal{D}$ for any $n$, cf. Cai and Tang (2004, Proposition 2.1), Watanabe and Yamamuro (2010, Lemma 7.1(iii)).

Similarly to Corollary 2.1, in the case of class $\mathcal{D}$ we obtain the following weak asymptotic equivalence relation.

**Corollary 2.2.** Let $\{(X_{2k-1}, X_{2k}), k \geq 1\}$ and $\{(\Theta_{2k-1}, \Theta_{2k}), k \geq 1\}$ be two independent sequences of random vectors satisfying all the basic conditions of Corollary 2.1. If $F_k \in \mathcal{D}$, $k \geq 1$, then for any fixed $n \geq 1$

$$\tilde{c}P(S^{\Theta+}_{2n} > x) \lesssim P(S^\Theta_{2n} > x) \leq P(M^\Theta_{2n} > x) \leq P(S^{\Theta+}_{2n} > x),$$
where $\tilde{c}$ is some positive constant, maybe depending on $n$. 

4
3 Proofs of main results

Before proving our main results, we firstly give two important lemmas. The first lemma is similar to Lemma 4.1 of Chen et al. (2011). The second lemma is similar to Theorem 2.39 of Foss et al. (2011). Both lemmas are not only at the core of the present study but also of independent interest in their own right.

**Lemma 3.1.** Let $X_1$ and $X_2$ be two real-valued r.v.s with distributions $F_1$ and $F_2$, respectively, and let relation (1.4) holds. If $F_k ∈ ℳ$, $k = 1, 2$, then for any $A > 0$ the relation

\[ P(w_1X_1 + w_2X_2 > x - A) \sim P(w_1X_1 + w_2X_2 > x) \]  \hspace{1cm} (3.1)

holds uniformly for all $(w_1, w_2) ∈ [a, b] × [a, b]$, $0 < a ≤ b < ∞$, i.e.

\[ \limsup \sup_{w_1, w_2 ∈ [a, b]} \left| \frac{P(w_1X_1 + w_2X_2 > x - A)}{P(w_1X_1 + w_2X_2 > x)} - 1 \right| = 0. \]

**PROOF.** Since $A$ is positive, relation (3.1) is equivalent to

\[ \limsup \sup_{w_1, w_2 ∈ [a, b]} \frac{P(w_1X_1 + w_2X_2 > x - A)}{P(w_1X_1 + w_2X_2 > x)} ≤ 1. \]  \hspace{1cm} (3.2)

Let $ε ∈ (0, 1)$. By $F_2 ∈ ℳ$, we have that

\[ \limsup \sup_{w_2 ∈ [a, b]} \frac{P(w_2X_2 > x - A)}{P(w_2X_2 > x)} ≤ \limsup \sup_{w_2 ∈ [a, b]} \frac{F_2(\frac{x - A}{w_2})}{F_2(\frac{x}{w_2})} ≤ \limsup_{z ≥ x/b} \frac{F_2(z - A/a)}{F_2(z)} = 1. \]

Hence, there exists $x_1 > A$ such that for all $x ≥ x_1$

\[ 1 ≤ \sup_{w_2 ∈ [a, b]} \frac{P(w_2X_2 > x - A)}{P(w_2X_2 > x)} ≤ 1 + ε. \]  \hspace{1cm} (3.3)

In addition, relation (1.4) implies that

\[ (1 - ε)h_1(y)F_2(x) ≤ P(X_2 > x | X_1 = y) ≤ (1 + ε)h_1(y)F_2(x) \]  \hspace{1cm} (3.4)

uniformly for $y ∈ ℜ$ and for all sufficiently large $x$ ($x ≥ x_2 > 2x_1$). If $x ≥ x_2$, then

\[
\frac{P(w_1X_1 + w_2X_2 > x - A)}{P(w_1X_1 + w_2X_2 > x)}
= \frac{\int_{-∞}^{∞} f_{(x-x_2-A)}(w_1) P(w_2X_2 > x - w_1 u - A | X_1 = u) F_1(du)}{\int_{-∞}^{∞} f_{(x-x_2)}(w_1) P(w_2X_2 > x - w_1 u | X_1 = u) F_1(du)}
= \frac{I_{11}(x) + I_{12}(x)}{I_{21}(x) + I_{22}(x)}
\leq \max \left\{ \frac{I_{11}(x)}{I_{21}(x)}, \frac{I_{12}(x)}{I_{22}(x)} \right\},
\]  \hspace{1cm} (3.5)
By (3.3) and (3.4), we have that

\[
\begin{align*}
\sup_{\omega_1, \omega_2 \in [a,b]} \frac{I_{11}(x)}{I_{21}(x)} & \leq \frac{1 + \epsilon}{1 - \epsilon} \sup_{\omega_1, \omega_2 \in [a,b]} \frac{\int_{-\infty}^{\infty} (x-x_2-A)/\omega_1 \cdot P(w_2X_2 > x-w_1u-A)h_1(u)F_1(du)}{\int_{-\infty}^{\infty} (x-x_2)/\omega_1 \cdot P(w_2X_2 > x-w_1u)h_1(u)F_1(du)} \\
& \leq \frac{1 + \epsilon}{1 - \epsilon} \sup_{\omega_1, \omega_2 \in [a,b]} \frac{P(w_2X_2 > x-w_1u-A)}{P(w_2X_2 > x-w_1u)} \\
& \leq (1 + \epsilon)^2 \frac{\epsilon}{1 - \epsilon}.
\end{align*}
\]  

(3.6)

Next we deal with \( I_{12}(x)/I_{22}(x) \). Clearly,

\[
\begin{align*}
\frac{I_{12}(x)}{I_{22}(x)} &= \frac{\left( \int_{-\infty}^{\infty} (x-x_2-A)/\omega_1 \right) P(w_2X_2 > x-w_1u-A|X_1 = u)F_1(du)}{\left( \int_{-\infty}^{\infty} (x-x_2)/\omega_1 + \int_{-\infty}^{\infty} (x-x_2)/\omega_1 \right) P(w_2X_2 > x-w_1u|X_1 = u)F_1(du)} \\
& \leq \frac{P((x-x_2-A)/\omega_1 < X_1 \leq (x-A)/\omega_1)}{\int_{-\infty}^{\infty} P(w_2X_2 > x-w_1u|X_1 = u)F_1(du)} \\
& \quad + \frac{\int_{-\infty}^{\infty} P(w_2X_2 > x-w_1u-A|X_1 = u)F_1(du)}{\int_{-\infty}^{\infty} P(w_2X_2 > x-w_1u|X_1 = u)F_1(du)} \\
& =: I_3(x) + I_4(x).
\end{align*}
\]  

(3.7)

To estimate \( I_3(x) \), note that for the above \( \epsilon > 0 \), there exists a large \( x_3 \geq x_2 \) such that for all \( x \geq x_3 \) and all \( \omega_1 \in [a,b] \)

\[
\begin{align*}
\int_{x/\omega_1}^{\infty} P(X_2 > 0|X_1 = u)F_1(du) &= P(X_1 > x/\omega_1, X_2 > 0) \\
& = \int_{0}^{\infty} P(X_1 > x/\omega_1|X_2 = u)F_2(du) \\
& \geq (1 - \epsilon^2) F_1(x/\omega_1) E_h(X_2)1_{\{X_2 > 0\}}.
\end{align*}
\]  

(3.8)

Here \( E_h(X_2)1_{\{X_2 > 0\}} > 0 \) because of heavy-tailedness of d.f. \( F_2 \). Hence, by (3.8) and \( F_1 \in \mathcal{L} \), there exists a large \( x_4 \geq x_3 \) such that for \( x \geq x_4 \)

\[
\begin{align*}
\sup_{\omega_1, \omega_2 \in [a,b]} I_3(x) & \leq \sup_{\omega_1 \in [a,b]} \frac{F_1((x-x_2-A)/\omega_1) - F_1((x-A)/\omega_1)}{\int_{x/\omega_1}^{\infty} P(X_2 > 0|X_1 = u)F_1(du)} \\
& \leq \frac{1}{(1 - \epsilon^2) E_h(X_2)1_{\{X_2 > 0\}}} \sup_{\omega_1 \in [a,b]} \frac{F_1((x-x_2-A)/\omega_1) - F_1((x-A)/\omega_1)}{F_1(x/\omega_1)} \\
& \leq \frac{\epsilon}{(1 - \epsilon^2) E_h(X_2)1_{\{X_2 > 0\}}}.
\end{align*}
\]  

(3.9)

For the denominator of \( I_4(x) \), by (1.4) we have that

\[
\begin{align*}
\int_{x/\omega_1}^{\infty} P(w_2X_2 > x-w_1u|X_1 = u)F_1(du) &= P(w_1X_1 + w_2X_2 > x, X_1 > x/\omega_1) \\
& = \int_{-\infty}^{0} P(w_1X_1 > x-w_2u|X_2 = u)F_2(du) \\
& \quad + \int_{0}^{\infty} P(w_1X_1 > x|X_2 = u)F_2(du).
\end{align*}
\]
Here, for large \( x \geq x_5 \geq x_4 \),

\[
\inf_{w_1, w_2 \in [a, b]} \inf_{u \leq 0} \frac{P(w_1 X_1 > x - w_2 u | X_2 = u)}{P(w_1 X_1 > x - w_2 u)|h_2(u)} \geq \inf_{z \geq x/b} \inf_{u \leq 0} \frac{P(X_1 > z | X_2 = u)}{P(X_1 > z)|h_2(u)} \geq 1 - \epsilon
\]

and

\[
\inf_{w_1 \in [a, b]} \inf_{u > 0} \frac{P(w_1 X_1 > x | X_2 = u)}{P(w_1 X_1 > x)|h_2(u)} \geq 1 - \epsilon,
\]

implying that

\[
\int_{x/w_1}^\infty P(w_2 X_2 > x - w_1 u | X_1 = u) F_1(du) \geq (1 - \epsilon) \left( \int_{-\infty}^0 P(w_1 X_1 > x - w_2 u) h_2(u) F_2(du) + P(w_1 X_1 > x) E h_2(X_2) \mathbf{1}_{\{X_2 > 0\}} \right)
\]

uniformly in \( w_1, w_2 \in [a, b] \). In the same way, the numerator of \( I_4(x) \) can be estimated for all \( x \geq x_6 \geq x_5 \) as follows

\[
\int_{(x-A)/w_1}^\infty P(w_2 X_2 > x - w_1 u - A | X_1 = u) F_1(du) \leq (1 + \epsilon) \left( \int_{-\infty}^0 P(w_1 X_1 > x - w_2 u - A) h_2(u) F_2(du) + P(w_1 X_1 > x - A) E h_2(X_2) \mathbf{1}_{\{X_2 > 0\}} \right)
\]

uniformly in \( w_1, w_2 \in [a, b] \). Thus, for large \( x \geq x_7 \geq x_6 \) it holds that

\[
\sup_{w_1, w_2 \in [a, b]} I_4(x) \leq 1 + \epsilon \frac{\int_{-\infty}^0 \mathcal{F}_1((x - w_2 u - A)/w_1) h_2(u) F_2(du) + \mathcal{F}_1((x - A)/w_1) E h_2(X_2) \mathbf{1}_{\{X_2 > 0\}}}{1 - \epsilon \sup_{w_1, w_2 \in [a, b]} \max \left\{ \frac{\mathcal{F}_1((x - w_2 u - A)/w_1)}{\mathcal{F}_1((x - w_2 u)/w_1)} , \frac{\mathcal{F}_1((x - A)/w_1)}{\mathcal{F}_1(x/w_1)} \right\}} \leq \frac{(1 + \epsilon)^2}{1 - \epsilon}
\]

by \( F_1 \in \mathcal{L} \). Plugging (3.9) and (3.11) into (3.7), we obtain that

\[
\sup_{x \geq x_7} \sup_{w_1, w_2 \in [a, b]} \frac{I_{12}(x)}{I_{22}(x)} \leq \frac{(1 + \epsilon)^2}{1 - \epsilon} + \frac{\epsilon}{(1 - \epsilon) E h_2(X_2) \mathbf{1}_{\{X_2 > 0\}}}.
\]

Thus, the desired relation (3.2) follows from (3.5), (3.6) and (3.12) by the arbitrariness of \( \epsilon \).  

\[\Box\]

**Lemma 3.2.** Suppose that \( G_1, G_2 \) and \( H \) are distributions on \( \mathbb{R} \), such that \( \mathcal{G}_1(x) \leq d_1 \mathcal{G}_2(x) \) for some finite \( d_1 > 0 \), and \( H \in \mathcal{D} \). Then there exists a finite constant \( d_2 > 0 \) such that \( G_1 \ast H(x) \leq d_2 G_2 \ast H(x) \).

**Proof.** The statement of the lemma follows observing that for all \( x > 2M, M > 1 \)

\[
\mathcal{G}_1 \ast H(x) \leq \int_{-\infty}^{x-M} \frac{\mathcal{G}_1(x - y)}{\mathcal{G}_2(x - y)} G_2(x - y) H(dy) + \frac{H(x - M)}{H(x + M)G_2(-M)} \int_{x+M}^\infty \mathcal{G}_2(x - y) H(dy) \leq \max \left\{ \sup_{z \geq M} \frac{\mathcal{G}_1(z)}{\mathcal{H}(3x/2)G_2(-M)} \right\} \frac{G_2 \ast H(x)}{G_2 \ast H(x)}.
\]
From (3.15) and (3.16) we obtain that

\[ P(S_2^\Theta > x - A) = \int_{-\infty}^b \int_{-\infty}^b P(w_1 X_1 + w_2 X_2 > x - A)P(\Theta_1 \in dw_1, \Theta_2 \in dw_2) \]

\[ \sim \int_{-\infty}^b \int_{-\infty}^b P(w_1 X_1 + w_2 X_2 > x)P(\Theta_1 \in dw_1, \Theta_2 \in dw_2) \]

\[ = P(S_2^\Theta > x), \]

which implies that the distribution of \( S_2^\Theta \) belongs to the class \( \mathcal{L} \).

Next we prove relation (2.1). Since \( S_2^\Theta \leq M_2^\Theta \leq \Theta_1 X_1^+ + \Theta_2 X_2^+ =: S_2^{\Theta^+} \), it suffices to show that

\[ P(S_2^\Theta > x) \geq P(S_2^{\Theta^+} > x). \] (3.13)

For any \( x > 0 \), we split the tail probability on the left-hand side of (3.13) as follows:

\[ P(S_2^\Theta > x) = P(\Theta_1 X_1 + \Theta_2 X_2^+ > x, X_1 \leq 0, X_2 > 0) \]

\[ + P(\Theta_1 X_1^+ + \Theta_2 X_2 > x, X_1 > 0, X_2 \leq 0) \]

\[ + P(\Theta_1 X_1^+ + \Theta_2 X_2^+ > x, X_1 > 0, X_2 > 0) \]

\[ =: J_1(x) + J_2(x) + J_3(x). \] (3.14)

We only estimate \( J_1(x) \). Assume \( P(X_1 \leq 0) > 0 \). (In the case \( P(X_1 \leq 0) = 0 \), we have \( J_1(x) = 0 \).) Clearly, the distribution of \( \Theta_2 X_2 \) belongs to \( \mathcal{L} \) (see also Lemma 4.2 of Chen et al. (2011)). Hence, from (1.4) and Fubini’s theorem we obtain that for arbitrary \( \epsilon \in (0, 1) \) and large \( x \)

\[ \frac{J_1(x)}{P(\Theta_2 X_2 > x)} \geq \frac{P(b X_1 + \Theta_2 X_2^+ > x, X_1 \leq 0, X_2 > 0)}{P(\Theta_2 X_2 > x)} \]

\[ = \frac{1}{P(\Theta_2 X_2 > x)} \int_{-\infty}^b \int_{-\infty}^0 P(w_2 X_2 > x - bu|X_1 = u)F_1(du)P(\Theta_2 \in dw_2) \]

\[ \geq \frac{1 - \epsilon}{P(\Theta_2 X_2 > x)} \int_{-\infty}^b \int_{-\infty}^0 P(w_2 X_2 > x - bu)h_1(u)F_1(du)P(\Theta_2 \in dw_2) \]

\[ = (1 - \epsilon) \int_{-\infty}^0 \frac{P(\Theta_2 X_2 > x - bu)}{P(\Theta_2 X_2 > x)} h_1(u)F_1(du) \] (3.15)

where, by the dominated convergence theorem, the integral in (3.15) tends to \( \text{E}h_1(X_1)1_{\{X_1 \leq 0\}} \) > 0. On the other hand, by (1.4) we have that for large \( x \)

\[ P(\Theta_2 X_2^+ > x, X_1 \leq 0, X_2 > 0) = P(\Theta_2 X_2 > x, X_1 \leq 0) \]

\[ = \int_{-\infty}^b \int_{-\infty}^0 P(w_2 X_2 > x|X_1 = u)F_1(du)P(\Theta_2 \in dw_2) \]

\[ \leq (1 + \epsilon) \int_{-\infty}^b \int_{-\infty}^0 P(w_2 X_2 > x)h_1(u)F_1(du)P(\Theta_2 \in dw_2) \]

\[ = (1 + \epsilon)P(\Theta_2 X_2 > x)\text{E}h_1(X_1)1_{\{X_1 \leq 0\}}. \] (3.16)

From (3.15) and (3.16) we obtain that

\[ J_1(x) \geq P(\Theta_2 X_2^+ > x, X_1 \leq 0, X_2 > 0) = P(S_2^{\Theta^+} > x, X_1 \leq 0, X_2 > 0). \] (3.17)
In the same way, we get that
\[ J_2(x) \gtrsim P(S_2^{\Theta^+} > x, X_1 > 0, X_2 \leq 0), \]
(3.18)
if \( P(X_2 \leq 0) > 0 \) and \( J_2(x) = 0 \) otherwise. Substituting (3.17) and (3.18) into (3.14) we derive (3.13), completing the proof of the theorem. \( \square \)

**Proof of Theorem 2.2.** Since
\[ S_2^{\Theta} \leq M_2^{\Theta} \leq S_2^{\Theta^+} := \sum_{k=1}^{2n} \Theta_k X_k^+, \]
it suffices to prove that
\[ P(S_2^{\Theta} > x) \sim P(S_2^{\Theta^+} > x) \]
for an arbitrary \( n \geq 1 \). We prove this equivalence by induction. Clearly, because of Theorem 2.1, (3.19) holds for \( n = 1 \). Suppose now that (3.19) holds for \( n = N \), i.e.
\[ P(S_2^{\Theta} > x) \sim P(S_2^{\Theta^+} > x). \]
(3.20)
By Theorem 2.1 we have that, for any \( k \geq 1 \),
\[ P(\Theta_{2k-1}X_{2k-1} + \Theta_{2k}X_{2k} > x) \sim P(\Theta_{2k-1}X_{2k-1}^+ + \Theta_{2k}X_{2k}^+ > x) \]
(3.21)
and the distributions of r.v.s \( \Theta_{2k-1}X_{2k-1} + \Theta_{2k}X_{2k} \) and \( \Theta_{2k-1}X_{2k-1}^+ + \Theta_{2k}X_{2k}^+ \) are in \( \mathcal{L} \). Let \( \mathcal{G} \) be a d.f. of r.v. \( \Theta_{2N+1}X_{2N+1} + \Theta_{2N+2}X_{2N+2} \). Since \( \mathcal{G} \) is long-tailed, relation (3.20) and Theorem 2.39 of Foss et al. (2011) imply, that
\[ P(S_2^{\Theta} > x) = P(S_2^{\Theta} + \Theta_{2N+1}X_{2N+1} + \Theta_{2N+2}X_{2N+2} > x) \sim P(S_2^{\Theta^+} + \Theta_{2N+1}X_{2N+1} + \Theta_{2N+2}X_{2N+2} > x). \]
(3.22)
Now let \( \mathcal{G} \) be a d.f. of \( S_2^{\Theta^+} \), which is long-tailed due to Corollary 2.42 of Foss et al. (2011). Therefore, by (3.21) and again by Theorem 2.39 of Foss et al. (2011), we get
\[ P(S_2^{\Theta^+} + \Theta_{2N+1}X_{2N+1} + \Theta_{2N+2}X_{2N+2} > x) \sim P(S_2^{\Theta^+} + \Theta_{2N+1}X_{2N+1}^+ + \Theta_{2N+2}X_{2N+2}^+ > x) = P(S_2^{\Theta^+} > x). \]
(3.23)
(3.22), (3.23) and induction imply that (3.19) holds for an arbitrary \( n \). \( \square \)

**Proof of Corollary 2.1.** Since
\[ \limsup P(\Theta_1X_1 + \Theta_2X_2 > x/2) \leq \infty. \]
(3.24)
For any \( x > 0 \) we have
\[ P\left(\Theta_1X_1 + \Theta_2X_2 > \frac{x}{2}\right) \leq P\left(b(X_1^+ + X_2^+) > \frac{x}{2}\right) \leq F_1\left(\frac{x}{4b}\right) + F_2\left(\frac{x}{4b}\right) \]
(3.25)
and
\[ P(\Theta_1X_1 + \Theta_2X_2 > x) = J_1(x) + J_2(x) + J_3(x) \]
(3.26)
as in (3.14). Without loss of generality assume $P(X_k \leq 0) > 0$, $k = 1, 2$.

According to (1.4),

$$J_2(x) \geq P(\Theta_1 X_1^+ + bX_2 > x, X_1 > 0, X_2 \leq 0)$$
$$= P(\Theta_1 X_1 + bX_2 > x, X_2 \leq 0)$$
$$= \int_{-\infty}^0 P(\Theta_1 X_1 > x - bu | X_2 = u) F_2(du)$$
$$\geq \int_{-x/b}^0 P(\Theta_1 X_1 > 2x | X_2 = u) F_2(du)$$
$$\geq \frac{1}{2} \int_{-x/b}^0 P\left(X_1 > \frac{2x}{a}\right) h_2(u) F_2(du)$$
$$= \frac{1}{2} F_1\left(\frac{2x}{a}\right) E h_2(X_2) 1_{\{-x/b < X_2 \leq 0\}}$$  \hspace{1cm} (3.27)

for $x$ sufficiently large. Similarly,

$$P(\Theta_1 X_1^+ > x, X_1 > 0, X_2 \leq 0) \leq P(bX_1 > x, X_2 \leq 0)$$
$$= \int_{-\infty}^0 P\left(X_1 > \frac{x}{b} \mid X_2 = u\right) F_2(du)$$
$$\leq 2F_1\left(\frac{x}{b}\right) E h_2(X_2) 1_{\{X_2 \leq 0\}}$$  \hspace{1cm} (3.28)

for $x$ sufficiently large. Thus, by (3.27)–(3.28),

$$\liminf \frac{J_2(x)}{P(\Theta_1 X_1^+ > x, X_1 > 0, X_2 \leq 0)} \geq \frac{1}{4} \liminf \frac{F_1\left(2x/a\right)}{F_1(x/b)} \liminf \frac{E h_2(X_2) 1_{\{-x/b < X_2 \leq 0\}}}{E h_2(X_2) 1_{\{X_2 \leq 0\}}}$$
$$\geq c_1 > 0.$$

Hence, for large $x$ and some positive constant $c_2$,

$$J_2(x) \geq c_2 P(\Theta_1 X_1^+ > x, X_1 > 0, X_2 \leq 0)$$
$$= c_2 P(\Theta_1 X_1^+ + \Theta_2 X_2^+ > x, X_1 > 0, X_2 \leq 0).$$

Analogously,

$$J_1(x) \geq c_3 P(\Theta_1 X_1^+ + \Theta_2 X_2^+ > x, X_1 \leq 0, X_2 > 0)$$

for some $c_3 > 0$. Now, from these estimates and (3.26) we obtain that

$$P(\Theta_1 X_1 + \Theta_2 X_2 > x) \geq \min\{1, c_2, c_3\} P(\Theta_1 X_1^+ + \Theta_2 X_2^+ > x)$$
$$\geq \min\{1, c_2, c_3\} P\left(X_1^+ + X_2^+ > \frac{x}{a}\right)$$
$$\geq \frac{1}{2} \min\{1, c_2, c_3\} \left( F_1\left(\frac{x}{a}\right) + F_2\left(\frac{x}{a}\right) \right).$$  \hspace{1cm} (3.29)

(3.25) and (3.29) imply that for large $x$

$$\frac{P(\Theta_1 X_1 + \Theta_2 X_2 > x/2)}{P(\Theta_1 X_1 + \Theta_2 X_2 > x)} \leq \frac{2}{\min\{1, c_2, c_3\}} \frac{F_1(x/(4b)) + F_2(x/(4b))}{F_1(x/a) + F_2(x/a)}$$
$$\leq \frac{2}{\min\{1, c_2, c_3\}} \max\left\{ \frac{F_1(x/(4b))}{F_1(x/a)}, \frac{F_2(x/(4b))}{F_2(x/a)} \right\}$$

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and the desired estimate (3.24) follows by $F_k \in \mathcal{D}$, $k = 1, 2$. Finally, the asymptotic relation in (2.2) follows from the proof.

**Proof of Corollary 2.2.** It suffices to prove, that

$$\hat{c}_{2n} P(S_{2n}^{\Theta^+} > x) \preceq P(S_{2n}^\Theta > x)$$

(3.30)

for some positive $\hat{c}_{2n}$.

We use induction again. By Theorem 2.2, (3.30) holds for $n = 1$. Suppose now that (3.30) holds for $n = N$, i.e.

$$\hat{c}_{2N} P(S_{2N}^{\Theta^+} > x) \preceq P(S_{2N}^\Theta > x)$$

(3.31)

with some $\hat{c}_{2N} > 0$. By Theorem 2.2 we have that, for any fixed $k \geq 1$,

$$\tilde{c}_k P(\Theta_{2k-1}X_{2k-1}^+ + \Theta_{2k}X_{2k}^+ > x) \preceq P(\Theta_{2k-1}X_{2k-1} + \Theta_{2k}X_{2k} > x)$$

(3.32)

with some positive constants $\tilde{c}_k$, $k \geq 1$. Moreover, by Theorem 2.2, distributions of r.v.s $\Theta_{2k-1}X_{2k-1} + \Theta_{2k}X_{2k}$ and $\Theta_{2k-1}X_{2k-1}^+ + \Theta_{2k}X_{2k}^+$ are in $\mathcal{D}$. Let $\hat{H}$ be a d.f. of $\Theta_{2N+1}X_{2N+1}^+ + \Theta_{2N+2}X_{2N+2}^+$. Since $\hat{H} \in \mathcal{D}$, relation (3.31) and Lemma 3.2 imply, that

$$P(S_{2(N+1)}^{\Theta^+} > x) = P(S_{2N}^{\Theta^+} + \Theta_{2N+1}X_{2N+1}^+ + \Theta_{2N+2}X_{2N+2}^+ > x) \preceq \frac{1}{\hat{c}_2 N} P(S_{2N}^{\Theta^+} + \Theta_{2N+1}X_{2N+1}^+ + \Theta_{2N+2}X_{2N+2}^+ > x)$$

(3.33)

with some $\hat{c}_{2N} > 0$. Now let $\tilde{H}$ be a d.f. of $S_{2N}^{\Theta^+}$, which belongs to $\mathcal{D}$ according to Remark 2.2. Therefore, applying (3.32) and Lemma 3.2 again, we get that

$$P(S_{2N}^{\Theta} + \Theta_{2N+1}X_{2N+1}^+ + \Theta_{2N+2}X_{2N+2}^+ > x) \preceq \frac{1}{\hat{c}_{2N+2}} P(S_{2N}^{\Theta} + \Theta_{2N+1}X_{2N+1} + \Theta_{2N+2}X_{2N+2} > x)$$

(3.34)

with some $\hat{c}_{2N+2} > 0$. Relations (3.33) and (3.34) imply that (3.30) holds for $n = 2(N + 1)$ with positive $\hat{c}_{2(N+1)} = \hat{c}_{2N} \hat{c}_{2N+2}$. Hence, by induction, relation (3.30) holds for an arbitrary $n$.

**References**


