Rational fixed radius rolling ball blends between natural quadrics

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Abstract

By applying results on canal surfaces, we study exact rational parametrizations of fixed radius rolling ball blends of pairs of natural quadrics. We classify all configurations where this kind of rational parametrization is possible, and describe a general algorithm for parametrizing fixed radius rolling ball blends. The algorithm is then applied to parametrize the fixed radius rolling ball blends of pairs of natural quadrics.

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1. Introduction

Simple primitive shapes play an important role in CAD, as building blocks of more complex shapes. According to Rossignac (1987), “99 percent of mechanical parts can be modelled exactly if one combines natural quadrics with the possibility of representing fillets and blends”, and while we might expect the percentage to be somewhat lower 25 years later, there is still a predominance of shapes built from these primitives. In this context planes are considered natural quadrics, along with spheres and right circular cylinders and cones. Fillets and blends (in the following we write “blends” for both) are usually generated by fixed radius rolling ball methods. Though the natural quadrics are rational, in general rolling ball blends between them are not, so in current CAD systems they are constructed by approximation in all but the simplest cases. On the other hand, if we consider the complete surface traced by a rolling ball, not just the patch giving the blend, it is self-evident that this is a canal surface: the envelope of a one-parameter family of spheres.

Canal surfaces have been studied extensively during the last 15 years by several authors (see, for example, Lü and Pottmann, 1996; Peternell and Pottmann, 1997; Landsmann et al., 2001; Cho et al., 2004; Krasauskas, 2007). It has been proved that a canal surface with a rational spine (the curve traced by the centres of the spheres) and rational radius function is itself rational. Constructions of canal surface parametrizations have been presented, together with their degree bounds. Unfortunately, there is still a gap in the literature in terms of applying the theoretical results on canal surfaces to practical applications in CAD. In part this is a result of the differing world views of mathematics and engineering: the non-trivial surfaces interesting to mathematicians studying canal surfaces, while having exact rational parametrization, are of too high degree to be of any interest in practical applications. But it turns out that some simple cases, which can be parametrized with reasonably low degrees, are in fact prevalent in CAD: rolling ball blends of two natural quadrics with rational offset intersections.

Although shape accuracy is important in current CAD, there is no requirement that adjacent surfaces match exactly, so gaps within fine tolerances are allowed. However, the introduction of Isogeometric Analysis (see e.g. Cottrell et al., 2009)
Fig. 1. Elliptic plane/cone intersection, rolling ball, spine, and touching curves.

Fig. 2. The orientations of the surfaces determine the placement of the rolling ball.

changes this as in Finite Element Analysis adjacent elements are required to match exactly. Consequently, there is growing interest in employing exact shape representations when possible to minimize the challenges related to approximation. The aim of our paper is to close the gap in the literature by applying theoretical results on canal surfaces, and by doing so extend the list of exact rational rolling ball blends of natural quadrics.

We start by introducing the necessary theoretical background in Section 2. In Section 3 we construct the parametrization of the blend for the simplest configuration of natural quadrics: plane/cone intersections. In Section 4 we present an algorithm for minimal degree parametrizations of fixed radius rolling ball blends of two surfaces with rational offset intersections, and in Section 5 we classify the remaining configurations of natural quadrics whose blends can be parametrized rationally by our approach. In Section 6 we show how the blend of two cones, and a cone and a sphere are parametrized. Finally, we sum up our results in Section 7.

2. Theoretical background and terminology

Fixed radius rolling ball blends between surfaces are a common feature in CAD programs. It is an easy concept to visualize (see Fig. 1): let a ball of radius \( R \) roll along the intersection of the two surfaces in such a way that at any point it is tangent to both surfaces. The two curves traced on the surfaces by the ball are called touching curves, and the surface traced by the ball between them is the rolling ball blend. The radius of the blend is the radius \( R \) of the rolling ball, and the path traced by the centre of the ball is called its spine curve. The complete surface traced by the ball is a pipe surface: a canal surface with constant radius.

Two intersecting surfaces have several possible blending surfaces: in Fig. 1 the blend can be placed above or below the plane, and inside or outside the cone. In order to make the positioning of the blend unambiguous, we assign surfaces orientations given by the direction of their unit normal vectors (for ease of notation, simply called normals in the rest of the text). For spheres orientation is encoded in the sign of the radius: a positive radius corresponds to normals oriented towards the inside of the sphere, a negative radius corresponds to normals oriented outwards. The blend is placed where the orientation of the rolling ball coincides with the orientation of the two surfaces, i.e. the rolling ball is in oriented contact with the two surfaces. In Fig. 1, if the radius of the rolling ball is positive, then the orientation of the plane and the half-cone it intersects is up- and outwards respectively (note that the orientations of the two half cones are opposite). By allowing the radius of the blend to be negative, we have reduced the number of cases to the two in Fig. 2.

The \( R \)-offset of a surface is constructed by moving each point on the surface the same length \( R \) along its normal. The natural quadrics are offset stable in the sense that their type is preserved when offsetting: the offset of a cylinder is still a cylinder, and so on. This offset stability is advantageous in e.g. Isogeometric analysis and applications in architecture. It is also useful when we determine the spine of a rolling ball blend:
Remark 1. If we intersect the $R$-offsets of two intersecting natural quadrics, we obtain a curve that is equidistant from the two surfaces. This is the spine of the rolling ball blend of radius $R$.

The touching curves of the blend are found by projecting the spine onto the two surfaces. To determine the projections onto the natural quadrics it is convenient to use some elements of Laguerre geometry.

2.1. Laguerre geometry

Laguerre geometry is a geometry of spheres – instead of considering points and distances between points, we consider oriented spheres and tangential distances between spheres. An oriented sphere $p$ is given by its centre $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and radius $x_4 \in \mathbb{R}$. Using the notation $p = (x; x_4) = (x_1, x_2, x_3; x_4)$, the space of all oriented spheres is identified with the Minkowski space $\mathbb{R}^{3,1}$, i.e. 4-dimensional space $\mathbb{R}^4$ equipped with the Minkowski scalar product of vectors:

$$\langle v, v' \rangle = v_1 v_1' + v_2 v_2' + v_3 v_3' - v_4 v_4'.$$

(1)

A point in $\mathbb{R}^3$ can be considered a sphere of zero radius, so in the following $(x_1, x_2, x_3) \in \mathbb{R}^3$ is identified with $(x_1, x_2, x_3; 0) \in \mathbb{R}^{3,1}$.

The tangential distance between two oriented spheres $p_0$ and $p_1$ is defined by the formula

$$d(p_0, p_1) = \|p_1 - p_0\| = \sqrt{\langle p_1 - p_0, p_1 - p_0 \rangle}.$$  
(2)

As long as $\langle p_1 - p_0, p_1 - p_0 \rangle > 0$, (2) has a geometric interpretation: it is the distance between the touching points of the given spheres with a common oriented tangent plane (see Fig. 3). When the tangential distance is zero, we say that the two spheres are in oriented contact.

In differential geometry a canal surface is the envelope of the family of spheres $x(t) = (x_1(t), x_2(t), x_3(t); x_4(t))$ with centres at $(x_1(t), x_2(t), x_3(t))$ and radius $x_4(t)$. Using the above representation of spheres as points in Minkowski space, a canal surface corresponds to a curve in Minkowski space.

Remark 2. Cones and cylinders are canal surfaces. Both have linear spines (their axes), and while a cylinder has a constant radius function (i.e. is a pipe surface), the radius function for a cone is linear.

It follows from Remark 2 that cones and cylinders correspond to lines in Minkowski space. A cylinder can be considered as a cone with apex at infinity, so in the following we write “cones” meaning “cylinders and cones”, and use the parametrization

$$p(u) = p_0 + u(p_1 - p_0), \quad p_0, p_1 \in \mathbb{R}^{3,1}, \quad u \in \mathbb{R}$$

(3)

for the line associated with the cone $\mathcal{C}$, where $p_0$ and $p_1$ are two spheres inscribed in $\mathcal{C}$. The apex of the cone corresponds to the point on the line with the fourth coordinate zero, i.e. the sphere with zero radius.

Remark 3. Not every line in Minkowski space describes a cone: the necessary and sufficient condition for a line through two points $p_0$ and $p_1$ to define a cone is $\langle p_1 - p_0, p_1 - p_0 \rangle > 0$, i.e. $d(p_1, p_0) \in \mathbb{R}^+$. This means that cones in $\mathbb{R}^3$ correspond to hyperbolic lines in $\mathbb{R}^{3,1}$, and the corresponding family of spheres is a hyperbolic pencil of spheres. When $\langle p_1 - p_0, p_1 - p_0 \rangle = 0$ (i.e. the line is parabolic) the line describes all spheres that are in oriented contact at one point, which is a parabolic pencil of spheres. The envelope of this family of spheres is a plane, which can be considered a degenerate cone with apex at the common touching point.

With the above elements of Laguerre geometry we determine the touching point of a sphere in oriented contact with a natural quadric. Two touching spheres generate a parabolic pencil of spheres (see Fig. 4), and the touching point is the apex of the corresponding degenerate cone (see Remark 3). Note that a sphere $q$ in oriented contact with a cone $\mathcal{C}$ is in oriented contact at a unique point, and is therefore in oriented contact with exactly one sphere $p(u_0)$ in the family associated with the cone. We find $u_0$ by solving the quadratic equation

$$u_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $a$, $b$, and $c$ are determined by the coefficients of the quadratic equation.
Lemma 4. A sphere \( q \in \mathbb{R}^{3,1} \) in oriented contact with the cone \( C \) associated with the line \( p(u) = p_0 + u(p_1 - p_0) \in \mathbb{R}^{3,1} \) is in oriented contact with the sphere

\[
p(u_0) = p_0 + \frac{(d - p_0, p_1 - p_0)}{\|p_1 - p_0\|^2} (p_1 - p_0).
\]

The touching point \( T \) of \( q \) and \( C \) is the apex of the parabolic pencil of spheres generated by \( q \) and \( p(u_0) \):

\[
T = \frac{\pi_4(p(u_0)) q - \pi_4(q) p(u_0)}{\pi_4(p(u_0)) - \pi_4(q)},
\]

where \( \pi_4 \) is the projection onto the fourth coordinate.

As the sphere \( q \) is in oriented contact with \( C \) at a unique point, (4) has a unique solution, so the discriminant of the quadratic equation is zero.

**Corollary 5.** A sphere \( x \in \mathbb{R}^{3,1} \) is in oriented contact with the cone \( C \) if and only if it satisfies

\[
(x - p_0, p_1 - p_0)^2 - \|x - p_0\|^2 \|p_1 - p_0\|^2 = 0.
\]

We call the quadratic hypersurface of \( \mathbb{R}^{3,1} \) defined in Corollary 5 the isotropic quadric of the cone \( C \).

3. Rolling ball blends of plane/cone intersections

We start with the simplest non-trivial case of configurations of natural quadrics: the intersection of a cone and a plane. In general, the spine of a cone/plane blend is a conic section: ellipse, hyperbola or parabola. These blends are inherently rational: for any given plane and cone, their R-offsets intersect in a conic section, thus we can find a rational parametrization of the blend for any fixed radius \( R \). In this section we show the details of the blend construction for elliptic blends, the results for hyperbolic and parabolic blends can be derived using the same approach. As the parametrizations of the touching curves and blending surfaces are rational, for ease of notation they are presented in projective coordinates where the first coordinate is the homogeneous coordinate. All parametrizations are in \( \mathbb{R}^3 \) or the projective space over \( \mathbb{R}^3 \), so we distinguish projective points from affine points by the number of coordinates, and define the projection of \( x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^3 \) from projective to affine coordinates: \( [x] = (x_1/x_0, x_2/x_0, x_3/x_0) \in \mathbb{R}^2 \).

3.1. Elliptic rolling ball blend

Consider a cone \( C \) and a plane \( \mathcal{P} \) whose R-offsets \( C_R \) and \( \mathcal{P}_R \) intersect in the ellipse \( \mathcal{E} \) (see Fig. 1) parametrized by

\[
\mathcal{E}(t) = \left( a \frac{2t}{1 + t^2}, b \frac{1-t^2}{1+t^2}, 0 \right)^T, \quad 0 < b < a, \quad t \in \mathbb{R}.
\]
We can assume that the ellipse is in canonical position without any loss of generality, as we can always consider the blend in the local coordinates of its spine. If the orientation of $\mathcal{C}$ is given by the normal vector $n = (0, 0, 1)^T$, the plane is given implicitly by $z = -R$. A classical result by G.P. Dandelin (see e.g. Theorem 4.1 in Shene and Johnstone, 1994 or the tangent ball theorems in Miller and Goldman, 2002) states that if a plane intersects an axial natural quadric in a non-degenerate conic, there are one or two spheres inscribed in the quadric and tangent to the plane, and the tangent points are the foci of the intersection conic. The inscribed tangent spheres are called focal spheres.

In the elliptic case, there are two spheres inscribed in the cone $\mathcal{C}$, tangent to the plane $\mathcal{P}_R$ at the foci of $\mathcal{E}$. One sphere is resting on the $xy$-plane at the focus $(c, 0, 0)^T$, where $c^2 = a^2 - b^2$, so if its centre is at $(c, 0, v)^T$ then the radius of the sphere is $\pm v$. If the half cone intersecting the plane is oriented outwards, the sphere is given by $(c, 0, v; -v)$. The tangential distance between the two focal spheres is $2a$ and the radius of the second sphere is equal to the $z$-coordinate of its centre, so solving $\| (c, 0, v; -v) - (-c, 0, 0; z) \| = 2a$ for $z$ gives us the second focal sphere $(-c, 0, -b^2/v; -b^2/v)$. The two focal spheres in $\mathcal{C}$ are $-R$-offsets of two spheres in $\mathcal{E}$: $p_0 = (c, 0, v; R - v)$ and $p_1 = (-c, 0, -b^2/v; R - b^2/v)$. We use $p_0$ and $p_1$ to parametrize the line corresponding to the cone $\mathcal{E}$ in Minkowski space:

$$p(u) = p_0 + u (p_1 - p_0) = (c, 0, v; R - v) - u(2c, 0, b^2/v + v; b^2/v - v).$$  

(9)

Given the plane $\mathcal{P}$ and the cone $\mathcal{C}$, we want to parametrize the rolling ball blend of radius $R$ with spine $\mathcal{E}$. We start by finding the touching curves on the plane and the cone. The touching curve on the plane is a translation of the spine ellipse $\mathcal{E}$ along $n$:

$$\left(a - \frac{2t}{1 + t^2}, b \frac{1 - t^2}{1 + t^2}, -R\right)^T.$$  

(10)

The rolling ball $x_t = (\mathcal{E}(t); R)$ is in oriented contact with the cone, so applying Lemma 4 to $x_t$ we find the tangent sphere $p_t$. Let $\pi_4$ be the projection onto the last coordinate. The touching point of $x_t$ and $\mathcal{C}$ is then given in Lemma 4:

$$T^+_v(t) = \frac{\pi_4(p_t) - \pi_4(x_t)}{\pi_4(p_t) - \pi_4(x_t)}.$$  

(11)

Expanding the expressions of $x_t$ and $p_t$, we find the parametrization of the touching curve on the cone:

$$T^+_v(t) = \left(\frac{a(v^2 + b^2)(1 + t^2) + 2c(v^2 - b^2)t(1 + t^2)}{b(a(v^2 + b^2 - 2Rv)(1 + t^2) + 2c(v^2 - b^2)t(1 - t^2))}, \frac{2((a^2(v^2 + b^2) - 2vb^2R)(1 + t^2) + 2ac(v^2 - b^2)t)t}{R(a(v^2 - b^2)(1 + t^2) + 2c(v^2 + b^2)t)(1 + t^2)}\right).$$  

(12)

Note that $\mathcal{C}$ is not the only cone that intersects the plane $\mathcal{P}$ in the ellipse $\mathcal{E}$ – in fact $p_0$ and $p_1$ generates a cone whose $R$-offsets go through $\mathcal{E}$ for any choice of $v$. Each of the cones has a quartic touching curve on the pipe surface (see Fig. 5). The pipe surface is the union of a one-parameter family of circles, each circle belonging to one of the spheres in the one-parameter family of spheres defining the pipe surface. A circle is a rational quadratic curve, so its parametrization by a line in parameter space is injective. By examining (12), we find that the circles are the isoparametric curves for $t$ constant. $T^+_v(t)$ is therefore an injective map onto the pipe surface, and as such a parametrization of the pipe surface. Evaluating $T^+_v(t)$ at $v = 0$ we recover the ellipse in (10), so both of our original touching curves are in the family of curves.

**Theorem 6.** The rolling ball blend $B_\mathcal{C}$ of radius $R$ of the plane $\mathcal{P}$ and cone $\mathcal{C}$ whose $R$-offsets intersects in the ellipse $\mathcal{E}$ in (8) is parametrized by

$$B_\mathcal{C}(t, v) = \left[T^+_v(t)\right], \quad t \in \mathbb{R}, \; v \in [0, v_0],$$  

(13)

where $-v_0$ is the radius of the focal sphere of the offset cone at the positive focal point. This parametrization is rational of bi-degree $(4, 2)$. 

Fig. 5. Touching curves on the pipe surface of the blend.
Remark 7. We have made a choice in the orientation of the plane and cone. The parametrization of the blend for other combinations of orientation is derived by changing \( n \) and/or the orientation of the focal spheres. By considering the cones whose offset cones have the opposite orientation of the family above, we find a second family of cones and quartic touching curves on the pipe surface.

The blend (Fig. 6) is a rationally parametrized surface patch of bi-degree \((4, 2)\), which can be given in tensor product Bézier form:

\[
B(t, v) = \sum_{i=0}^{4} \sum_{j=0}^{2} Q_{i,j} B_i^4(t) B_j^2(v),
\]

where \( B_i^4(t) \) and \( B_j^2(v) \) are Bernstein basis functions, and \( Q_{i,j} \) the weight points of the patch. Weight points are the rational equivalent to control points: a weight point \( Q_{i,j} = (w, wq_{i,j}) \) corresponds to a control point \( q_{i,j} \) with weight \( w \). A weight point with the first coordinate zero corresponds to a control point at infinity.

Consider the inner quarter of the elliptic pipe surface parametrized by \( B(t, v) \) with \( t \in [-1, 1] \) and \( v \in \mathbb{R}^+ \). To retrieve the weight points of this surface patch, we reparametrize \( B(t, v) \) by replacing \( t \) with \( t = (t + 1)/2 \) and \( v \) with \( v = v/(b + v) \) to send the parameter intervals to \([0, 1] \). A change of polynomial basis from monomial to Bernstein basis then gives us the following weight points:

\[
Q = \begin{pmatrix}
(a + c)(e_0 - ae_1 - Re_3) & bRe_1 & (a - c)(e_0 - ae_1 + Re_3) \\
\frac{1}{2}b(a + c)e_2 & -\frac{1}{2}aRe_2 & \frac{1}{2}b(a - c)e_2 \\
\frac{1}{2}a(e_0 + ae_1 - Re_3) & 0 & \frac{1}{2}a(e_0 - ae_1 + Re_3) \\
\frac{1}{2}b(a - c)e_2 & -\frac{1}{2}aRe_2 & \frac{1}{2}b(a + c)e_2 \\
(a - c)(e_0 + ae_1 - Re_3) & -bRe_1 & (a + c)(e_0 + ae_1 + Re_3)
\end{pmatrix},
\]

where \( e_i \) are unit coordinate vectors and \( e_0 \) the homogeneous coordinate.

The weight points of the rolling ball blend of radius \( R \) of the plane \( \mathcal{P} \) and cone \( \mathcal{C} \) are found by subdividing \( Q \) at \( \bar{v}_0 = v_0/(v_0 + b) \). In Fig. 7, we see a quarter of the blend, together with its control mesh.

3.2. Hyperbolic and parabolic blends

The above construction of the elliptic rolling ball blend applied to hyperbolic and parabolic blends gives us the parametrizations for these cases. Consider the radius \( R \) blend of a cone and a plane, with hyperbolic spine

\[
\mathcal{H}(t) = \left( a \frac{1 + t^2}{2t}, b \frac{1 - t^2}{2t}, 0 \right)^T, \quad 0 < b, a, t \in \mathbb{R}.
\]
By parametrizing the cone by the offsets of the focal spheres of the spine
\[ p_0 = (c, 0; R - v), \quad p_1 = (-c, 0; R + b^2/v), \quad c^2 = a^2 + b^2 \] (17)
we find the parametrization of the blend:

**Theorem 8.** Let \( B_H \) be the plane/cone rolling ball blend of radius \( R \) with spine \( H \). Then the blend is parametrized by

\[
B_H(t, v) = \left[ \begin{array}{c}
2(c(v^2 + b^2)(1 + t^2) + 2a(v^2 - b^2)t)t \\
(ac(v^2 + b^2)(1 + t^2) + 2(a^2(v^2 - b^2) + 2b^2 vR)t)(1 + t^2) \\
bc(v^2 + b^2)(1 + t^2) + 2a(v^2 - b^2 - 2vR)t)(1 - t^2) \\
2RC(v^2 - b^2)(1 + t^2) + 2a(v^2 + b^2)t)t
\end{array} \right] (18)
\]
\[ t \in \mathbb{R}, \quad v \in [0, v_0], \text{where } -v_0 \text{ is the radius of the focal sphere of the spine at the positive focal point.} \]

By parametrizing the cone by the offsets of the focal sphere of the spine, and the offset of the apex of the offset of the cone.

\[ p_0 = (a, 0; R - v), \quad p_1 = (a - v^2/a, 0, 2v; R) \] (20)

we find the parametrization of the blend:

**Theorem 9.** Let \( B_P \) be the plane/cone rolling ball blend of radius \( R \) with parabolic spine \( P \). Then \( B_P \) is parametrized by

\[
B_P(t, v) = \left[ \begin{array}{c}
(v^2 + a^2 + t^2)a \\
(v^2 + a^2 + t^2)a^2 + 2vRa^2 \\
2a(v^2 + a^2 + t^2 - 2vR)t \\
Ra(v^2 - a^2 - t^2)
\end{array} \right] (21)
\]
\[ t \in \mathbb{R}, \quad v \in [0, v_0] \text{ where } -v_0 \text{ is the radius of the focal sphere of the spine.} \]

**Remark 10.** The pipe surfaces in the above illustrations of quadratic rolling ball blends have no self-intersections. However, if we increase the radius \( R \) of the pipe surface, this may no longer be the case. The hyperbolic blend in Fig. 8 is smooth, but as the radius of the blend increases, a self-intersection will appear below the apex of the cone.

3.3. **Parametrizing cone/cone blends using the families of touching curves**

From the above derivation of the parametrization of the plane/cone blend, it is tempting to use the same approach to parametrize the blend of two cones in the same family (see Remark 7). For a given radius \( R \) it is indeed possible to blend two such cones whose \( R \)-offsets intersect in a given conic, however this is the unique radius for which the offsets of the two cones intersect in a conic section. The two offset cones circumscribe a common sphere (the condition for having a conic intersection), but when the cones are from the same family the two spheres have opposite orientations, so their offsets have no common spheres.

In this section we have considered only one family of cones through a conic. By changing the orientation of the focal sphere(s) of the intersection, we get the second family of cones, and two cones from different families do indeed have a common oriented sphere, so the intersection of the offsets of the cones will remain a conic. As the cones belong to different families, we cannot get the blend by moving seamlessly from one touching curve to the other as we did here, but we will use the touching curves and methods from canal surfaces to parametrize the blend.
4. Parametrizing rational patches on pipe surfaces

A rational pipe surface of radius \( R \) has a parametrization of the form

\[
F(t, u) = s(t) + RN(t, u) \tag{22}
\]
where \( s(t) \) is its spine, \( R \) its radius, and \( N(t, u) \) the Gaussian map of \( F(t, u) \) (i.e. the image of the pipe surface on the Gaussian sphere). When we construct a rolling ball blend of two surfaces, the blend is the patch on the pipe surface between the two touching curves traced by the rolling ball. Likewise the touching curves can be traced on the rolling ball, giving us a patch on the Gaussian sphere \( S \). Parametrizing the blend is thus reduced to parametrizing the patch on \( S \) limited by the Gaussian images \( \beta^0(t) \) and \( \beta^1(t) \) of the touching curves. Krasauskas (2007) takes advantage of this to formulate an algorithm for parametrizing tensor product Bézier patches on canal surfaces. In this section we will present a simplified algorithm for the case of pipe surfaces.

The isoparametric curve \( F_t(u) \) on the pipe surface is an arc of a circle in the intersection with the normal plane of the spine curve. Its Gaussian image \( N_t(u) \) is the arc of the large circle between \( \beta^0(t) \) and \( \beta^1(t) \) in the intersection of the Gaussian sphere with the plane \( Bx^T = 0 \), where \( B = (0, \delta(t)\dot{s}_1(t), \delta(t)\dot{s}_2(t), \delta(t)\dot{s}_3(t)) \) and \( x = (1, x_1, x_2, x_3) \). \( \dot{s}(t) \) is the tangent vector of the spine, and \( \delta(t) \) the common denominator of \( \dot{s}_1(t) \), \( \dot{s}_2(t) \), and \( \dot{s}_3(t) \).

The Gaussian sphere can be parametrized using the generalized stereographic projection \( P_S \), mapping points in \( \mathbb{R}^3 \) onto the sphere (Dietz et al., 1993). If we identify the four homogeneous coordinates of a point in \( \mathbb{R}^3 \) with a point in \( \mathbb{C}^2 \), we get a compact formulation in complex numbers:

\[
P_S(U_0, U_1) = (U_0\overline{U}_0 + U_1\overline{U}_1, 2 \text{Re}(U_0\overline{U}_1), 2 \text{Im}(U_0\overline{U}_1), U_0\overline{U}_0 - U_1\overline{U}_1)^T. \tag{23}
\]
This is also known as the universal rational parametrization of the sphere (see e.g. Krasauskas, 2007).

Remark 11. We can also parametrize the Gaussian sphere by the stereographic projection from the real plane \( \mathbb{R}^2 \) onto \( S \). However, the inverse stereographic projection maps circles on \( S \) onto both circles and lines in \( \mathbb{R}^2 \). This is impractical as we want to recover the parametrizations of circles on \( S \) from the parametrizations of their projections in the plane. On the other hand, any circle on \( S \) is the projection of a line in \( \mathbb{C}^2 \) by the generalized stereographic projection (Krasauskas, 2007, Lemma 2). Furthermore, the universality of this parametrization ensures that its degree is minimal. The level of abstraction of the parametrization construction is increased as it takes place in \( \mathbb{C}^2 \) before it is projected onto the Gaussian sphere. However, the end results are the compact explicit parametrizations of the rolling ball blends given in Theorems 16–22.

We define the lifting of a curve \( x(t) = (x_0, x_1, x_2, x_3) \in \mathbb{R}P[t]^3 \) on the Gaussian sphere as

\[
L(x(t)) = \left( U_0, U_0 \frac{x_0 - x_3}{x_1 + ix_2} \right)^T \in \mathbb{C}P[t]^2, \quad U_0 = \gcd(x_0 + x_3, x_1 + ix_2) \tag{24}
\]
(see Eq. (10) in Krasauskas, 2007). As \( x(t) \) lies on the Gaussian sphere it satisfies \( x^2_0 - x^2_3 = x^2_1 + x^2_2, \) i.e. \( (x_0 + x_3)(x_0 - x_3) = (x_1 + ix_2)(x_1 - ix_2) \), so \( x_1 + ix_2 \) divides \( U_0(x_0 - x_3) \) and \( L(x(t)) \) is polynomial rather than rational. As long as the projective representation of the curve is of minimal degree, i.e. \( \gcd(x_0, x_1, x_2, x_3) = 1 \), we also have \( \gcd(L(x(t))) = 1 \). We find that \( L \) sends a point \( x(t) \) to a point in its pre-image by \( P_S \), as \( P_S(L(x(t))) = x(t) \). Note that the lifting is well defined outside the constant curves \( x(t) = (1, 0, 0, \pm 1) \) (e.g. the Gaussian image of the touching curve on a horizontal plane). In this case we perform a change of coordinates to move the constant curve away from these exceptional points.

Consider the lifting of \( N_t(u) \) in \( \mathbb{C}[t, u]^2 \). By parametrizing the lifting and then applying \( P_S \), we recover a parametrization of \( N(t, u) \). Furthermore, the universality of \( P_S \) means that if we minimize the bi-degree of the lifting, the parametrization will likewise have minimal bi-degree.

Lemma 2 in Krasauskas (2007) gives the lifting of \( N_t(u) \) as the line between the liftings \( X \) and \( Y \) of \( \beta^0(t) \) and \( \beta^1(t) \) parametrized by \( (1 - u)\lambda_0X + u\lambda_1Y \) where the lifting coefficient \( \lambda_0\lambda_1 \) is the unique solution of a set of linear equations. In the case of pipe surfaces, the system is reduced to a single equation:

**Lemma 12.** The system of linear complex equations determining the lifting coefficient \( \lambda = \lambda_0\lambda_1 \in \mathbb{C}[t] \) has a unique solution up to multiplication by a real number:

\[
\lambda = \frac{(\beta^0_1 + i\beta^0_2)(\beta^1_1 - i\beta^1_2) + (\beta^0_0 + \beta^0_2)(\beta^1_0 + \beta^1_2)}{X_0\overline{Y}_0} \tag{25}
\]
if \( |\beta^0(t)| \neq |-\beta^1(t)| \). Otherwise \( \beta^0(t) = (\pm \beta_0, \beta_1, \beta_2, \beta_3) \) and

\[
\lambda = \frac{i(\beta_1 + i\beta_2)(B_1 - iB_2) + (\beta_0 + \beta_3)B_3}{X_0\overline{Y}_0} \tag{26}
\]
Here \( X_0 = \gcd(\beta^0_0 + \beta^0_2, \beta^1_0 + i\beta^1_2), \overline{Y}_0 = \gcd(\beta^0_0 + \beta^1_3, \beta^1_0 - i\beta^1_2), \) and \( B = (0, B_1, B_2, B_3) = (0, \dot{s}_1(t), \dot{s}_2(t), \dot{s}_3(t)) \).
Proof. In Lemma 2 in Krasauskas (2007), the lifting coefficient λ is determined by the linear system of complex equations

\begin{align}
\lambda X_1 \vec{Y}_1 - \bar{X}_1 Y_1 &= iB_3, \\
\bar{X}_0 Y_1 - \lambda X_1 \bar{Y}_0 &= iB_1 + B_2, \\
\lambda X_0 \bar{Y}_1 - \bar{X}_1 Y_0 &= -iB_1 + B_2, \\
\lambda X_0 \bar{Y}_0 - \bar{X}_0 Y_0 &= -iB_3,
\end{align}

where \( \mathbf{X} = (X_0, X_1)^T \) and \( \mathbf{Y} = (Y_0, Y_1)^T = \mathbf{L}(\beta^0(t)) \) are the lifting of the touching curves on the Gaussian sphere, and \( \mathbf{B} = (0, B_1, B_2, B_3) = (0, \dot{s}_1(t), \dot{s}_2(t), \dot{s}_3(t)) \) defines the plane whose intersection with the Gaussian sphere contains the isoparametric curve \( \mathbf{N}_i(u) \). (29) is the complex conjugate of (28), so it can be eliminated. By construction of the lifting we have

\begin{align}
X_1 &= X_0 \frac{\beta_0^0 - \beta_2^0}{\beta_1^0 + i\beta_2^0}, \\
Y_1 &= Y_0 \frac{\beta_0^1 - \beta_2^1}{\beta_1^1 + i\beta_2^1}
\end{align}

and by inserting these equations into the remaining three equations in the system above, and separating the real and imaginary part of (28) ((27) and (30) are purely imaginary), we arrive at a system of four real linear equations in \( X = \Re(\lambda X_0 \bar{Y}_0) \) and \( Y = \Im(\lambda X_0 \bar{Y}_0) \):

\begin{align}
(\beta_1^0 \beta_2^0 - \beta_1^1 \beta_2^1) X + (\beta_1^0 \beta_1^0 + \beta_2^0 \beta_2^0) Y &= \alpha_0 \alpha_1 B_3/2, \\
(\beta_1^0 \alpha_0 - \beta_1^1 \alpha_1) X - (\beta_1^0 \alpha_0 + \beta_2^0 \alpha_1) Y &= \alpha_0 \alpha_1 B_2, \\
(\beta_2^0 \alpha_0 - \beta_2^1 \alpha_1) X + (\beta_1^0 \alpha_0 + \beta_2^0 \alpha_1) Y &= -\alpha_0 B_1, \\
2Y &= -B_3,
\end{align}

where \( \alpha_0 = \beta_0^0 + \beta_2^0 \) and \( \alpha_1 = \beta_1^1 + \beta_2^1 \).

Recall that \( \beta^0 \) and \( \beta^1 \) lie on the plane \( \mathbf{Bx} \mathbf{T} = 0 \), and so \( \mathbf{B} \beta^0 = 0 \) and \( \mathbf{B} \beta^1 = 0 \). This gives two linear equations in \( B_i \), which are linearly independent if and only if \( [\beta^0] \neq -[\beta^1] \). We can then express \( B_1 \) and \( B_2 \) as functions of \( B_3 \):

\begin{align}
B_1 &= \frac{\beta_3^1 \beta_2^0 - \beta_3^0 \beta_2^1}{\beta_1^0 \beta_2^0 - \beta_1^1 \beta_2^1} B_3, \\
B_2 &= \frac{\beta_3^0 \beta_2^0 - \beta_3^1 \beta_2^1}{\beta_1^0 \beta_2^0 - \beta_1^1 \beta_2^1} B_3
\end{align}

and by Eq. (35), we have \( B_3 = -2Y \). By replacing the \( B_i \)s in (32)–(35) by their expressions as functions of \( Y \), we arrive at three homogeneous polynomials in \( X \) and \( Y \). Closer examination reveals that the equations are in fact equal, giving us

\begin{align}
\lambda X_0 \bar{Y}_0 &= X + iY = (\beta_0^0 + i\beta_2^0)(\beta_1^1 - i\beta_2^1) + (\beta_0^1 + \beta_2^1)(\beta_1^0 + \beta_2^0)
\end{align}

which proves (25). Note that \( X + iY \) is unique up to multiplication with a real number, which is what we required for \( \lambda \). As \( X_0 = \gcd(\beta_0^0, \beta_0^1, \beta_2^0, \beta_2^1) \) and \( Y_0 = \gcd(\beta_1^0, \beta_1^1, \beta_2^0, \beta_2^1) \), \( \lambda \) is a polynomial.

When \( \beta^i(t) = (\pm \beta_0, \beta_1, \beta_2, \beta_3) \), (32)–(35) are reduced to

\begin{align}
Y &= -1/2B_3, \\
\beta_0 \beta_1 X - \beta_3 \beta_2 Y &= -1/2(\beta_0^2 - \beta_2^2) B_2, \\
\beta_0 \beta_2 X + \beta_1 \beta_3 Y &= 1/2(\beta_0^2 - \beta_2^2) B_1.
\end{align}

The determinant of the corresponding matrix is zero, so the system is linearly independent and we may chose to solve (38) and (39) to find \( X \) and \( Y \). After some simplifications, this gives us

\begin{align}
\lambda X_0 \bar{Y}_0 &= i((\beta_1 + i\beta_2)(B_1 - iB_2) + (\beta_0 + \beta_3)B_3),
\end{align}

where \( \lambda \) is a polynomial by the same argument as before. This proves (26). \( \square \)

The procedure outlined above is summarized in the following algorithm:

**Algorithm 13.** Consider two surfaces whose \( R \)-offsets intersect in the rational curve \( \mathbf{s}(t) \), and let \( \mathbf{T}^i(t) \) be the rational touching curves on the two surfaces, of the pipe surface with spine \( \mathbf{s}(t) \) and radius \( R \). The rolling ball blend of radius \( R \) of these two surfaces is parametrized by executing the following steps:

1. The Gaussian images \( \beta^0(t) \) and \( \beta^1(t) \) of the touching curves

\begin{align}
\beta^i(t) &= (\beta_0^i, \beta_1^i, \beta_2^i, \beta_3^i)^T, \\
[\beta^i(t)] &= \frac{\mathbf{T}^i(t) - \mathbf{s}(t)}{R}, \quad i = 1, 2.
\end{align}


2. The liftings of the Gaussian touching curves
\[ X = (X_0, X_1)^T = L(\beta^0(t)), \quad Y = (Y_0, Y_1)^T = L(\beta^1(t)). \] (43)

3. The lifting coefficient \( \lambda = \lambda_0 \lambda_1 \in \mathbb{C}[t] \) from Lemma 12. Note that \( \lambda \) is unique up to multiplication by a real scalar, so any real factors may be eliminated.

4. The lifting of the blend in \( \mathbb{C}[t]^2 \)
\[ (1 - u)\lambda_0 X + u \lambda_1 Y, \] (44)
where \( \lambda = \lambda_0 \lambda_1 \), and \( \lambda_0 X \) and \( \lambda_1 Y \) are of equal degree (or as close as possible).

5. The parametrization of the blend
\[ B(t, u) = s(t) + R \left[ P_S ((1 - u)\lambda_0 X + u \lambda_1 Y) \right], \] (45)
where \( P_S \) is the generalized stereographic projection in (23).

This parametrization is of minimal bi-degree \( (n, 2) \).

5. Classification of pairs of natural quadrics

Before we apply Algorithm 13 to cone/cone and cone/sphere blends, we need to determine which configurations of natural quadrics can be blended rationally. A canal surface is rational if its spine and radius function are rational (Paternelel and Pottmann, 1997). The pipe surface of a fixed radius rolling ball blend has a constant radius function, so two surfaces can be blended rationally at a fixed radius if their \( R \)-offsets intersect in a rational spine curve. We say that a configuration of two surfaces is \textit{rationally stable} if this is true for any \( R \). The intersection of two quadratic surfaces is a quartic curve, which is not in general rational. If the quartic is reducible, the intersection is a combination of lines, conics, and space cubics, all of which are rational. If the quartic is irreducible, it is rational if and only if it is singular. Pairs of natural quadrics have been completely classified from the point of view of Laguerre geometry in Kazakevičiūtė (2005), which also gives rational parametrizations of their bisector surfaces in Minkowski space. By intersecting the bisector surface with the hyperplane \( x_4 = R \) we get the spine of the rolling ball blend of radius \( R \).

Remark 14. The bisector surface of two natural quadrics is always rational. However, the hyperplane sections of this surface are not necessarily so. There are only two types of algebraic surfaces whose hyperplane sections are always rational: rational ruled surfaces and the quartic Steiner surface (see Moore, 1887).

There are two configurations of cones where the intersection of the hyperplane \( x_4 = R \) with the bisector surface is always rational: cones with one or two touching points (when the bisector surface is respectively the Steiner quartic or a pair of planes). A touching point is a point where the cones are in oriented contact. When the cones have two touching points, their intersection is a pair of quadratic curves. If the intersection is degenerate, i.e. contains one or more lines, the linear components may be blended by a cylinder. Cylindrical (and torical, corresponding to circular intersections) blends are natural quadrics can be blended rationally. A canal surface is rational if its spine and radius function are rational (Peternell

When the cones have one touching point, their intersection is a quartic curve with a singularity at the touching point. Consider the two cones \( \mathcal{Q} \) and \( \mathcal{S} \), which correspond to the lines
\[ p(u) = p_0 + u(p_1 - p_0), \quad q(v) = q_0 + v(q_1 - q_0) \] (46)
in Minkowski space. If \( \mathcal{Q} \) and \( \mathcal{S} \) are in oriented contact, exactly one sphere \( q(v) \) is in oriented contact with \( \mathcal{Q} \), i.e. \( q(v_0) \) is in the isotropic quadric of \( \mathcal{Q} \) defined in Corollary 5. We find \( v_0 \) by solving \( (q(v) - p_0, p_0)^2 - \|q(v) - p_0\|^2 \|p_0\|^2 = 0 \), where \( p_0 = p_1 - p_0 \) and \( q_0 = q_1 - q_0 \):
\[ v_0^2 \left( (q_0, p_0)^2 - \|q_0\|^2 \|p_0\|^2 \right) + 2v \left( (q_0 - p_0, p_0)(q_0, p_0) - (q_0, p_0, q_0, p_0)(q_0, p_0)^2 \right) (q_0 - p_0, p_0)^2 - \|q_0 - p_0\|^2 \|p_0\|^2 = 0. \] (47)
As \( v_0 \) is unique we find
\[ v_0 = \frac{(q_0 - p_0, p_0)(q_0, p_0)^2 - (q_0, p_0, q_0, p_0)(q_0, p_0)^2}{(q_0, p_0)^2 - \|q_0\|^2 \|p_0\|^2}, \] (48)
and \( q(v_0) = q_0 + v_0 q_0 \). Likewise we find the sphere \( p(u_0) \) in oriented contact with \( \mathcal{S} \). These two spheres are in oriented contact, and lets us categorize the rationally stable configurations of two cones:
Theorem 15. Two cones $\mathcal{P}$ and $\mathcal{Q}$ are in oriented contact if the discriminant of \((47)\) is zero. Let $p(u_0)$ and $q(v_0)$ be the spheres in the cones in oriented contact. Then the intersection of $\mathcal{P}$ and $\mathcal{Q}$ is rationally stable, and is categorized by comparing the radii of $p(u_0)$ and $q(v_0)$:

1. $\pi_4(p(u_0)) = \pi_4(q(v_0))$: the cones have two touching points, and the intersection is a pair of quadratic curves.
2. $\pi_4(p(u_0)) \neq \pi_4(q(v_0))$: the cones intersect in an isolated point.
3. $\pi_4(p(u_0)) = \pi_4(q(v_0))$ and $\pi_4(p(u_0)) \gtrless 0$: the cones have one touching point at the apex of the parabolic pencil of spheres generated by $p(u_0)$ and $q(v_0)$, and intersect in a quartic curve with a singularity at the touching point.

Likewise there is a rationally stable configuration of a cone and a sphere: when they have one touching point. This is the case if the sphere belongs to the isotropic quadric of the cone, as determined in Corollary 5, and not to the cone itself.

6. Rolling ball blends of rationally stable pairs of natural quadrics

The classification in Theorem 15 gives us the two rationally stable configurations of cones: the intersection curve is either a pair of quadric curves or a singular quartic. We also have a rationally stable configuration of a cone and a sphere with one touching point, intersecting in a singular quartic curve. In this section we will apply Algorithm 13 to these three cases to parametrize their fixed radius rolling ball blends.

6.1. Quadratic cone/cone blends

Using the cone/cone blend with an elliptic spine as an example, we show how Algorithm 13 is applied.

We start by calculating the Gaussian images of the touching curves: In Section 3 we determined the touching curves for one family of cones in oriented contact with an elliptic pipe surface, giving the touching curves $T_v(t)$ in \((12)\). A cone in this family is parametrized by the spheres

$$p_0 = (c, 0, v; R - v), \quad p_1 = (-c, 0, -b^2/v; R - b^2/v).$$

A cone from the other family in oriented contact with the pipe surface is parametrized by the spheres

$$q_0 = (c, 0, -v; R - v), \quad q_1 = (-c, 0, b^2/v; R - b^2/v).$$

The touching curve on this cone is

$$T_v(t) = (\pi_0(T_v^+(t)), \pi_1(T_v^+(t)), \pi_2(T_v^+(t)), -\pi_3(T_v^+(t)))^T.$$

Then the Gaussian images of $T_v^+(t)$ and $T_v^-(t)$ are

$$\beta_v^\pm(t) = \begin{pmatrix} a(v^2 + b^2)(1 + t^2) + 2c(v^2 - b^2)t \\ -4b^2vt \\ \pm(a(v^2 - b^2)(1 + t^2) + 2c(v^2 + b^2)t) \end{pmatrix}.$$

Note that $v$ is fixed for each of the cones. Let $v_0$ (resp. $v_1$) be the value associated with the cone with touching curve $\beta_0 = \beta_{v_0}^\pm$ (resp. $\beta_1 = \beta_{v_1}^\pm$).

The second step is to calculate the liftings of $\beta_0$ and $\beta_1$. In the elliptic case, we find $X = (X_0, X_1) = L(\beta_0)$:

$$X_0 = \gcd(2ct + a(1 + t^2), 2bt + ia(1 - t^2)) = v_0(t + \frac{c + ib}{a}),$$

$$X_1 = -\frac{b}{v_0} \frac{2ct + a(1 + t^2)}{2bt + ia(1 - t^2)} = -ib\left(t - \frac{c + ib}{a}\right)$$

and $Y = (Y_0, Y_1) = L(\beta_1)$:

$$Y_0 = \gcd(-2ct + a(1 + t^2), 2bt + ia(1 - t^2)) = b\left(t - \frac{c - ib}{a}\right),$$

$$Y_1 = -\frac{v_1}{b} \frac{2ct + a(1 + t^2)}{2bt + ia(1 - t^2)} = -iv_1\left(t + \frac{c - ib}{a}\right).$$

We then calculate the lifting coefficient $\lambda$

$$\lambda = \frac{2}{bv_0} \frac{(b^2 - c^2)t^2 + a^2(1 + t^4)}{(t + \frac{c + ib}{a})(t - \frac{c + ib}{a})} \equiv \left(t - \frac{c - ib}{a}\right)\left(t + \frac{c - ib}{a}\right).$$
and factor to retrieve $\lambda_0$ and $\lambda_1$

$$
\lambda_0 = t - \frac{c - ib}{a}, \quad \lambda_1 = t + \frac{c + ib}{a}.
$$

(56)

Combining $X$, $Y$, $\lambda_0$, and $\lambda_1$, we calculate the lifting $(1-u)\lambda_0X + u\lambda_1Y$ of the arc of the circle $N_t(u)$ between $\beta^0(t)$ and $\beta^1(t)$ on the Gaussian sphere:

$$
\begin{align*}
&\left(\frac{v_0(1-u)+bu}{1-u}, \frac{b(1-u)(t^2 - \frac{2t}{a} + 1) + v_1u(t^2 + \frac{2t}{a} + 1)}{1-u}\right).
\end{align*}
$$

(57)

By applying $P_S$ to the lifting, we get the parametrization of the patch on the Gaussian sphere.

We summarize the results of these calculations in the following theorem:

\textbf{Theorem 16.} Let $\mathcal{P}$ and $\mathcal{Q}$ be two cones whose $R$-offsets intersect in the ellipse $\mathcal{E}$. If the offset cones belong to opposite families, the rolling ball blend of radius $R$ of $\mathcal{P}$ and $\mathcal{Q}$ has a minimal rational parametrization $B_{\mathcal{E}}(t, u)$ given by

$$
B_{\mathcal{E}}(t, u) = \mathcal{E}(t) + R \left[ P_S((1-u)\lambda_0X + u\lambda_1Y) \right],
$$

(58)

where

$$
X = \begin{pmatrix} v_0(t + \alpha) \\ -ib(t - \alpha) \end{pmatrix}, \quad Y = \begin{pmatrix} b(t - \alpha) \\ -iv_1(t + \alpha) \end{pmatrix},
$$

(59)

$$
\lambda = \begin{pmatrix} t - \alpha \\ t + \alpha \end{pmatrix}, \quad \alpha = \frac{c + ib}{a}.
$$

(60)

The parametrization has minimal bi-degree $(6, 2)$.

\textbf{Proof.} $X$, $Y$ and $\lambda$, are of degree 1, so $(1-u)\lambda_0X + u\lambda_1Y$ is of bi-degree $(2, 1)$, and $P_S((1-u)\lambda_0X + u\lambda_1Y)$ is of bi-degree $(4, 2)$. Adding the rational function $\mathcal{E}(t)$ of bi-degree $(2, 0)$ gives us a parametrization of bi-degree $(6, 2)$. \hfill \Box

In order to parametrize the blend in the hyperbolic case, consider a cone in the family described in Section 3.2, parametrized by the spheres

$$
p_0 = (c, 0, v_0; R - v_0), \quad p_1 = (-c, 0, b^2/v_0; R + b^2/v_0).
$$

(61)

A cone in the opposite family is parametrized by

$$
q_0 = (c, 0, -v_1; R - v_1), \quad q_1 = (-c, 0, -b^2/v_1; R + b^2/v_1).
$$

(62)

Applying the algorithm, we find

\textbf{Theorem 17.} Let $\mathcal{P}$ and $\mathcal{Q}$ be two cones whose $R$-offsets intersect in the hyperbola $\mathcal{H}$. If the offset cones belong to opposite families, the rolling ball blend (Fig. 9) of radius $R$ of $\mathcal{P}$ and $\mathcal{Q}$ has a minimal rational parametrization given by

$$
B_{\mathcal{H}}(t, u) = \mathcal{H}(t) + R \left[ P_S((1-u)\lambda_0X + u\lambda_1Y) \right],
$$

(63)

where

$$
X = \begin{pmatrix} v_0(\alpha t + 1) \\ -ib(\alpha t - 1) \end{pmatrix}, \quad Y = \begin{pmatrix} b(\alpha t - 1) \\ -iv_1(\alpha t + 1) \end{pmatrix},
$$

(64)

$$
\lambda = \begin{pmatrix} t - \alpha \\ t + \alpha \end{pmatrix}, \quad \alpha = \frac{a + ib}{c}.
$$

(65)

The parametrization has minimal bi-degree $(6, 2)$.

In the parabolic case, consider a cone in the family described in Section 3.2, parametrized by the spheres

$$
p_0 = (a, 0, v_0; R - v_0), \quad p_1 = \left(a - v_0^2/a, 0, 2v_0; R\right).
$$

(66)

A cone in the opposite family is parametrized by

$$
q_0 = (a, 0, -v_1; R - v_1), \quad q_1 = \left(a - v_1^2/a, 0, -2v_1; R\right).
$$

(67)

Applying the algorithm, we find
Theorem 18. Let \( \mathcal{P} \) and \( \mathcal{Q} \) be two cones whose \( R \)-offsets intersect in the parabola \( \mathcal{P} \). If the offset cones belong to opposite families, the rolling ball blend (Fig. 9) of radius \( R \) of \( \mathcal{P} \) and \( \mathcal{Q} \) has a minimal rational parametrization \( B_P(t, u) \) given by

\[
B_P(t, u) = \mathcal{P}(t) + R \left[ P_S((1 - u)\lambda_0 X + u\lambda_1 Y) \right],
\]

where

\[
X = \begin{pmatrix} v_0 \\ a + it \end{pmatrix}, \quad Y = \begin{pmatrix} a - it \\ v_1 \end{pmatrix}, \quad \lambda = \begin{pmatrix} a - it \\ a + it \end{pmatrix}.
\]

The parametrization has minimal bi-degree \((6, 2)\).

6.2. Quartic cylinder/cylinder blends

By a careful selection of the local coordinate system of the quartic spine, we can also give the general parametrization of the blend of two cylinders with one touching point (see the supplementary materials for the construction of the parametrization of the spine). Let \( \mathcal{P} \) and \( \mathcal{Q} \) be two cylinders whose \( R \)-offsets have a common touching point in the origin, such that the common tangent plane in that point is \( z = 0 \) oriented in the positive \( z \)-direction. The offset cylinders are parametrized by the spheres \( p_0 = (a, b, -r_0, -r_0) \) and \( p_1 = (0, 0, -r_0, -r_0) \), and \( q_0 = (1, 0, -r_1, -r_1) \) and \( q_1 = (0, 0, -r_1, -r_1) \), and without loss of generality we can assume that \( r_1 \geq r_0 \geq 0 \), \( b > 0 \), and \( a^2 + b^2 = 1 \).

The intersection curve of the two offset cylinders can then be parametrized by

\[
s(t) = \begin{pmatrix} -b(r_1(1 + t^4) + 2(r_1 - 2r_0)t^2) \\ 2r_0r_1(4b_0^2(1 + t^2) + 2\alpha_1 t(1 - t^2)) \\ 2r_0r_1b(1 - t^2)^2 \end{pmatrix}.
\]

By applying Algorithm 13, we find the touching curves \( \beta^0 \) and \( \beta^1 \) on the Gaussian sphere

\[
\beta^0 = \begin{pmatrix} r_1(1 + t^4) + 2(r_1 - 2r_0)t^2 \\ -42r_0r_1(1 - t^2)t \\ 4ar_1\alpha_1(1 - t^2)t \\ r_1(1 + t^4) - 2(3r_1 - 2r_0)t^2 \end{pmatrix},
\]

\[
\beta^1 = \begin{pmatrix} -r_1(1 + t^4) - 2(r_1 - 2r_0)t^2 \\ 0 \\ 2r_0\alpha_0(1 - t^4) + 2r_1t^2 \end{pmatrix}
\]

and the parameters of the blend:

Theorem 19. Let \( \mathcal{P} \) and \( \mathcal{Q} \) be two cylinders whose \( R \)-offsets have a common touching point in the origin and common tangent plane \( z = 0 \), as given above. The rolling ball blend of radius \( R \) of \( \mathcal{P} \) and \( \mathcal{Q} \) has a minimal rational parametrization

\[
B(t, u) = s(t) + R \left[ P_S((1 - u)\lambda_0 X + u\lambda_1 Y) \right].
\]
Using the same approach as above, it is possible to parametrize the intersection of two cones with one touching point (see the supplementary materials for the construction of the parametrization of the spine). However, the lack of symmetry in cone/cone configurations with only one touching point drastically increases the length of the expressions. Let \( P = (a, b, r_0, r_1, R) = (0.5, \sqrt{1 - 0.3^2}, 0.7, 1.0, 0.3) \), and corner blend with three different edge radii.

\[
X = \left( \frac{-i(1 - t^2)}{2\alpha_1(a - ib)t} \right), \quad Y = \left( \frac{i(1 - t^2)}{\alpha_0(1 + t^2)} \right), \quad \lambda = \begin{pmatrix} 1 + t^2 - (\alpha - \sqrt{\alpha^2 + 4})t \\ 1 + t^2 - (\alpha + \sqrt{\alpha^2 + 4})t \end{pmatrix}. 
\]

where
\[
\alpha_0 = \sqrt{\frac{r_1 - r_0}{r_0}}, \quad \alpha_1 = \sqrt{\frac{r_1 - r_0}{r_1}}, \quad \alpha = (a + ib)\alpha_0\alpha_1.
\]

The parametrization has minimal bi-degree \((12, 2)\).

**Remark 20.** Quartic cylinder blends occur e.g. in corner blends. Consider the corner in Fig. 10, where three planes intersect at right angles. Each of the three edges of the corner are blended by cylinders of different radii. The corner itself is blended between the middle and the smallest cylinder. The latter is a patch on a torus, the cylinder/cylinder blend is a quartic patch as described above.

### 6.3. Quartic cone/cone blends

Using the same approach as above, it is possible to parametrize the intersection of two cones with one touching point (see the supplementary materials for the construction of the parametrization of the spine). However, the lack of symmetry in cone/cone configurations with only one touching point drastically increases the length of the expressions. Let \( P \) and \( Q \) be two cones whose \( R \)-offsets have a common touching point in the origin, such that the common tangent plane in that point is \( z = 0 \) oriented in the positive \( z \)-direction. The offset cones are parametrized by the spheres \( p_0 = (0, 0, -r, -r) \) and \( q_0 = (0, 0, -1, -1) \) and \( q_1 = (A, 0, 0, 0) \), and without loss of generality we can assume that \( 0 < r < 1, \ b > 0, \) and \( A > 0 \). The intersection curve of the two offset cones can then be parametrized by

\[
s(t) = \begin{pmatrix} \alpha_2^+ + \alpha_2(1 - t^2)t - (\alpha_5^+ + \alpha_5_+ + 2(1 - r)\alpha_1)t^2 + \alpha_5^- t^4 \\ A(\alpha_4^+ + \alpha_4_+ + 2(1 - t)(1 - t^2)) \\ A(\alpha_4^- + \alpha_4^- + 2(1 - t^2)) \\ A\alpha_2^+ a^2 r(1 - t^2) \\ A^2b^2 r(1 - t^2) \end{pmatrix},
\]

where
\[
v_0 = a + ib, \quad v_1 = A - a - ib, \quad \alpha_0 = \sqrt{1 - r\sqrt{A}} - r, \quad \alpha_1 = \alpha_5 = \pm \alpha_0 (A - a), \quad \alpha_2^\pm = \pm r(1 - r) - \alpha_1, \quad \alpha_3 = \frac{1}{2}((v_0^2 - A)^2 r(1 - r) - \alpha_1) \pm \alpha_0 A.
\]

By applying Algorithm 13, we find the parametrization of the fixed radius rolling ball blend of the two cones (see Fig. 11):

**Theorem 21.** Let \( P \) and \( Q \) be two cones whose \( R \)-offsets have a common touching point in the origin and common tangent plane \( z = 0 \), as given above. The rolling ball blend of radius \( R \) of \( P \) and \( Q \) has a minimal rational parametrization

\[
B(t, u) = s(t) + R \left[ \begin{pmatrix} p_s((1 - u)\lambda_0 X + u\lambda_1 Y) \end{pmatrix} \right].
\]
Fig. 11. Rolling ball blend of two cones with one common touching point (parameters \((a, b, A, r, R) = (3, 4, 5, 0.75, 0.25)\)).

Fig. 12. Sphere/cylinder blend (parameters \((r, R) = (0.7, 0.5)\)), and its Viviani type quartic spine curve.

\[
X = \left( \begin{array}{c} \alpha_0 \sqrt{r} v_0(1 - t^2) \\ i(2\alpha_0|v_0|t + r^{3/2}v_0v_1(1 - t^2)) \end{array} \right),
\]

\[
Y = \left( \begin{array}{c} A\sqrt{r} \alpha_0 r(1 - t^2) \\ i(\alpha_0 - \bar{\alpha}_1 r + (\alpha_0 + \bar{\alpha}_1)r^2) \end{array} \right),
\]

\[
\lambda = (1, t, t^2, t^3, t^4) \left( \begin{array}{c} r^{3/2}v_0(\alpha_1 - v_1\alpha_0) \\ 2|v_0|\alpha_0(\bar{\alpha}_1 r - \alpha_0) \\ -2r^{3/2}v_0\alpha_1 \\ -2|v_0|\alpha_0(\bar{\alpha}_1 r + \alpha_0) \\ r^{3/2}v_0(\alpha_1 + v_1\alpha_0) \end{array} \right),
\]

where
\[
\alpha_0 = \sqrt{1 - r^{1/2}}\alpha_1, \quad \alpha_1 = r|v_1|^2 + A^2b^2, \quad v_0 = a + ib, \quad v_1 = A - a - ib.
\]

The parametrization has minimal bi-degree \((12, 2)\).

**Proof.** \(X_1\) and \(Y_1\) are of degree two in \(t\), and since \(\lambda\) is of degree four, \(\lambda_0\) and \(\lambda_1\) of degree two in \(t\). Thus the lifting \((1 - u)\lambda_0 X + u\lambda_1 Y\) is of degree four in \(t\) and one in \(u\). As \(P_S\) doubles the degrees, and the spine of the blend is also of degree four in \(t\), this gives the blend \(B(t, u)\) the minimal bi-degree \((12, 2)\).

6.4. Quartic sphere/cylinder blends – Viviani’s Curve

A classic example of a sphere/cone intersection is Viviani’s Curve – the intersection of a sphere and cylinder with one touching point, where the radius of the cylinder is half of the radius of the sphere (see Fig. 12). Consider the sphere \(S = (0, 0, -1, R - 1)\) and the cylinder parametrized by the two spheres \(P_0 = (0, 0, -r, R - r)\) and \(P_1 = (1, 0, -r, R - r)\), where \(0 < r < 1\). Then the spine curve for the rolling ball blend of fixed radius \(R\) is parametrized by

\[
s(t) = \begin{pmatrix} -2\sqrt{r(1-r)} \frac{1 - t^2}{1 + t^2}, -4rt \frac{1 - t^2}{1 + t^2}, -2r \frac{(1 - t^2)^2}{(1 + t^2)^2} \end{pmatrix}^T
\]

using the same approach as in the parametrization of the singular quartic intersection of two cylinders. By applying Algorithm 13, we parametrize the rolling ball blend with spine \(s(t)\):
Theorem 22. Consider a sphere and a cylinder whose R-offsets intersect in the quartic curve \( s(t) \). The rolling ball blend of radius R of the intersection has a minimal rational parametrization \( B(t, u) \) given by

\[
B(t, u) = s(t) + R \left[ P_S((1 - u)\lambda_0 X + u\lambda_1 Y) \right],
\]

where

\[
X = \left( \frac{1 - t^2}{-2it} \right), \quad Y = \left( \frac{\sqrt{t}(1 - t^2)}{\sqrt{1-t}(1 + t^2) - 2i\sqrt{t}} \right),
\]

\[
\lambda = \left( \frac{\sqrt{t}i + (1 + \sqrt{1-t})}{\sqrt{t}i + (1 - \sqrt{1-t})} \right).
\]

The parametrization has minimal bi-degree \((8, 2)\).

Proof. \( X \) and \( Y \) are of degree 2 in \( t \), and \( \lambda \) of degree 1, so \((1 - u)\lambda_0 X + u\lambda_1 Y\) is of bi-degree \((3, 1)\), and \( P_S((1 - u)\lambda_0 X + u\lambda_1 Y)\) of bi-degree \((6, 2)\). Adding the rational function \( s(t) \) of bi-degree \((4, 0)\) gives us a parametrization of bi-degree \((10, 2)\). However, \(1 + t^2\) is a factor of the denominators of both \( s(t) \) and \( P_S((1 - u)\lambda_0 X + u\lambda_1 Y)\), so the minimal bi-degree of the blend is reduced to \((8, 2)\). \( \Box \)

The general case of sphere/cone blends can be calculated following Algorithm 13.

7. Conclusions

Four configurations of natural quadrics have fixed radius rolling ball blends with rational parametrizations: plane/cone intersections, two cones with one or two touching points, and cone/sphere intersections with one touching point. We have presented closed expressions for the rational parametrizations of the fixed radius rolling ball blends in these cases, as well as a general algorithm for rational parametrizations of fixed radius rolling ball blends. The blends are all of minimal bi-degree.

Further research will investigate parametrizations of variable radius rolling ball blends, as well as volumetric representations of blends in e.g. isogeometric analysis.

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Supplementary material

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References


