ASYMPTOTICS FOR RANDOMLY WEIGHTED AND STOPPED DEPENDENT SUMS*

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Abstract

The paper deals with the tail probability of \( \max_{1 \leq k \leq \tau} \sum_{i=1}^{k} \Theta_i X_i \), where \( \{X_1, X_2, \ldots\} \) is a sequence of extended negatively upper orthant dependent or bivariate upper tail independent, identically distributed random variables with dominatedly-varying tails, \( \{\Theta_1, \Theta_2, \ldots\} \) is a sequence of nonnegative nondegenerate at zero random variables (not necessarily independent and identically distributed), \( \tau \) is a random variable taking values in \( \{1, 2, \ldots\} \cup \{\infty\} \). In addition, \( \{X_1, X_2, \ldots\} \), \( \{\Theta_1, \Theta_2, \ldots\} \) and \( \tau \) are mutually independent. Under some mild conditions, the weak asymptotic equivalence relations for the probability \( P(\max_{1 \leq k \leq \tau} \sum_{i=1}^{k} \Theta_i X_i > x) \) are established. An application to the random-time ruin probability in the discrete time ruin risk model is provided.

Keywords: asymptotic tail probability; randomly weighted and stopped sums; dominated variation; extended negative upper orthant dependence; bivariate upper tail independence; random-time ruin probability.

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1 Introduction

Let \( \{X_1, X_2, \ldots \} \) be a sequence of identically distributed (not necessarily independent) random variables (r.v.s) with a common distribution \( F \); let \( \{\Theta_1, \Theta_2, \ldots \} \) be a sequence of nonnegative (not necessarily independent and identically distributed) r.v.s; and let \( \tau \) be a positive integer-valued extended (which can take also infinite value) r.v. Assume that \( \{X_1, X_2, \ldots \}, \{\Theta_1, \Theta_2, \ldots \} \) and \( \tau \) are mutually independent. We are interested in the tail behavior of the maximum of randomly weighted and stopped sums

\[
M_\tau = \max_{1 \leq k \leq \tau} S_k,
\]

where

\[
S_k := \sum_{i=1}^{k} \Theta_i X_i, \quad k = 1, 2, \ldots
\]

A large number of papers have been devoted to the study of randomly weighted sums \( S_n, S_\infty \) or their maxima \( M_n, M_\infty \) in their relation to non-life insurance mathematics models. In the discrete time risk model, r.v. \( X_i \) can be interpreted as the net loss of an insurance company within the time period \( i \), while the weight \( \Theta_i \) can be regarded as the discount factor from time \( i \) to time 0. In such case, the tail probabilities \( P(M_n > x) \) and \( P(M_\infty > x) \) can be understood as the probability of ruin by time \( n \) and of ultimate ruin, respectively, where \( x \geq 0 \) is the initial capital reserve of the insurance company. Some earlier results on the asymptotics of \( P(M_n > x) \) or \( P(M_\infty > x) \) with independent and identically distributed (i.i.d.) heavy-tailed r.v.s \( X_1, X_2, \ldots \) can be found in Tang and Tsitsiashvili (2003a), Tang and Tsitsiashvili (2004), Wang et al. (2005), Wang and Tang (2006), among others. Similar results under some dependence structures were derived in Zhang et al. (2009), Gao and Wang (2010), Chen and Yuen (2009), Yi et al. (2011), Hazra and Maulik (2012) and Yang et al. (2012).

Recently, Olvera-Cravioto (2012) assumed that \( X_1, X_2, \ldots \) are i.i.d. r.v.s, the number of terms in the sum is random (including infinity) and studied the asymptotic tail behavior for the randomly weighted and randomly stopped sums and their maxima when d.f. of the \( X_i \)'s is consistently varying-tailed. In the insurance context, \( P(M_\tau > x) \) can be interpreted as the random-time ruin probability. Such randomly weighted and stopped sums appear also in other areas, such as analysis of information ranking algorithms, e.g. Google PageRank (see Volkovich and Litvak (2010), Jelenković and Olvera-Cravioto (2010) and references therein). Simultaneously, comparing with some existing results for \( \tau = \infty \), Olvera-Cravioto (2012) weakened some tedious moment conditions on the weights by properly strengthening the moment condition on the generic r.v. \( X_1 \).

In this paper, we investigate the tail behavior of \( M_\tau \) with \( X_1, X_2, \ldots \) being dominatedly varying-tailed and possessing some dependence structure, which is more realistic in applications, \( \Theta_1, \Theta_2, \ldots \) and \( \tau \) satisfying some mild conditions. In Theorems 2.1 and 2.2, we obtain the
following asymptotic relation

\[ P(M_\tau > x) \asymp E \sum_{i=1}^\tau P(\Theta_i X_i > x), \quad x \to \infty, \]

where we use \( a(x) \asymp b(x) \) to denote \( 0 < \liminf_{x \to \infty} a(x)/b(x) \leq \limsup_{x \to \infty} a(x)/b(x) < \infty. \)

In Section 2, after introducing some heavy-tailed distribution classes and dependence structures, we present our main results. In Section 3 we collect auxiliary lemmas. In particular, one of useful results is Lemma 3.1, which is taken from Gao and Wang (2010). In Sections 4 and 5, we give the proofs of, respectively, the lower and upper bounds for our main result. In Section 6 we consider the case of extended r.v. \( \tau \). In Section 7 we provide an application to the random-time ruin probability in the discrete time risk model with financial and insurance risk.

## 2 Preliminaries and main results

Throughout this paper, without special statement, all the limit relationships hold for \( x \) tending to \( \infty \). For two positive functions \( a(x) \) and \( b(x) \), we write \( a(x) \sim b(x) \) if \( \lim a(x)/b(x) = 1 \); write \( a(x) \lesssim b(x) \) or \( b(x) \gtrsim a(x) \) if \( \limsup a(x)/b(x) \leq 1 \); \( a(x) = o(b(x)) \) if \( \lim a(x)/b(x) = 0 \).

For an event \( A \) we denote its indicator function by \( 1_I \). For two real numbers \( x \) and \( y \), denote \( x \wedge y = \min\{x, y\} \) and \( x \vee y = \max\{x, y\} \); for every real number \( x \) denote its positive part by \( x^+ = x \vee 0 \).

### 2.1 Heavy-tailed distribution classes

A d.f. \( V \) belongs to the dominatedly varying-tailed class, denoted by \( \mathcal{D} \), if \( \limsup_{y \to 1} \liminf_{x \to \infty} V(xy)/V(x) < 1 \) for any \( y > 0 \). A slightly smaller class \( \mathcal{C} \) contains all consistently varying-tailed d.f.s: a d.f. \( V \) belongs to the class \( \mathcal{C} \), if \( \lim_{y \downarrow 1} \liminf_{x \to \infty} V(xy)/V(x) = 1 \). Finally, the well-known heavy-tailed classes are \( \text{ERV}(\alpha, -\beta) \), consisting of d.f.s with extended regularly varying tails, and \( \mathcal{R}_{-\alpha} \) of d.f.s with regularly varying tails. For the sake of notational clarity, we recall that a d.f. \( V \) belongs to the class \( \text{ERV}(\alpha, -\beta) \), if there exist some constants \( 0 < \alpha \leq \beta < \infty \) such that \( y^{-\beta} \leq \liminf_{y \to 1} \liminf_{x \to \infty} V(xy)/V(x) \leq \limsup_{y \to 1} \limsup_{x \to \infty} V(xy)/V(x) \leq y^{-\alpha} \) for any \( y \geq 1 \). If, in the above relation, \( \alpha = \beta \), then we say that d.f. \( V \) belongs to the class \( \mathcal{R}_{-\alpha} \). The following inclusion relationship between the aforementioned classes holds:

\[ \mathcal{R}_{-\alpha} \subset \text{ERV}(\alpha, -\beta) \subset \mathcal{C} \subset \mathcal{D}. \]

Furthermore, for a d.f. \( V \), denote, respectively, the upper and lower Matuszewska indices of \( V \):

\[
J_V^+ = -\lim_{y \to \infty} \frac{\log V_+(y)}{\log y} \quad \text{with} \quad V_+(y) := \liminf_{x \to \infty} \frac{V(xy)}{V(x)} \quad \text{for} \quad y > 1,
\]

\[
J_V^- = -\lim_{y \to \infty} \frac{\log V_-(y)}{\log y} \quad \text{with} \quad V_-(y) := \limsup_{x \to \infty} \frac{V(xy)}{V(x)} \quad \text{for} \quad y > 1.
\]
Additionally, denote $L_V := \lim_{y \searrow 1} V_s(y)$. It is clear that $0 \leq L_V \leq 1$. The presented definitions imply the equivalence of the following statements (for details, see Bingham et al. (1987)):

(i) $V \in \mathcal{D}$, (ii) $V_s(y) > 0$ for some $y > 1$, (iii) $L_V > 0$, (iv) $J_V^+ < \infty$.

It also holds that $V \in \mathcal{C}$ if and only if $L_V = 1$. The following useful lemma origins from Proposition 2.2.1 of Bingham et al. (1987) and Lemma 3.5 of Tang and Tsitsiashvili (2003a).

**Lemma 2.1** Let $V \in \mathcal{D}$. For any $p_1 < J_V^+$, there exist positive constants $C_1 = C_1(V)$ and $D_1 = D_1(V)$, such that for all $x \geq y \geq D_1$ it holds

$$\frac{V(y)}{V(x)} \geq C_1 \left( \frac{y}{x} \right)^{-p_1}.$$  

For any $p_2 > J_V^+$, there exist positive constants $C_2 = C_2(V)$ and $D_2 = D_2(V)$, such that for all $x \geq y \geq D_2$ it holds

$$\frac{V(y)}{V(x)} \leq C_2 \left( \frac{y}{x} \right)^{-p_2}.$$  

In addition, for every $p > J_V^+$ it holds that $x^{-p} = o(V(x))$.

By the last item of this lemma, it is easy to see that if d.f. $V^+(x) \equiv V(x) \mathbb{I}_{\{x \geq 0\}}$ has finite mean $\mu_V^+ = \int_0^\infty x dV(x)$ then $J_V^+ \geq 1$. On the other hand, if $J_V^- > 1$ then $\mu_V^- < \infty$.

### 2.2 Dependence structures

In this paper we consider two classes of dependence.

**Extended negative upper orthant dependence (ENUOD).** A sequence of real-valued r.v.s $\{\xi_1, \xi_1, \ldots\}$ is said to be extended negatively upper orthant dependent (see Liu (2009)), if there exists a positive constant $\kappa$ such that, for each $n \geq 1$ and all $x_1, \ldots, x_n$ it holds

$$P(\xi_1 > x_1, \ldots, \xi_n > x_n) \leq \kappa \prod_{i=1}^n P(\xi_i > x_i).$$

Similarly, a sequence of real-valued r.v.s $\{\xi_1, \xi_1, \ldots\}$ is said to be extended negatively lower orthant dependent (ENLOD), if there exists a positive constant $\kappa$ such that, for each $n \geq 1$ and all $x_1, \ldots, x_n$ it holds

$$P(\xi_1 \leq x_1, \ldots, \xi_n \leq x_n) \leq \kappa \prod_{i=1}^n P(\xi \leq x_i).$$

If a sequence of real-valued r.v.s $\{\xi_1, \xi_1, \ldots\}$ is both ENUOD and ENLOD then this sequence is said to be extended negatively orthant dependent (ENOD).

If $\kappa = 1$, then these dependence structures are called negative upper orthant dependence (NUOD), negative lower orthant dependence (NLOD) and negative orthant dependence (NOD),
respectively, which were introduced by Ebrahimi and Ghosh (1981) and Block et al. (1982). According to Liu (2009), the ENOD structure can reflect not only a negative dependence structure but, to some extent, also a positive one (see also Wang et al. (2013)).

The following lemma is due to Block et al. (1982) (see also Lemma 2.2 of Chen et al. (2010) or Lemma 3.1 in Liu (2009)).

**Lemma 2.2** (i) If r.v.s $\xi_1, \ldots, \xi_n$ are ENUOD with some dominating constant $\kappa > 0$, then

$$E \prod_{i=1}^{n} \xi_i^+ \leq \kappa \prod_{i=1}^{n} E \xi_i^+.$$  

(ii) Assume that r.v.s $\xi_1, \ldots, \xi_n$ are ENLOD/ENUOD/ENOD with some dominating constant $\kappa$. If functions $f_1, \ldots, f_n$ are all nondecreasing then r.v.s $f_1(\xi_1), \ldots, f_n(\xi_n)$ are still ENLOD/ENUOD/ENOD. If $f_1, \ldots, f_n$ are all nonincreasing then r.v.s $f_1(\xi_1), \ldots, f_n(\xi_n)$ are ENUOD/ENLOD/ENOD. For each case, the dominating constant $\kappa$ remains unchanged.

Bivariate upper tail independence. A sequence of real-valued and identically distributed r.v.s $\{\xi_i, \ i \geq 1\}$ is said to be **bivariate upper tail independent** if

$$\lim \frac{P(\xi_i > x, \xi_j > x)}{P(\xi_1 > x)} = 0, \quad i \neq j.$$  

(2.1)

For a comprehensive study of asymptotic tail independence, see Sibuya (1960), De Carvalho and Ramos (2012), Liu et al. (2012). Note that the introduced dependence allows a wide range of dependence structures between r.v.s. For example, if the nonnegative r.v.s $\xi_1, \xi_2$ are assumed to be identically distributed with absolutely continuous joint distribution and to be dependent according to the Clayton family copulas of the form

$$C(u,v; \rho) = \left( u^{-\rho} + v^{-\rho} - 1 \right)^{-\frac{1}{\rho}}, \quad \rho > 0,$$

then (2.1) is satisfied with $i = 1, j = 2$. In fact, the corresponding pair of r.v.s linked by such bivariate Clayton copula is ENOD, although not NOD (see, e.g., Remark 3.1 in Ko and Tang (2008)).

Clearly, by the definition, the bivariate upper tail independence is weaker and more easily verifiable than the commonly used notion of extended negative upper orthant dependence. Note that the bivariate upper tail independence structure is strictly larger than the ENUOD structure. To see this, one can consider two positive r.v.s $\xi_1$ and $\xi_2$ with the joint tail probability

$$P(\xi_1 > x, \xi_2 > y) = \frac{1}{(x \vee 1)(y \vee 1)(1 + x + y)}, \quad x \geq 0, \ y \geq 0.$$  

Such a pair $(\xi_1, \xi_2)$ is bivariate upper tail independent, but not ENUOD, see Liu et al. (2012, Example 3.1). Hence, obtained results under bivariate upper tail independence are meaningful.
2.3 Main results

The following theorems are the main results of the paper. First we state the assumptions concerning the sequences \( \{X_1, X_2, \ldots \} \), \( \{\Theta_1, \Theta_2, \ldots \} \) and r.v. \( \tau \).

**Assumption A.** \( \{X_1, X_2, \ldots \} \) is a sequence of bivariate upper tail independent real-valued r.v.s with common d.f. \( F \in \mathcal{D} \), such that \( J_F > 0 \) and \( F(-x) = o(\bar{F}(x)) \). \( \{\Theta_1, \Theta_2, \ldots \} \) is a sequence of nonnegative nondegenerate at zero (not necessarily independent and identically distributed) r.v.s. \( \tau \) is a positive integer-valued r.v. The r.v.s \( \{X_1, X_2, \ldots \} \), \( \{\Theta_1, \Theta_2, \ldots \} \) and \( \tau \) are mutually independent.

**Assumption B.** \( \{X_1, X_2, \ldots \} \) is a sequence of ENUOD (with dominating constant \( \kappa \) ) real-valued r.v.s with common d.f. \( F \in \mathcal{D} \), such that \( J_F > 0 \) and \( F(-x) = o(\bar{F}(x)) \). \( \{\Theta_1, \Theta_2, \ldots \} \) is a sequence of nonnegative nondegenerate at zero (not necessarily independent and identically distributed) r.v.s; \( \tau \) is a positive integer-valued r.v. The r.v.s \( \{X_1, X_2, \ldots \} \), \( \{\Theta_1, \Theta_2, \ldots \} \) and \( \tau \) are mutually independent.

**Theorem 2.1** (a) Assume that Assumption A is satisfied. If \( J_F^+ < 1 \) and for some \( \epsilon \in (0, J_F^+) \) it holds that

\[
\mathbb{E} \sum_{i=1}^{\tau} \Theta_i J_F^{-\epsilon} < \infty, \quad \mathbb{E} \sum_{i=1}^{\tau} \Theta_i J_F^{+\epsilon} < \infty,
\]

then

\[
L_F \mathbb{E} \sum_{i=1}^{\tau} \mathbb{P}(\Theta_i X_i > x) \lesssim \mathbb{P}(M_\tau > x) \lesssim L_F^{-1} \mathbb{E} \sum_{i=1}^{\tau} \mathbb{P}(\Theta_i X_i > x). \tag{2.3}
\]

(b) Assume that Assumption B is satisfied. If d.f. of r.v. \( Z_\tau := \sum_{i=1}^{\tau} \Theta_i \) satisfies \( \mathbb{P}(Z_\tau > x) = o(\bar{F}(x)) \) and there exists \( \epsilon \in (0, J_F^-) \) such that \( \mathbb{E}(X_1^+)^{1+\epsilon} < \infty \) and condition (2.2) is satisfied, then (2.3) holds.

The proof of the theorem immediately follows from the asymptotic lower and upper bounds given in Theorems 4.1 and 5.1, respectively.

**Remark 2.1** Note that under condition \( \mathbb{E} \tau < \infty \) the first requirement in (2.2) can be dropped because

\[
\mathbb{E} \sum_{i=1}^{\tau} \Theta_i J_F^{-\epsilon} = \mathbb{E} \left( \sum_{i=1}^{\tau} \Theta_i J_F^{-\epsilon} \mathbb{I}(\Theta_i \leq 1) \right) + \mathbb{E} \left( \sum_{i=1}^{\tau} \Theta_i J_F^{-\epsilon} \mathbb{I}(\Theta_i > 1) \right) 
\]

\[
\leq \mathbb{E} \tau + \mathbb{E} \sum_{i=1}^{\tau} \Theta_i J_F^{+\epsilon}.
\]

**Remark 2.2** Theorem 2.1 (b) involves dominance condition \( \mathbb{P}(Z_\tau > x) = o(\bar{F}(x)) \), which holds under appropriate asymptotic relations among the tail probabilities of \( \Theta \)'s, \( \tau \) and \( X \)'s. This issue was addressed in the note Dindiené and Leipus (2015). In particular, their Theorem 2 says that if Assumption B is satisfied, \( \mathbb{E}(X_1^+)^{1+\epsilon} < \infty \) and \( \Theta_1, \Theta_2, \ldots \) are identically distributed with
\( \mathbb{E} \Theta_i^{J_F^+ - \epsilon} < \infty \), then sufficient conditions for relation \( P(Z_r > x) = o(F(x)) \) (and, so, for (2.3) to hold) are (i) \( F_\Theta \in \mathcal{D} \) and \( F_r(x) = o(F_\Theta(x)) \) or (ii) \( F_r \in \mathcal{D}, \; E \tau < \infty \) and \( \overline{F_\Theta}(x) = o(F_r(x)) \), where \( \overline{F_\Theta}(x) := P(\Theta_1 > x), \; F_r(x) := P(\tau > x) \).

Next consider the case where \( \tau \) is an extended random variable, i.e. \( \tau \in \{1, 2, \ldots \} \cup \{\infty\} \) and \( 0 \leq P(\tau = \infty) \leq 1 \). For rigorousness, in this case we rename Assumption A by Assumption A’ and Assumption B by Assumption B’. As we can see in (2.4), the result in the latter case differs from that in Theorem 2.1 by the slightly worse asymptotic upper bound.

**Theorem 2.2**

(a) Assume that Assumption A’ is satisfied. If \( J_F^+ < 1 \) and there exists \( \epsilon \in (0, J_F^-) \) such that (2.2) holds, then

\[
L_F E \sum_{i=1}^\tau P(\Theta_i X_i > x) \lesssim P(M_r > x) \lesssim L_F^{-2} E \sum_{i=1}^\tau P(\Theta_i X_i > x). \tag{2.4}
\]

(b) Assume that Assumption B’ is satisfied. If \( P(Z_r > x) = o(F(x)) \) and there exists \( \epsilon \in (0, J_F^-) \) such that \( E(X_1^+)^{1+\epsilon} < \infty \) and condition (2.2) is satisfied, then relation (2.4) holds.

**Remark 2.3** Note that in the case \( P(\tau = \infty) > 0 \) assumption (2.2) can be replaced by the equivalent condition

\[
\sum_{i=1}^\infty E \Theta_i^{J_F^- - \epsilon} < \infty, \quad \sum_{i=1}^\infty E \Theta_i^{J_F^+ + \epsilon} < \infty.
\]

**Remark 2.4** It seems that, under bivariate upper tail independence structure, asymptotic relations (2.3) and (2.4) hold if condition \( E(X_1^+)^{1+\epsilon} < \infty \) is replaced by more general assumption \( J_F^+ \geq 1 \) and condition (2.2) is replaced by

\[
E \sum_{i=1}^\tau (E \Theta_i^{J_F^- - \epsilon})^{J_F^-_i + \epsilon} < \infty, \quad E \sum_{i=1}^\tau (E \Theta_i^{J_F^+ + \epsilon})^{J_F^+_i} < \infty \tag{2.5}
\]

for some \( \epsilon \in (0, J_F^-) \). However, (2.5) is much stronger than (2.2) if \( J_F^+ \geq 1 \) (see also Remark on p. 1146 in Olvera-Cravioto (2012)) and would require a different proof technique. In view of this limitation, we decided to use similar technique as was used in Olvera-Cravioto (2012).

Restricting the d.f. \( F \) to some smaller distribution class, some refinements in the previous result can be obtained.

**Corollary 2.1** Assume that the conditions of Theorem 2.2 hold and take \( F \in \mathcal{C} \) instead of \( F \in \mathcal{D} \). In such case it holds that

\[
P(M_r > x) \sim E \sum_{i=1}^\tau P(\Theta_i X_i > x). \tag{2.6}
\]
Proof. The statement of the corollary follows immediately from (2.4) because \( L_F = 1 \) for d.f. \( F \in \mathcal{C} \).

**Corollary 2.2** Assume that the conditions of Theorem 2.2 hold and take \( F \in \text{ERV}(\alpha_1, \alpha_2) \) for some \( 0 < \alpha_1 \leq \alpha_2 < \infty \) instead of \( F \in \mathcal{D} \). Then

\[
\mathbb{F}(x) \leq \sum_{i=1}^{\tau} (\Theta_i^{\alpha_1} \wedge \Theta_i^{\alpha_2}) \leq \mathbb{P}(M_\tau > x) \leq \mathbb{F}(x) \sum_{i=1}^{\tau} (\Theta_i^{\alpha_1} \vee \Theta_i^{\alpha_2}).
\]

(2.7)

In particular, if \( F \in \mathcal{R}_- \), then

\[
\mathbb{P}(M_\tau > x) \sim \mathbb{F}(x) \sum_{i=1}^{\tau} \Theta_i^\alpha.
\]

**Proof.** We have \( J_F = \alpha_1, J_{F^+} = \alpha_2 \). By Lemma 3.2 below, for fixed \( 0 < \alpha_1 - \epsilon < \alpha_2 + \epsilon < \infty \) there exist some positive constants \( x^* \) and \( c_0 \), irrespective to \( \Theta_1, \Theta_2, \ldots \), such that for each \( i \geq 1 \) and all \( x \geq x^* \) it holds

\[
\mathbb{P}(\Theta_i X_i > x) \leq c_0 \mathbb{F}(x) (E\Theta_i^{\alpha_1-\epsilon} \vee E\Theta_i^{\alpha_2+\epsilon}).
\]

Hence, from (2.4), using the Fatou’s lemma and Theorem 3.5(v) of Cline and Samorodnitsky (1994), we obtain the right-hand side of (2.7). Analogously, the Fatou’s lemma and Theorem 3.5 of Cline and Samorodnitsky (1994) imply the left-hand side of (2.7). □

**Remark 2.5** The results above are related to the corresponding results in Olvera-Cravioto (2012), although there is no exact inclusion between the two papers. Olvera-Cravioto (2012) considered the case of i.i.d. r.v.s. \( X_1, X_2, \ldots \) with \( F \in \mathcal{C} \), \( 0 < J_F \leq J_{F^+} < \infty \), \( E|X_1|^{1+\epsilon} \) with some \( 0 < \epsilon < J_F \), \( (\tau, \Theta_1, \Theta_2, \ldots) \) being nonnegative random vector independent of the \( \{X_i\} \) with \( \tau \in \mathbb{N} \cup \{\infty\} \), satisfying (2.2). Theorem 2.1 of Olvera-Cravioto (2012) states that relation (2.6) holds under: (a) \( E X_1 < 0 \), (b) \( E X_1 = 0 \) and \( P(Z_\tau > x) = O(\mathbb{F}(x)) \), (c) \( E X_1 > 0 \) and \( P(Z_\tau > x) = o(\mathbb{F}(x)) \). Our Theorems 2.1, 2.2 allow more general dependence structure (ENUOD or bivariate upper tail independent structure), larger class, \( \mathcal{D} \), of heavy-tailed distributions; moreover, in the case \( J_{F^+} < 1 \), neither condition \( E(X_1^+)^{1+\epsilon} < \infty \) nor \( P(Z_\tau > x) = o(\mathbb{F}(x)) \) is required. Also, these conditions are not needed when proving the lower bound (see Theorem 4.1 below). Though, we assume the independence between \( \tau \) and \( (\Theta_1, \Theta_2, \ldots) \), and the left tail condition \( F(-x) = o(\mathbb{F}(x)) \).

## 3 Auxiliary lemmas

In this section we present some auxiliary lemmas which are used in the proofs of our main results.

The first lemma deals with the maximum of randomly weighted sums with a finite/infinite (nonrandom) number of increments and is due to Theorem 2.1 of Gao and Wang (2010).
Lemma 3.1 Let \( \{X_1, X_2, \ldots\} \) be a sequence of bivariate upper tail independent real-valued r.v.s with common distribution \( F \in \mathcal{D} \), for which \( J_F^- > 0 \) and \( F(-x) = o(F(x)) \). Let \( \{\Theta_1, \Theta_2, \ldots\} \) be a sequence of nonnegative nondegenerate at zero r.v.s such that \( E\Theta_i^{J_F^-+\epsilon} < \infty \) for some \( \epsilon > 0 \) and all \( i \geq 1 \). Assume that \( \{X_1, X_2, \ldots\} \) and \( \{\Theta_1, \Theta_2, \ldots\} \) are mutually independent. Then

\[
L_F \sum_{i=1}^{n} P(\Theta_i X_i > x) \leq P(M_n > x) \leq L_F^{-1} \sum_{i=1}^{n} P(\Theta_i X_i > x)
\]

for any fixed \( n \geq 1 \). If, in particular, \( J_F^- < 1 \) and for some \( 0 < \epsilon < \min\{J_F^-, 1 - J_F^+\} \) it holds

\[
\sum_{i=1}^{\infty} E\Theta_i^{J_F^- - \epsilon} < \infty, \quad \sum_{i=1}^{\infty} E\Theta_i^{J_F^+ + \epsilon} < \infty,
\]

then

\[
L_F \sum_{i=1}^{\infty} P(\Theta_i X_i > x) \leq P(M_\infty > x) \leq L_F^{-2} \sum_{i=1}^{\infty} P(\Theta_i X_i > x).
\]

The next three lemmas describe some useful properties of the product of independent r.v.s in the presence of heavy tails. The first lemma can be found in Yi et al. (2011, Lemma 1).

Lemma 3.2 Let \( X \) be a real-valued r.v. with d.f. \( F \in \mathcal{D} \) such that \( J_F^- > 0 \). Let \( \Theta \) be a nonnegative r.v., nondegenerate at zero and independent of \( X \). Then, for any fixed \( 0 < p_1 < J_F^- \leq J_F^+ < p_2 < \infty \), there exists some constants \( c_1, d_1 \), irrespective to \( \Theta \), such that for all \( x \geq d_1 \)

\[
P(\Theta X > x) \leq c_1 F(x)(E\Theta^{p_1} \vee E\Theta^{p_2}).
\]

The second lemma is due to Theorem 3.3 (iv) of Cline and Samorodnitsky (1994).

Lemma 3.3 Let \( X \) be a real-valued r.v. with d.f. \( F \in \mathcal{D} \) and let \( \Theta \) be a nonnegative nondegenerate at zero r.v. independent of \( X \). If, in addition, \( E\Theta^{J_F^+ + \epsilon} < \infty \) for some \( \epsilon > 0 \) then

\[
0 < \liminf \frac{P(\Theta X > x)}{F(x)} \leq \limsup \frac{P(\Theta X > x)}{F(x)} < \infty.
\]

The last lemma is due to Yang et al. (2012) (see also Yi et al. (2011, Lemma 2)).

Lemma 3.4 Let \( X \) and \( \Theta \) be two independent nonnegative r.v.s, where \( F \in \mathcal{D} \) and \( \Theta \) is nondegenerate at zero. Then, for any \( p > J_F^+ \), there exists a constant \( c_2 = c_2(F, p) > 0 \) such that for any \( x > 0 \)

\[
E(\Theta X)^p 1_{\Theta X \leq x} \leq c_2 x^p P(\Theta X > x).
\]
4 Asymptotic lower bound

In this section, we obtain the asymptotic lower bound for $P(M > x)$ under the bivariate upper tail independence structure, which is wider than the ENUOD structure. Therefore the lower asymptotic bound in Theorem 2.1 in both cases $(a)$ and $(b)$ follows immediately from Theorem 4.1 below. Note also that condition $P(Z > x) = o(F(x))$ as well as the moment condition on $X_i^+$ are not necessary for this bound.

**Theorem 4.1** Suppose that Assumption A is satisfied and (2.2) holds for some $\epsilon \in (0, J_F)$. Then

$$P(M > x) \geq L_F E \sum_{i=1}^{\tau} P(\Theta_i X_i > x).$$

(4.1)

Proof of Theorem 4.1. Take $N^* \geq 1$ such that $P(\tau = N^*) > 0$. By independence of $\{X_1, X_2, \ldots\}$, $\{\Theta_1, \Theta_2, \ldots\}$ and $\tau$, for every $N > N^*$ it holds

$$\hat{\pi} := \liminf \frac{P(M > x)}{E \sum_{i=1}^{\tau} P(\Theta_i X_i > x)} \geq \liminf \frac{P(M > x, \tau \leq N)}{E \sum_{i=1}^{\tau} P(\Theta_i X_i > x)} \geq (1 - \pi_N) \liminf \frac{P(M > x, \tau \leq N)}{E \sum_{i=1}^{\tau} P(\Theta_i X_i > x)} \geq (1 - \pi_N) \liminf \min_{1 \leq n \leq N} \frac{P(M_n > x)}{\sum_{i=1}^{\tau} P(\Theta_i X_i > x)},$$

where

$$\pi_N := \limsup \frac{E \sum_{i=1}^{\tau} P(\Theta_i X_i > x)}{E \sum_{i=1}^{\tau} P(\Theta_i X_i > x)}.$$  

Hence, according to Lemma 3.1,

$$\hat{\pi} \geq (1 - \pi_N)L_F.$$  

(4.2)

Lemmas 3.2 and 3.3 imply that

$$\pi_N \leq c_3 \limsup \frac{P(\tau > N) \sum_{i=1}^{\tau} (E \Theta_i^{\bar{F}} - \epsilon \vee E \Theta_i^{\bar{F}} + \epsilon)}{P(\tau = N^*) P(\Theta_1 X_1 > x)} \leq c_4 E \sum_{i=1}^{\tau} (\Theta_i^{\bar{F}} - \epsilon + \Theta_i^{\bar{F}} + \epsilon) \rightarrow 0, \quad N \rightarrow \infty,$$

by (2.2). This, together with (4.2), implies that $\hat{\pi} \geq L_F$. Hence, the desired lower bound (4.1) holds.  \[\square\]
5 Asymptotic upper bound

In this section, we obtain the upper bound for the tail probability of $M_r$.

**Theorem 5.1** (a) Suppose that Assumption A is satisfied and condition (2.2) holds with some $\epsilon \in (0, J_F^-)$. If, in addition, $J_F^- + F < 1$, then

$$P(M_r > x) \lesssim L_F^{-1} E \sum_{i=1}^{\tau} P(\Theta_i X_i > x). \tag{5.1}$$

(b) Suppose that Assumption B is satisfied. Assume, in addition, that $P(Z_r > x) = o(F(x))$ and there exists $\epsilon \in (0, J_F^+)$ such that $E(X_i^+)^{1+\epsilon} < \infty$ and condition (2.2) is satisfied. Then (5.1) holds.

**Remark 5.1** Without loss of generality, we can assume that (2.2) holds for $0 < \epsilon < \min\{J_F^-, 1 - J_F^+\}$. This follows from the following inequalities:

$$E \sum_{i=1}^{\tau} \Theta_i^{J_F^- - \epsilon_1} = E \sum_{i=1}^{\tau} \left( \Theta_i^{J_F^- - \epsilon_1} I(\Theta_i \leq 1) + \Theta_i^{J_F^- - \epsilon_1} I(\Theta_i > 1) \right) \leq E \sum_{i=1}^{\tau} \Theta_i^{J_F^- - \epsilon_2} + E \sum_{i=1}^{\tau} \Theta_i^{J_F^+ + \epsilon_2},$$

$$E \sum_{i=1}^{\tau} \Theta_i^{J_F^+ + \epsilon_1} \leq E \sum_{i=1}^{\tau} \Theta_i^{J_F^- - \epsilon_2} + E \sum_{i=1}^{\tau} \Theta_i^{J_F^+ + \epsilon_2},$$

where $0 < \epsilon_1 < \epsilon_2 < J_F^-.$

**Remark 5.2** Note that in the case $J_F^+ < 1$ the condition $P(Z_r > x) = o(F(x))$ is satisfied. Indeed, taking $J_F^+ + \epsilon < 1$, we have by Markov’s inequality

$$P(\sum_{i=1}^{\tau} \Theta_i > x) \leq \frac{E(\sum_{i=1}^{\tau} \Theta_i)^{J_F^+ + \epsilon}}{x^{J_F^+ + \epsilon} F(x)} \leq \frac{E \sum_{i=1}^{\tau} \Theta_i^{J_F^+ + \epsilon}}{x^{J_F^+ + \epsilon} F(x)} \rightarrow 0,$$

where the last relations are due to c.r.-inequality $(\sum_{i=1}^{n} a_i)^r \leq \sum_{i=1}^{n} a_i^r$ with $a_i \geq 0$, $i = 1, \ldots, n$, $0 < r \leq 1$, assumption (2.2) and (see Lemma 2.1) convergence $x^{J_F^+ + \epsilon} F(x) \rightarrow \infty$.

**Proof of Theorem 5.1** (a). Take $N^* \geq 1$ such that $P(\tau = N^*) > 0$. Then for any $N > N^*$ we have

$$P(M_r > x) \leq \frac{P(M_r > x, \tau \leq N)}{E \sum_{i=1}^{\tau} P(\Theta_i X_i > x)} + \frac{P(M_r > x, \tau > N)}{E \sum_{i=1}^{\tau} P(\Theta_i X_i > x)} \leq \max_{1 \leq n \leq N} \frac{P(M_n > x)}{\sum_{i=1}^{n} P(\Theta_i X_i > x)} + \frac{P(M_r > x, \tau > N)}{P(\tau = N^*) P(\Theta_1 X_1 > x)}.$$
Thus, applying Lemmas 3.1 and 3.3 we get

\[
\limsup \frac{P(M_T > x)}{E \sum_{i=1}^{\tau} P(\Theta_i, X_i > x)} \leq \frac{1}{L_F} + c_5 \limsup \frac{\sum_{n=N+1}^{\infty} P(\tau = n) P(M_n > x)}{F(x)}, \tag{5.2}
\]

where \(c_5\) is some positive constant.

Next, for \(x > 0\), write

\[
P(M_n > x) \leq P\left(\sum_{i=1}^{n} \Theta_i, X_i > x\right) = P\left(\sum_{i=1}^{n} \Theta_i, X_i > x, \bigcup_{k=1}^{n} \{\Theta_k, X_k > x\}\right) + P\left(\sum_{i=1}^{n} \Theta_i, X_i > x, \bigcap_{k=1}^{n} \{\Theta_k, X_k \leq x\}\right)
\]

\[
\leq \sum_{i=1}^{n} P(\Theta_i, X_i > x) + P\left(\sum_{i=1}^{n} \Theta_i, X_i \mathbb{1}_{(\Theta_i, X_i \leq x)} > x\right). \tag{5.3}
\]

By Remark 5.1, without loss of generality, we can assume that \(0 < \epsilon < \min\{J_F^{-1}, 1 - J_F^{-1}\}\). Hence, the second term in (5.3) can be estimated using Markov’s inequality as follows:

\[
P\left(\sum_{i=1}^{n} \Theta_i, X_i \mathbb{1}_{(\Theta_i, X_i \leq x)} > x\right) \leq \frac{E\left(\sum_{i=1}^{n} \Theta_i, X_i \mathbb{1}_{(\Theta_i, X_i \leq x)}\right)^{J_F^{-1} + \epsilon}}{x^{J_F^{-1} + \epsilon}} \leq \frac{\sum_{i=1}^{n} E(\Theta_i, X_i)J_F^{J_F^{-1} + \epsilon} \mathbb{1}_{(\Theta_i, X_i \leq x)}}{x^{J_F^{-1} + \epsilon}}, \tag{5.4}
\]

where in the last step we have used the \(c_r\)-inequality.

For the expectation in (5.4), we have by Lemma 3.4

\[
E(\Theta_i, X_i)^{J_F^{-1} + \epsilon} \mathbb{1}_{(\Theta_i, X_i \leq x)} \leq c_6 x^{J_F^{-1} + \epsilon} P(\Theta_i, X_i > x).
\]

Hence, from (5.3) and then from Lemma 3.2 we get

\[
P(M_n > x) \leq (1 + c_6) \sum_{i=1}^{n} P(\Theta_i, X_i > x)
\]

\[
\leq (1 + c_6) c_7 F(x) \sum_{i=1}^{n} (E\Theta_i^{J_F^{-1} - \epsilon} + E\Theta_i^{J_F^{-1} + \epsilon}),
\]

where \(c_6, c_7\) are positive constants, that do not depend on \(n\).

Substituting the last inequality into (5.2), we obtain

\[
\limsup \frac{P(M_T > x)}{E \sum_{i=1}^{\tau} P(\Theta_i, X_i > x)} \leq \frac{1}{L_F} + c_5 c_7 (1 + c_6) \sum_{n=N+1}^{\infty} P(\tau = n) \sum_{i=1}^{n} (E\Theta_i^{J_F^{-1} - \epsilon} + E\Theta_i^{J_F^{-1} + \epsilon})
\]

for all \(N > N^*\). This estimate and condition (2.2) imply (5.1). \(\square\)
PROOF OF THEOREM 5.1 (b). Let, as above, $N^*$ be a positive integer with $P(\tau = N^*) > 0$. Then, by (5.2), for any $N > N^*$ we have

$$\limsup_{n \to \infty} \frac{P(M_n > x)}{E_n \sum_{i=1}^{N} P(\Theta_i X_i > x)} \leq \frac{1}{L_F} + c_3 \limsup_{n \to \infty} \frac{\sum_{n=N+1}^{\infty} P(\tau = n)P(\sum_{i=1}^{n} \Theta_i X_i^+ > x)}{F(x)}. \quad (5.5)$$

In order to estimate $P(\sum_{i=1}^{n} \Theta_i X_i^+ > x)$ uniformly in $n$, we use similar ideas as used in Olvera-Cravioto (2012). By $E(X_i^+)^{1+\epsilon} < \infty$, we have that $\mu_+ := \int_0^\infty xdF(x) < \infty$, which implies $J_F \geq 1$. For all $n \geq 1$, all $x \geq 2$ and all $\Delta_1, \Delta_2 \in (0, 1)$ we have

$$P\left(\sum_{i=1}^{n} \Theta_i X_i^+ > x\right) = P\left(\sum_{i=1}^{n} \Theta_i X_i^+ > x, \max_{1 \leq i \leq n} \Theta_i X_i^+ \geq \Delta_1 x\right)$$

$$+ P\left(\sum_{i=1}^{n} \Theta_i X_i^+ > x, \max_{1 \leq i \leq n} \Theta_i X_i^+ \leq \Delta_1 x, \sum_{i=1}^{n} \Theta_i > \frac{x}{\mu_+ + \Delta_1}\right)$$

$$+ P\left(\sum_{i=1}^{n} \Theta_i X_i^+ > x, \max_{1 \leq i \leq n} \Theta_i X_i^+ \leq \Delta_1 x, \sum_{i=1}^{n} \Theta_i \leq \frac{x}{\mu_+ + \Delta_1}, \max_{1 \leq i \leq n} \Theta_i \geq x \Delta_2\right)$$

$$+ P\left(\sum_{i=1}^{n} \Theta_i X_i^+ > x, \max_{1 \leq i \leq n} \Theta_i X_i^+ \leq \Delta_1 x, \sum_{i=1}^{n} \Theta_i \leq \frac{x}{\mu_+ + \Delta_1}, \max_{1 \leq i \leq n} \Theta_i \leq x \Delta_2\right)$$

$$=: s_1 + s_2 + s_3 + s_4.$$ 

For the first three terms we have:

$$s_1 \leq \sum_{i=1}^{n} P(\Theta_i X_i^+ > \Delta_1 x) \leq c_5 \bar{F}(\Delta_1 x) \sum_{i=1}^{n} (\Theta_i^{J_F-\epsilon} \vee \Theta_i^{J_F+\epsilon}), \quad (5.6)$$

because of Lemma 3.2, where $x$ is large enough ($x \geq c_9 = c_9(J_F, J_F^+, \epsilon)$);

$$s_2 \leq P\left(Z_n > \frac{x}{\mu_+ + \Delta_1}\right) \quad (5.7)$$

with $Z_n := \sum_{i=1}^{n} \Theta_i$, $n \geq 1$, and, by Markov’s inequality,

$$s_3 \leq \sum_{i=1}^{n} P(\Theta_i > x \Delta_2) \leq \frac{1}{x^{J_F+\epsilon/2}} \sum_{i=1}^{n} \Theta_i^{J_F+\epsilon} \quad (5.8)$$

provided $\Delta_2 \geq (J_F^+ + \epsilon/2)/(J_F^+ + \epsilon)$ and $x \geq 1$.

Now we will estimate (most difficult) term $s_4$. Write

$$s_4 \leq P\left(\sum_{i=1}^{n} \Theta_i X_i^+ \mathbb{1}_{\{\Theta_i X_i^+ \leq \Delta_1 x\}} > x, \sum_{i=1}^{n} \Theta_i \leq \frac{x}{\mu_+ + \Delta_1}, \max_{1 \leq i \leq n} \Theta_i \leq x \Delta_2\right).$$

For any real $u$ and any nonnegative $v$ denote $u^{(v)} := u \mathbb{1}_{\{u \leq v\}} + v \mathbb{1}_{\{u > v\}}$. Obviously,

$$s_4 \leq P\left(\sum_{i=1}^{n} (\Theta_i X_i^+)^{\Delta_1 x} > x, \sum_{i=1}^{n} \Theta_i \leq \frac{x}{\mu_+ + \Delta_1}, \max_{1 \leq i \leq n} \Theta_i \leq x \Delta_2\right)$$

$$= \int_{D_n} \cdots \int_{D_n} P\left(\sum_{i=1}^{n} (t_i X_i^+)^{\Delta_1 x} > x\right) dP(\Theta_1 \leq t_1, \ldots, \Theta_n \leq t_n), \quad (5.9)$$

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where
\[ D_n = \left\{ 0 \leq t_j \leq x^{\Delta^2}, j = 1, \ldots, n, \sum_{i=1}^n t_i \leq \frac{x}{\mu_+ + \Delta^1} \right\}. \]

By the ENUOD property of r.v.s \( X_1, X_2, \ldots \) and Lemma 2.2, for any \( h > 0 \)
\[
P\left( \sum_{i=1}^n (t_iX_i^+(\Delta_1x)) > x \right) \leq e^{-hx} \exp \left\{ h \sum_{i=1}^n (t_iX_i^+(\Delta_1x)) \right\}
\leq \kappa e^{-hx} \prod_{i=1}^n Ee^{h(t_iX_i^+(\Delta_1x))}.
\]

If \( h > (\Delta_1x)^{-1} \) then
\[ Ee^{h(t_iX_i^+(\Delta_1x))} = P(t_iX_i \leq 0) + e^{h\Delta_1x} P(t_iX_i > \Delta_1x) + \left( \int_{(0,1/h]} + \int_{(1/h,\Delta_1x]} \right) e^{hy} dP(t_iX_i \leq y). \]

Here,
\[ \int_{(0,1/h]} e^{hy} dP(t_iX_i \leq y) \leq \int_{(0,1/h]} (e^{hy} - 1 - hy) dP(t_iX_i \leq y) + P(0 < t_iX_i \leq 1/h) + ht_i\mu_+ \]
and, using integration by parts,
\[ \int_{(1/h,\Delta_1x]} e^{hy} dP(t_iX_i \leq y) = -e^{h\Delta_1x} P(t_iX_i > \Delta_1x) + eP(t_iX_i > 1/h) \]
\[ + h \int_{(1/h,\Delta_1x]} e^{hy} P(t_iX_i > y) dy. \]

The last three relations imply
\[ Ee^{h(t_iX_i^+(\Delta_1x))} \leq 1 + ht_i\mu_+ + eP(t_iX_i > 1/h) + \int_{(0,1/h]} (e^{hy} - 1 - hy) dP(t_iX_i \leq y) \]
\[ + h \int_{1/h}^{\Delta_1x} e^{hy} P(t_iX_i > y) dy. \]

Applying the inequality
\[ e^z - 1 - z \leq z^{1+\epsilon} e^z \quad \text{for any } \epsilon \in (0,1) \text{ and } z \geq 0, \]
we obtain
\[ \int_{(0,1/h]} (e^{hy} - 1 - hy) dP(t_iX_i \leq y) \leq e(h \mu_+)^{1+\epsilon} E(X_i^+)^{1+\epsilon} \]
with \( E(X_i^+)^{1+\epsilon} < \infty \) by the assumption of the theorem.

By Markov’s inequality, we have
\[ eP(t_iX_i > 1/h) + h \int_{1/h}^{\Delta_1x} e^{hy} P(t_iX_i > y) dy \leq t_i^{1+\epsilon} E(X_i^+)^{1+\epsilon} \left( eh^{1+\epsilon} + h \int_{1/h}^{\Delta_1x} y^{-(1+\epsilon)} e^{hy} dy \right) \]
\[ \leq c_1 t_i^{1+\epsilon} \frac{e^{h\Delta_1x}}{(\Delta_1x)^{1+\epsilon}} E(X_i^+)^{1+\epsilon}, \]
\[ 14 \]
because for $h > (\Delta_1 x)^{-1}$

$$eh^{1+\epsilon} + h \int_{1/h}^{\Delta_1 x} y^{-(1+\epsilon)} e^{hy} dy = eh^{1+\epsilon} + h \left( \int_{1/h}^{\Delta_1 x/2} y^{-(1+\epsilon)} dy + \int_{\Delta_1 x/2}^{\Delta_1 x} y^{-(1+\epsilon)} dy \right)$$

$$\leq eh^{1+\epsilon} + h e^{h\Delta_1 x/2} \int_{1/h}^{\Delta_1 x/2} y^{-(1+\epsilon)} dy + h \left( \frac{\Delta_1 x}{2} \right)^{-(1+\epsilon)} \int_{\Delta_1 x/2}^{\Delta_1 x} e^{hy} dy$$

$$\leq eh^{1+\epsilon} + h e^{h\Delta_1 x/2} \frac{h^{1+\epsilon}}{\epsilon} + \left( \frac{\Delta_1 x}{2} \right)^{-(1+\epsilon)} e^{h\Delta_1 x}$$

$$= \frac{e^{h\Delta_1 x}}{(\Delta_1 x)^{1+\epsilon}} \left( e \left( \frac{h\Delta_1 x}{\epsilon} \right)^{1+\epsilon} + e^{-h\Delta_1 x/2} \left( \frac{h\Delta_1 x}{\epsilon} \right)^{1+\epsilon} + 2^{1+\epsilon} \right)$$

$$\leq c_{10} \frac{e^{h\Delta_1 x}}{(\Delta_1 x)^{1+\epsilon}},$$

where $c_{10} = c_{10}(\epsilon) := \sup_{r \geq 1} \left( v^{1+\epsilon} e^{-v+1} + v^{1+\epsilon} e^{-v/2} + 2^{1+\epsilon} \right) < \infty$. Combining (5.10), (5.11) and (5.12) we obtain that for all $i \geq 1$ and $h > (\Delta_1 x)^{-1}$

$$E e^{h(t_i X_i^+)^{\Delta_1 x}} \leq 1 + h t_i \mu_+ + e^{(ht_i)^{1+\epsilon} E(X_i^+)^{1+\epsilon}} + c_{10} t_i^{1+\epsilon} \frac{e^{h\Delta_1 x}}{(\Delta_1 x)^{1+\epsilon}} E(X_i^+)^{1+\epsilon}$$

$$\leq 1 + h t_i \mu_+ + c_{11} h t_i \left( (ht_i)^{\epsilon} + \frac{t_i^{\epsilon} e^{h\Delta_1 x}}{(\Delta_1 x)^{1+\epsilon}} \right), \quad (5.13)$$

where $c_{11}$ is a constant irrespective to $x$, $h$ and $t_i$.

Next, split the set $D_n$ in (5.9) into two nonintersecting subsets $D_n^{(1)}$ and $D_n^{(2)}$, where

$$D_n^{(1)} = \left\{ 0 \leq t_j \leq x^{\Delta_2}, j = 1, \ldots, n, \sum_{i=1}^{n} t_i \leq \frac{x}{\log x} \right\},$$

$$D_n^{(2)} = \left\{ 0 \leq t_j \leq x^{\Delta_2}, j = 1, \ldots, n, \frac{x}{\log x} < \sum_{i=1}^{n} t_i \leq \frac{x}{\mu_+ + \Delta_1} \right\}$$

and denote

$$s_{4k} := \int_{D_n^{(k)}} \cdots \int P \left( \sum_{i=1}^{n} (t_i X_i)^{\Delta_1 x} > x \right) dP(\Theta_1 \leq t_1, \ldots, \Theta_n \leq t_n), \quad k = 1, 2.$$ 

Clearly, $s_4 \leq s_{41} + s_{42}$.

On the set $D_n^{(1)}$, (5.13) and inequality $1 + z \leq e^z$, $z \in \mathbb{R}$ yield that

$$e^{-hx} \prod_{i=1}^{n} E e^{h(t_i X_i^+)^{\Delta_1 x}} \leq \exp \left\{ -hx + \frac{hx}{\log x} \mu_+ + c_{11} \frac{hx}{\log x} \left( (hx)^{\epsilon} + \frac{x^{\Delta_2 e^{h\Delta_1 x}}}{(\Delta_1 x)^{1+\epsilon}} \right) \right\}. \quad (5.14)$$

Until now, $\Delta_1$ was an arbitrary constant in $(0,1)$. Assume from now that $\Delta_1 \leq \epsilon (1 - \Delta_2)/(\ell_{F^+} + \epsilon)$ and take $h = h_1 = (\ell_{F^+} + \epsilon) \log x/x$ (obviously, $h_1 > (\Delta_1 x)^{-1}$ for large $x$). In such a case, for $x > 1$, we have

$$\frac{x^{\Delta_2 e^{h_1 \Delta_1 x}}}{(\Delta_1 x)^{1+\epsilon} h_1} \leq \frac{c_{12}}{\log x}$$
with constant $c_{12} = c_{12}(\epsilon, \Delta_1, J_F^+)$, irrespective to $x$. Hence, for large $x$, 

$$-h_1 x + h_1 \frac{x}{\log x} \mu_+ + c_{11} h_1 \frac{x}{\log x} \left( (h_1 x)_{\Delta_2}^\epsilon + \frac{x^{2+\epsilon} e^{h_1 x} e^{\Delta_1 x}}{(\Delta_1 x)^{1+\epsilon h_1}} \right)$$

$$= - (J_F^+ + \epsilon) \log x + (J_F^+ + \epsilon) \mu_+ + c_{11} (J_F^+ + \epsilon) \left( \frac{(J_F^+ + \epsilon) \log x}{x^{1-\Delta_2}} \right) + \frac{c_{12}}{\log x}$$

$$\leq (J_F^+ + \epsilon) \left( - \log x + \mu_+ + \frac{c_{13}}{\log x} \right)$$

$$\leq (J_F^+ + \epsilon) (- \log x + 2 \mu_+) \quad (5.15)$$

and from (5.14), (5.15) we obtain that for sufficiently large $x$

$$s_{41} \leq \kappa e^{-h_1 x} \int \cdots \int \prod_{i=1}^n E e^{h_1 (t_i X_i^+)^{(\Delta_1 + \epsilon)}} dP(\Theta_1 \leq t_1, \ldots, \Theta_n \leq t_n)$$

$$\leq c_{14} x^{-(J_F^+ + \epsilon)}. \quad (5.16)$$

Similarly, we get from (5.13) that on the set $D_n^{(2)}$

$$e^{-h_2 x} \prod_{i=1}^n E e^{h_2 (t_i X_i^+)^{(\Delta_1 + \epsilon)}}$$

$$\leq \exp \left\{ - h_2 x + \frac{h_2 x}{\mu_+ + \Delta_1} \mu_+ + c_{11} \frac{h_2 x}{\mu_+ + \Delta_1} \left( (h_2 x)_{\Delta_2}^\epsilon + \frac{x^{2+\epsilon} e^{h_2 x} e^{\Delta_1 x}}{(\Delta_1 x)^{1+\epsilon h_1}} \right) \right\}. \quad (5.17)$$

Choose $h = h_2 = \epsilon (\Delta_1 x)^{-1} \log (\Delta_1 x^{-1} - \Delta_2)$ (obviously, $h_2 > (\Delta_1 x)^{-1}$ for large $x$). Then the expression under curly brackets in (5.17) equals

$$\frac{- \frac{\epsilon}{\Delta_1} \log (\Delta_1 x^{-1} - \Delta_2) + \frac{\epsilon \mu_+}{\Delta_1 (\mu_+ + \Delta_1)} \log (\Delta_1 x^{-1} - \Delta_2)}{\epsilon \log (\Delta_1 x^{-1} - \Delta_2) + \frac{1}{\epsilon \log (\Delta_1 x^{-1} - \Delta_2)}}$$

$$= - \frac{\epsilon}{\Delta_1} \log (\Delta_1 x^{-1} - \Delta_2) \left( 1 - \frac{\mu_+}{\mu_+ + \Delta_1} - \epsilon(x) \right), \quad (5.18)$$

where $\epsilon(x)$ is some vanishing function. From (5.17) and (5.18) we obtain that for sufficiently large $x$

$$s_{42} \leq \kappa e^{-h_2 x} \int \cdots \int \prod_{i=1}^n E e^{h_2 (t_i X_i^+)^{(\Delta_1 + \epsilon)}} dP(\Theta_1 \leq t_1, \ldots, \Theta_n \leq t_n)$$

$$\leq c_{15} P \left( Z_n > \frac{x}{\log x} \right) x^{-c_{16}}, \quad (5.19)$$

where $c_{15}$ and $c_{16}$ are some positive constants depending on $\kappa, \epsilon, \Delta_1, \Delta_2, \mu_+$.

Estimates (5.16) and (5.19) imply

$$s_4 \leq c_{14} x^{-(J_F^+ + \epsilon)} + c_{15} P \left( Z_n > \frac{x}{\log x} \right) x^{-c_{16}}. \quad (5.20)$$
Collecting the estimators for \( s_1, s_2, s_3, s_4 \), given in (5.6), (5.7), (5.8), (5.20), and substituting them into (5.5) we obtain

\[
\limsup \frac{P(M_\tau > x)}{E^{\sum_{i=1}^{\tau} P(\Theta_i X_i > x)}} \leq \frac{1}{L_F} + c_5c_8 \limsup \frac{F(\Delta_1 x)}{F(x)} \left( E\mathbb{I}_{\{\tau > N+1\}} \sum_{i=1}^{\tau} \Theta_i J_\tau^{+\epsilon} - E\mathbb{I}_{\{\tau > N+1\}} \sum_{i=1}^{\tau} \Theta_i J_\tau^{-\epsilon} \right) + c_5 \limsup \frac{P(Z_\tau > x/(\mu_+ + \Delta_1))}{F(x)} + c_5 \limsup \frac{1}{x^{\epsilon_1 + 2/4}F(x)} E\sum_{i=1}^{\tau} \Theta_i J_\tau^{+\epsilon} + c_5 c_{14} \limsup \frac{1}{x^{\epsilon_1 + 2/4}F(x)} + c_5 c_{15} \limsup \frac{P(Z_\tau > x/\log x)}{F(x/\log x)} \frac{F(x/\log x)}{x^{\epsilon_1 + 2/4}F(x)}
\]

provided that \((J_\tau^+ + \epsilon/2)/(J_\tau^+ + \epsilon) \leq \Delta_2 < 1\) and \(0 < \Delta_1 \leq \epsilon(1 - \Delta_2)/(J_\tau^+ + \epsilon)\). Since all the constants \(c_5, c_8, c_{14}, c_{15}, c_{16}\) are positive and do not depend on \(x\), the desired estimate (5.1) follows from conditions of the theorem and Lemma 2.1.

\section{The case of extended r.v. \( \tau \)}

**Proof of Theorem 2.2.** (a) In the case \(P(\tau = \infty) = 1\), the statement of the theorem follows from Lemma 3.1, because without a loss of generality we can assume that \(0 < \epsilon < \min\{J_\tau^- - 1, -J_\tau^+\}\). If \(P(\tau = \infty) = 0\), then the statement follows from Theorem 4.1 and Theorem 5.1 (a).

Consider the case \(0 < P(\tau = \infty) < 1\) and define \(\tau^* := \tau \mathbb{I}_{\{\tau < \infty\}}\). If \(x > 0\), then \(P(M_\tau > x) = P(M_\tau > x) + P(\tau = \infty)P(M_\tau > x)\) and \(E\sum_{i=1}^{\tau} P(\Theta_i X_i > x) = E\sum_{i=1}^{\tau^*} P(\Theta_i X_i > x) + P(\tau = \infty)\sum_{i=1}^{\infty} P(\Theta_i X_i > x)\). Hence, the upper bound follows from Lemma 3.1 and Theorem 5.1 (a), whereas the lower bound follows from Lemma 3.1 and Theorem 4.1.

(b) Assume \(0 < P(\tau = \infty) < 1\), because the results in cases \(P(\tau = \infty) = 0\) and \(P(\tau = \infty) = 1\) follow immediately from Theorem 4.1, Theorem 5.1 (b) and Lemma 3.1. In view of Theorems 4.1 and 5.1 (b), it suffices to prove that

\[
L_F \leq \liminf \sum_{i=1}^{\infty} P(\Theta_i X_i > x) \leq \limsup \sum_{i=1}^{\infty} P(\Theta_i X_i > x) \leq \frac{L_F^2}{2}.
\]

For any \(x > 0\), \(\delta \in (0, 1)\) and \(N \geq 1\) we have

\[
P(M_\infty > x) \leq P\left( \sum_{i=1}^{N} \Theta_i X_i^+ > (1 - \delta)x \right) + P\left( \sum_{i=N+1}^{\infty} \Theta_i X_i^+ > \delta x \right).
\]

Hence, using Lemma 3.1 and Lemma 3.3 we have

\[
\limsup \frac{P(M_\infty > x)}{\sum_{i=1}^{\infty} P(\Theta_i X_i > x)} \leq L_F^{-1} \limsup \max_{1 \leq i \leq N} \frac{P(\Theta_i X_i > (1 - \delta)x)}{P(\Theta_i X_i > x)} + c_{17} \limsup \frac{P(\sum_{i=N+1}^{\infty} \Theta_i X_i^+ > \delta x)}{F(x)},
\]

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where \( c_{17} \) is some positive constant. Since \( P(\Theta_i > x) = o(F(x)) \) for any \( i \geq 1 \), by Lemma 3.3 (iii) in Yang et al. (2012) we have

\[
\limsup \max_{1 \leq i \leq N} \frac{P(\Theta_i X_i > (1 - \delta)x)}{P(\Theta_i X_i > x)} \leq \lim sup \frac{F(1 - \delta)x}{F(x)}. \tag{6.3}
\]

On the other hand,

\[
P \left( \sum_{i=N+1}^{\infty} \Theta_i X_i^+ > \delta x \right) = \lim_{n \to \infty} \min \left\{ \sum_{i=N+1}^{n} \Theta_i X_i^+ > \delta x \right\}, \tag{6.4}
\]

where the probability on the right-hand side of (6.4) can be estimated using (5.6), (5.7), (5.8), (5.20). Hence, for large \( x \),

\[
P \left( \sum_{i=N+1}^{\infty} \Theta_i X_i^+ > \delta x \right) \leq c_{18} \left( F(\Delta_1 \delta x) \sum_{i=N+1}^{\infty} (E\Theta_i^{J_F - \epsilon} \vee E\Theta_i^{J_F + \epsilon}) \right. \\
+ P \left( Z_\infty > \frac{\delta x}{\mu_+ + \Delta_1} \right) + (\delta x)^{- (J_F + \epsilon/2)} \sum_{i=N+1}^{\infty} E\Theta_i^{J_F + \epsilon} \\
\left. + (\delta x)^{- (J_F + \epsilon)} + P \left( Z_\infty > \frac{\delta x}{\log(\delta x)} \right) (\delta x)^{- c_{19}} \right) \tag{6.5}
\]

with some positive constants \( c_{18}, c_{19} \). Estimates (6.2), (6.3) and (6.5), together with Lemma 2.1, imply that for some positive constant \( c_{20} \)

\[
\limsup \sum_{i=1}^{\infty} \frac{P(M_i > x)}{P(\Theta_i X_i > x)} \leq L_F^{-1} \limsup \frac{F(1 - \delta)x}{F(x)} + \sum_{i=N+1}^{\infty} (E\Theta_i^{J_F - \epsilon} \vee E\Theta_i^{J_F + \epsilon}) \tag{6.7}
\]

because \( P(\tau = \infty)P(Z_\infty > x) \leq P(Z_\tau > x) = o(F(x)) \). Noting that \( P(\tau = \infty) \sum_{i=1}^{\infty} (E\Theta_i^{J_F - \epsilon} \vee E\Theta_i^{J_F + \epsilon}) \leq E \sum_{i=1}^{\pi} (E\Theta_i^{J_F - \epsilon} \vee E\Theta_i^{J_F + \epsilon}) < \infty \), we obtain the upper bound in (6.1).

For the lower bound, similarly to the proof of Theorem 4.1, we have for \( N \geq 1 \)

\[
\liminf \sum_{i=1}^{\infty} \frac{P(M_i > x)}{P(\Theta_i X_i > x)} \geq \liminf \frac{P(M_N > x)}{P(\Theta_i X_i > x)} \left( 1 - \limsup \sum_{i=N+1}^{\infty} \frac{P(\Theta_i X_i > x)}{\sum_{i=1}^{\infty} P(\Theta_i X_i > x)} \right), \tag{6.8}
\]

where, according to Lemma 3.1,

\[
\liminf \sum_{i=1}^{\infty} \frac{P(M_i > x)}{P(\Theta_i X_i > x)} \geq L_F, \tag{6.9}
\]

and, by Lemmas 3.2, 3.3,

\[
\limsup \sum_{i=N+1}^{\infty} \frac{P(\Theta_i X_i > x)}{\sum_{i=1}^{\infty} P(\Theta_i X_i > x)} \leq c_{21} \sum_{i=N+1}^{\infty} \left( E\Theta_i^{J_F - \epsilon} \vee E\Theta_i^{J_F + \epsilon} \right) \to 0 \text{ as } N \to \infty. \tag{6.10}
\]

Thus, the lower bound in (6.1) follows. \( \square \)
7 Application: the random-time ruin probability

In this section we provide an application of obtained results to risk theory. Consider a discrete time risk model with insurance and financial risks which satisfies the following assumptions:

**Assumption A**<sub>1</sub>. The successive net losses for an insurance company (insurance risk) \( X_1, X_2, \ldots \) constitute a sequence of real-valued r.v.s with common d.f. \( F \). (A net loss \( X_i \) is understood as the total claim amount minus the total premium income within period \( i \) both calculated at the end of period \( i \).)

**Assumption A**<sub>2</sub>. The discount factors (financial risk) \( \Theta_1, \Theta_2, \ldots \) are nonnegative and nondegenerate at zero r.v.s, which may be arbitrarily dependent and nonidentically distributed r.v.s. \( (\Theta_i \text{ is a discount factor from period } i \text{ to period } 0.) \)

**Assumption A**<sub>3</sub>. \( \tau \) is a random variable taking values in \( \{1, 2, \ldots \} \cup \{\infty\} \).

**Assumption A**<sub>4</sub>. \( \{X_1, X_2, \ldots\}, \{\Theta_1, \Theta_2, \ldots\} \) and \( \tau \) are mutually independent.

Let \( x \geq 0 \) be the initial capital reserve. Then the discounted value of the surplus accumulated by the insurer till time \( n \) is \( U_n(x) = x - \sum_{i=1}^{n} \Theta_i X_i \) and the probability of ruin till the random time \( \tau \) can be defined as

\[
\Psi(x, \tau) = P\left( \inf_{1 \leq n \leq \tau} U_n(x) < 0 \right)
= P\left( \sup_{1 \leq n \leq \tau} \sum_{i=1}^{n} \Theta_i X_i > x \right).
\]

If \( \tau = k \) for some fixed positive integer \( k \), then \( \Psi(x, k) \) is the so-called finite-time ruin probability; if \( \tau = \infty \), then \( \Psi(x, \infty) \) is the infinite-time ruin probability. Both finite-time and infinite-time ruin probabilities in the discrete time risk model have been widely investigated, see Tang and Tsitsiashvili (2003a, 2003b, 2004), Wang et al. (2005), Wang and Tang (2006), Zhang et al. (2009), Yang et al. (2012), Zhou et al. (2012), Yang and Wang (2013) among others.

The random-time ruin probability, which is of more universal and practical value in finance and insurance environments, in such a discrete time model was not considered earlier and is partially motivated by works of Tang (2004), Wang et al. (2009), Wang et al. (2012), Bai and Song (2012a, 2012b), where both the standard renewal risk model and generalized renewal risk model (possessing some dependence structure) were studied and random-time ruin probability was defined as

\[
\psi(x, \tau) = P\left( \sup_{1 \leq n \leq N(\tau)} \sum_{k=1}^{n} (X_k - c \theta_k) > x \right).
\]

Here, \( X_1, X_2, \ldots \) form a sequence of (nonnegative) claim sizes, \( \theta_1, \theta_2, \ldots \) are corresponding interarrival times, \( N(t) = \sup\{n \geq 1 : \sum_{k=1}^{n} \theta_k \leq t\} \), \( t \geq 0 \) is (generalized) renewal counting process, \( c \geq 0 \) is the constant premium rate, \( x > 0 \) is the insurer’s initial surplus, and \( \tau \) is a nonnegative random variable, which represents the economic cycle, whose timing is random and which is usually unpredictable.
For the discrete time risk model, postulated by Assumptions A$_1$–A$_4$ and satisfying conditions of Theorem 2.1 or Theorem 2.2, we have that the probability of insurer’s ruin during an economic cycle $\tau \in \{1, 2, \ldots\} \cup \{\infty\}$ can be approximated (as $x \to \infty$) using the asymptotic relation

$$\Psi(x, \tau) \simeq \mathbb{E} \sum_{i=1}^{\tau} P(\Theta_i X_i > x) = \sum_{i=1}^{\infty} P(\Theta_i X_i > x) P(\tau \geq i).$$

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References


